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A Davey–Stewartson-Related Scattering Problem with Nonsmall Potential

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1. INTRODUCTION

In this paper, we solve the scattering problem of the following linear system

$$\begin{pmatrix} \frac{2}{i}\partial_{\bar{x}} & 0 \\ 0 & \frac{2}{i}\partial_x \end{pmatrix} \Psi(x) = \begin{pmatrix} 0 & q_1+1 \\ q_2+1 & 0 \end{pmatrix} \Psi(x) \tag{1.1}$$

in $\mathbf{R}^2$, where q_1 and q_2 are small, $x = x_1 + ix_2$, $\partial_{\bar{x}} = \frac{1}{2}(\partial/\partial x_1 + i(\partial/\partial x_2))$, $\partial_x = \frac{1}{2}(\partial/\partial x_1 - i(\partial/\partial x_2))$, and generate a corresponding hierachy of nonlinear evolutions of $q_1 + 1$, $q_2 + 1$. For a direct application, we see that the initial value problem of the Davey–Stewartson [6] system with nonsmall potential could be solved explicitly.

We follow the argument of the $\bar{\partial}$-method to solve the above problem, which was introduced by R. Beals and R. R. Coifman [1, 5]. By recasting the scattering data as $\bar{\partial}$-data, they gave complete and systematic analysis of the scattering problems for general classes of linear systems in one and higher dimensional spaces [2, 3].

The inverse scattering problem of the Davey–Stewartson related linear systems with small potential was solved by A. S. Fokas and M. J. Ablowitz [7] and R. Beals and R. R. Coifman [3, 4]. In their papers, the smallness of the potential assures the compactness of some integral operators. Here by setting our potential approximately constant, we still have the compactness property. However, our analysis has a very different feature due to the occurrence of the discontinuous $\bar{\partial}$-data. The nonsmoothness relates to the soliton solutions in 2 + 1, as was pointed out by A. S. Fokas and M. J. Ablowitz in their paper [7].

The plan of this paper is as follows. In Sections 2 and 3 we deal with the direct problem of finding the class of potentials in which the scattering transform may exist. We also construct the scattering transform on them.

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In Section 4 we deal with the inverse problem of finding the potentials of scattering transforms. In Section 5 we derive a hierachy of evolution equations of the potentials in terms of the evolutions of their scattering transforms. And in Section 6 we apply our theory to prove the solvability of the Davey–Stewartson equations.

2. The Existence of Eigenfunctions

In this section, we show that if q_i are sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, we can find a λ-parametrized family of solutions of (1.1), $\lambda \in \mathbf{C}$, with $|\lambda| \neq 0, 1$.

If we denote

$$P(D)=\begin{pmatrix} \frac{2}{i}\partial_{\bar{x}} & -1 \\ -1 & \frac{2}{i}\partial_x \end{pmatrix}, \qquad Q(x)=\begin{pmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{pmatrix}$$

and set

$$\Psi_\lambda(x)=\Psi(x,\lambda)=\Phi(x,\lambda)\, m(x,\lambda),$$

with

$$\Phi(x,\lambda)=e^{i(x_1((\lambda+1/\lambda)/2)+x_2((\lambda-1/\lambda)/2i))}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad |\lambda| \neq 0, 1,$$

then (1.1) reduces to

$$P_\lambda(D)\, m=\begin{pmatrix} \frac{2}{i}\partial_{\bar{x}}+\lambda & -1 \\ -1 & \frac{2}{i}\partial_x+\frac{1}{\lambda} \end{pmatrix} m=Qm. \tag{2.1}$$

By posing the boundary condition

$$m \to \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \qquad \text{as} \quad |x| \to \infty$$

and taking the Fourier transform on both sides of (2.1), we obtain the integral equation

$$m(x,\lambda)=\begin{pmatrix} 1 \\ \lambda \end{pmatrix}+G * Qm(x,\lambda), \tag{2.2}$$

where

$$G(x, \lambda)=\frac{1}{(2\pi)^2}\int \frac{\begin{pmatrix} \bar{\xi}+1/\lambda & 1 \\ 1 & \xi+\lambda \end{pmatrix}}{|\xi|^2+\lambda\bar{\xi}+\xi/\lambda} e^{i(x_1\xi_1+x_2\xi_2)}\, d\xi_1\, d\xi_2. \tag{2.3}$$

For simplicity, we define $d\xi = d\xi_1\, d\xi_2$ and $x\cdot\xi = x_1\xi_1 + x_2\xi_2$, for every pair of complex numbers for the rest of the paper.

The existence of $m(x, \lambda)$ follows from the following lemmas.

LEMMA 1. *If f, g are differentiable functions on the interval* $[-1, 1]$, $g(0)\neq 0$, *then*

$$\left|\int_{-1}^{1} \frac{f(r)}{r+ig(r)}\, dr\right| \leqslant C(\|f\|_\infty + \|f'\|_\infty + \|f\|_\infty \|g'\|_\infty + \|f'\|_\infty \|g'\|_\infty (1+\|g'\|_\infty)),$$

for some positive constant C.

Proof. This lemma follows by elementary argument. We omit the details.

LEMMA 2. *Suppose $\omega, \beta \in \mathbf{C}$, $|\omega|=1$, $|\beta|<1$, $\omega\cdot\beta=0$, and*

$$g_{\omega,\beta}(x)=\int \frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\, d\xi.$$

Then $g_{\omega,\beta}\in C^\infty(R^2\backslash 0)$, and

$$|g_{\omega,\beta}(x)| \leqslant \begin{cases} C(1-\log|x|), & \text{as } |x|\to 0, \\ C\dfrac{1}{|x|}, & \text{as } |x|\to\infty \text{ and } |\beta|\geqslant \dfrac{1}{2}, \\ C\dfrac{1}{|x|^{1/2}}, & \text{as } |x|\to\infty \text{ and } |\beta|\leqslant\dfrac{1}{2}, \end{cases}$$

for some positive constant C independent of ω and β.

Proof. We choose $\chi(\xi)$ to be a smooth function with support contained in $\{|\xi|<1\}$, satisfying $\chi=1$ on $\{|\xi|\leqslant\frac{1}{2}\}$, and $0\leqslant\chi\leqslant 1$. Moreover, set $\chi_k=\chi(\xi/k)$.

$$\begin{aligned} g_{\omega,\beta}(x) &= \int \frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}(1-\chi_3(\xi))\, d\xi \\ &\quad + \int \frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\chi_3(\xi)\, d\xi \\ &= I + II. \end{aligned}$$

Case 1. $|x|\to 0$. First of all, we can use polar coordinate and Lemma 1 to show that II is uniformly bounded. Second, note that away from singularities,

$$\left|(\partial^k/\partial\xi^k)\frac{1}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\right|\sim\frac{1}{r^{2+k}},\qquad k=0,1,2.$$

Then by integration by parts,

$$\begin{aligned}|I|&=\left|\int\frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}(1-\chi_3)\chi_N\right.\\&\quad\left.+\int\frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}(1-\chi_3)(1-\chi_N)\right|\\&\leqslant C\int_0^{2\pi}\int_{3/2}^N\frac{1}{r}\,dr\,d\theta+\frac{1}{|x|^2}\left|\int(\Delta_\xi e^{ix\cdot\xi})\frac{1-\chi_N}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\,d\xi\right|\\&\leqslant C\log N+\frac{1}{|x|^2}\left(\int(1-\chi_N)\left|\Delta_\xi\frac{1}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\right|d\xi\right.\\&\quad+\int|\nabla_\xi(1-\chi_N)|\left|\nabla_\xi\frac{1}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\right|d\xi\\&\quad\left.+\int|\Delta_\xi(1-\chi_N)|\left|\frac{1}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\right|d\xi\right)\\&\leqslant C\log N+\frac{C}{|x|^2}\left(\int_{N/2}^\infty\frac{r}{r^4}\,dr+\frac{1}{N}\int_{N/2}^N\frac{r}{r^3}\,dr+\frac{1}{N^2}\int_{N/2}^N\frac{r}{r^2}\,dr\right)\\&\leqslant -C\log|x|,\end{aligned}$$

if we take $N=1/|x|$.

Case 2. $|x|\to\infty$, $\beta\geqslant\frac{1}{2}$. For $\forall N$, $|I|<C_N|x|^{-N}$ follows from a standard argument of integration by parts. For part II, since $\beta\geqslant\frac{1}{2}$, $|\xi-\omega|^2-1+2i\beta\cdot\xi=0$ has two roots, say 0, and ξ_0, where $|\xi_0|=2$. As a result, we can reduce the estimate of II to those of II_1 and II_2, where

$$\begin{aligned}II=II_1+II_2&=\int\frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\chi_3\chi_1\\&\quad+\int\frac{e^{ix\cdot\xi}}{|\xi-\omega|^2-1+2i\beta\cdot\xi}\chi_3(1-\chi_1).\end{aligned}$$

Now let us define $p^0(\xi)=|\xi-\omega|^2-1+2i\beta\cdot\xi$, and write $p^0(\xi)=\eta_1+i\eta_2$, $\eta_i\in\mathbf{R}$. Suppose that $\chi_\varepsilon^0(\xi)$ is a localized function with respect to η at $\eta=0$; that is, support of $\chi_\varepsilon^0(\xi)$ is contained in $\{\xi:|p^0(\xi)|\leqslant\varepsilon\}$. Then if we change coordinate ξ to η, the Jacobian's at 0 and ξ_0 will be uniformly bounded. So to estimate II_1 (respectively, II_2), we can split II_1 with χ_ε^0 and change coordinate ξ to η. Then by similar method as in the above case (see [11] for details), we obtain

$$|II_1| < C\varepsilon + \frac{C}{|x|^2\,\varepsilon} < C\,\frac{1}{|x|},$$

if we take $\varepsilon = 1/|x|$.

Case 3. $|x| \to \infty$, $|\beta| \leqslant \frac{1}{2}$. $|I| < C_N(1/|x|^N)$ as in the above case. For part II, we use a similar method again, except that we take the split function $\chi_\phi(\theta)$ as a localized function on the θ domain after using the polar coordinate $\xi = re^{i\theta}$. For details, see [11]. Thus we get

$$|I| \leqslant C\varepsilon + \frac{C}{|x \log \varepsilon|} + \frac{C}{|x|\,\varepsilon} \leqslant C\,\frac{1}{|x|^{1/2}},$$

if we take $\varepsilon = 1/|x|^{1/2}$. ∎

LEMMA 3. *$G(x, \lambda) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ defined as in* (2.3) *satisfies*

$$|G_{12}(x, \lambda)| = |G_{21}(x, \lambda)| \leqslant \frac{C}{|\lambda| + 1/|\lambda|} \max\left(\frac{1}{|x|}, \frac{1}{|x|^{1/2}}\right),$$

$$|G_{11}(x, \lambda)|, |G_{22}(x, \lambda)| \leqslant C \max\left(\frac{1}{|x|}, \frac{1}{|x|^{1/2}}\right),$$

for some positive constant C independent of λ.

Proof. Note that $|\xi|^2 + \lambda\bar{\xi} + \xi/\lambda = |\xi + a|^2 - |a|^2 + 2ib \cdot \xi$, where $a = \frac{1}{2}(1 + 1/|\lambda|^2)\,\lambda$, $b = \frac{1}{2}(-1 + 1/|x|^2)(-\lambda_2 + i\lambda_1)$. So if we choose $\omega = -a/|a|$, $\beta = b/|a|$, then $G_{12} = G_{21} = (2\pi)^2\, g_{\omega,\beta}(|a|\,x)$, and the results for G_{12}, G_{21} follow immediately from Lemma 2. Estimate for G_{11} and G_{22} can be obtained by standard calculation and similar argument. ∎

Then we are ready to prove the existence of our eigenfunctions $m(x, \lambda)$.

THEOREM 1. *If the norm of Q is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, then for any complex number $|\lambda| \neq 0, 1$,* (2.2) *has a unique solution in L^∞.*

Proof. We first write $G = G^1 + G^2$, with $G^1 = G\chi_1$, and χ_1 is defined as in Lemma 2. By Lemma 3, $G^1 \leqslant C\,|x|^{-1}$, $G^2 \leqslant C\,|x|^{-1/2}$, so for every ε, $0 < \varepsilon < 1$, $G^1 \in L^{2-\varepsilon}$, $G^2 \in L^{4+\varepsilon}$. Consequently for every $Q \in L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, $G_\lambda * Q\cdot : L^\infty \to L^\infty$ has a norm less than $C(\|Q\|_{2+\varepsilon} + \|Q\|_{4/3-\varepsilon})$, which is independent of λ.

Therefore if $(\|Q\|_{2+\varepsilon}+\|Q\|_{4/3-\varepsilon})\leqslant 1/C$, then $(1-G*Q\cdot)$ is invertible in L^∞. So

$$m=(1-G*Q\cdot)^{-1}\begin{pmatrix}1\\ \lambda\end{pmatrix}\in L^\infty$$

is the unique solution of (2.2). ∎

3. The $\bar\partial$-Data and the Scattering Transform

In 1985, P. G. Grinevich and S. V. Manakov [9] formally solved the scattering problem of the Schrödinger operator

$$L=\partial_x\partial_{\bar x}+q(x),$$

where x is a complex number. In this section, we show how similar ideas can be exploited to obtain the $\bar\partial$-data in our case. Then we define the scattering transform and conclude the direct problem of (1.1).

Lemma 4. *If* $|\lambda|\neq 0, 1$, *and* $G=\begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}$, *then*

$$\frac{\partial}{\partial\bar\lambda}G(x,\lambda)=-\frac{1}{4\pi}\frac{\operatorname{sgn}(|\lambda|-1)}{\bar\lambda}e^{-ix\cdot(\lambda+1/\lambda)}\begin{pmatrix}-\bar\lambda & 1\\ 1 & -1/\bar\lambda\end{pmatrix}.$$

Proof. Let $p_\lambda(\xi)=|\xi|^2+\lambda\bar\xi+\xi/\lambda$. Then if $\xi=re^{i\theta}$, $p_\lambda(\xi)=(\bar\xi/\lambda)(\lambda-\frac12(r-\sqrt{r^2-4})\,e^{i(\theta+\pi)})(\lambda-\frac12(r+\sqrt{r^2-4})\,e^{i(\theta+\pi)})$. Suppose ϕ is a test function supported outside the unit circle. Since $(r-\sqrt{r^2-4})\leqslant 2$, $(r+\sqrt{r^2-4})\geqslant 2$, by residue theorem, we have

$$\begin{aligned}\int_\lambda\phi(\lambda)\frac{\partial G_{12}}{\partial\bar\lambda}\,d\lambda_1\,d\lambda_2&=-\frac{1}{(2\pi)^2}\int_\lambda\frac{\partial\phi(\lambda)}{\partial\bar\lambda}\int_\xi\frac{e^{ix\cdot\xi}}{|\xi|^2+\lambda\bar\xi+\xi/\lambda}\,d\xi\,d\lambda_1\,d\lambda_2\\&=-\frac{1}{(2\pi)^2}\int_{\xi(r,\theta)}\int_\lambda\frac{1}{\lambda-\frac12(r+\sqrt{r^2-4})\,e^{i(\theta+\pi)}}\\&\quad\times\frac{\partial}{\partial\bar\lambda}\left(\frac{\lambda e^{ix\cdot\xi}\phi(\lambda)}{\bar\xi(\lambda-\frac12(r-\sqrt{r^2-4})\,e^{i(\theta+\pi)})}\right)d\lambda_1\,d\lambda_2\,d\xi\\&=\frac{1}{4\pi}\int_{\xi(r,\theta)}\frac{e^{ix\cdot\xi}\phi(\lambda)}{\partial p_\lambda(\xi)/\partial\lambda}\bigg|_{\lambda=(1/2)(r+\sqrt{r^2-4})\,e^{i(\theta+\pi)}}r\,dr\,d\theta.\end{aligned}$$

By a change of variables: $r = R + 1/R$, $R \geqslant 1$, $\theta = u - \pi$, we get $\xi = -(\eta + 1/\bar{\eta})$, where $\eta = Re^{iu}$ and its Jacobian $= 1 - 1/R^2$. The above expression then becomes

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_{R\geqslant 1} \frac{e^{-ix\cdot(\eta+1/\eta)}\phi(\eta)}{-(\eta+1/\bar{\eta}) - (1/\eta^2)(-(\eta+1/\bar{\eta}))}\left(R+\frac{1}{R}\right)\left(1-\frac{1}{R^2}\right) dR\, du$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_{R\geqslant 0} \frac{e^{-ix\cdot(\eta+1/\eta)}\phi(\eta)}{-(\eta+1/\bar{\eta}) - (1/\eta^2)(-(\eta+1/\bar{\eta}))}\left(R+\frac{1}{R}\right)\left(1-\frac{1}{R^2}\right) dR\, du$$

$$= -\frac{1}{4\pi}\int_\eta \frac{1}{\bar{\eta}} e^{-ix\cdot(\eta+1/\bar{\eta})}\phi(\eta)\, d\eta_1\, d\eta_2.$$

Consequently if $|\lambda| > 1$,

$$\frac{\partial}{\partial\bar{\lambda}} G_{12}(x, \lambda) = -\frac{1}{4\pi}\frac{1}{\bar{\lambda}} e^{-ix\cdot(\lambda+1/\lambda)}.$$

On the other hand if we take ϕ to be a test function supported inside the unit circle, by a similar argument as above, but with $\lambda - \frac{1}{2}(r - \sqrt{r^2-4})\, e^{i(\theta+\pi)}$, $\lambda - \frac{1}{2}(r + \sqrt{r^2-4})\, e^{i(\theta+\pi)}$ replaced by $\lambda - \frac{1}{2}(r + \sqrt{r^2-4})\, e^{i(\theta+\pi)}$, $\lambda - \frac{1}{2}(r - \sqrt{r^2-4})\, e^{i(\theta+\pi)}$, and with the condition $R \geqslant 1$ replaced by the condition $R \leqslant 1$, Jacobian $= -(1 - 1/R^2)$, we can prove that for every $|\lambda| \neq 0, 1$,

$$\frac{\partial}{\partial\bar{\lambda}} G_{12}(x, \lambda) = -\frac{1}{4\pi}\frac{\operatorname{sgn}(|\lambda| - 1)}{\bar{\lambda}} e^{-ix\cdot(\lambda+1/\lambda)}.$$

Similarly we can prove for the cases G_{21}, G_{11}, and G_{22}. ▮

For simplicity, we write

$$\partial_{\bar{\lambda}} = \frac{\partial}{\partial\bar{\lambda}},$$

$$G_\lambda(x) = G(x, \lambda),$$

$$\xi_\lambda = -\left(\lambda + \frac{1}{\bar{\lambda}}\right),$$

for the rest of the paper.

LEMMA 5. *If* $|\lambda| \neq 0, 1$, *then*

$$e^{-ix\cdot\xi_\lambda} G_\lambda * e^{ix\cdot\xi_\lambda} f = G_{-1/\bar{\lambda}} * f.$$

Proof. This follows directly by a change of variables. ▮

THEOREM 2. *If Q is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, $Q \in L^1$, $|\lambda| \neq 0, 1$, then*

$$\partial_{\bar{\lambda}} m(x, \lambda) = -\frac{1}{4\pi} \frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}} e^{ix\xi_\lambda}(-\bar{\lambda}\widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda)\, m\left(x, -\frac{1}{\bar{\lambda}}\right).$$

Proof. Let $m = \binom{m_1}{m_2}$. Then by (2.2), Lemma 4, and Lemma 5,

$$\begin{aligned}
\partial_{\bar{\lambda}} m(x, \lambda) &= -\frac{1}{4\pi} \frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}} (1 - G_\lambda * Q\cdot)^{-1} e^{ix\cdot\xi_\lambda} \\
&\quad \times \begin{pmatrix} -\bar{\lambda} & 1 \\ 1 & -1/\bar{\lambda} \end{pmatrix} \widehat{Qm}(\xi_\lambda, \lambda) \\
&= -\frac{1}{4\pi} \frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}} e^{ix\cdot\xi_\lambda}(1 - G_{-1/\bar{\lambda}} * Q\cdot)^{-1} \\
&\quad \times \begin{pmatrix} -\bar{\lambda} & 1 \\ 1 & -1/\bar{\lambda} \end{pmatrix} \widehat{Qm}(\xi_\lambda, \lambda) \\
&= -\frac{1}{4\pi} \frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}} e^{ix\cdot\xi_\lambda}(1 - G_{-1/\bar{\lambda}} * Q\cdot)^{-1} \\
&\quad \times (-\bar{\lambda}\widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda) \begin{pmatrix} 1 \\ -1/\bar{\lambda} \end{pmatrix} \\
&= -\frac{1}{4\pi} \frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}} e^{ix\xi_\lambda}(-\bar{\lambda}\widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda)\, m(x, -1/\bar{\lambda}). \quad \blacksquare
\end{aligned}$$

Theorem 2 suggests that there is a jump of $m(x, \lambda)$ across the unit circle. In fact, the jump is the discontinuous part of the $\bar{\partial}$-data of $m(x, \lambda)$. In order to show this, we need some more lemmas.

LEMMA 6. *Let $\lambda = \mathrm{Re}^{i\phi}$. Then*

$$\lim_{R \to 1\mp} e^{i(x_1((\lambda + 1/\lambda)/2) + x_2((\lambda - 1/\lambda)/2i))} G(x, \lambda)$$

$$= \frac{1}{(2\pi)^2} \lim_{\varepsilon \to 0^+} \int \frac{e^{ix\cdot\xi} \begin{pmatrix} \bar{\xi} & 1 \\ 1 & \xi \end{pmatrix}}{|\xi|^2 - 1 \pm (e^{-i\phi}\xi - e^{i\phi}\bar{\xi})\,\varepsilon}\, d\xi.$$

Proof. By a change of variables $\omega = \xi + \frac{1}{2}(\lambda + 1/\bar{\lambda})$,

$$\lim_{R \to 1\mp} e^{i(x_1((\lambda + 1/\lambda)/2) + x_2((\lambda - 1/\lambda)/2i))} G(x, \lambda)$$

$$= \frac{1}{(2\pi)^2} \lim_{R \to 1\mp} \int \frac{e^{i\{x_1(\xi_1 + (\lambda + 1/\lambda)/2) + x_2(\xi_2 + (\lambda - 1/\lambda)/2i)\}}}{|\xi|^2 + \lambda\bar{\xi} + \xi/\lambda} \begin{pmatrix} \bar{\xi} + 1/\lambda & 1 \\ 1 & \xi + \lambda \end{pmatrix} d\xi$$

$$= \frac{1}{(2\pi)^2} \lim_{R \to 1\mp}$$

$$\times \int \frac{e^{i\{x_1(\omega_1 + (\lambda + 1/\lambda - \bar{\lambda} - 1/\bar{\lambda})/4) + x_2(\omega_2 + (\lambda - 1/\lambda + \bar{\lambda} - 1/\bar{\lambda})/4i)\}}}{|\omega|^2 - 1 + (e^{-i\phi}\omega - e^{i\phi}\bar{\omega})\frac{1}{2}(1/R - R) + \frac{1}{4}(2 - R^2 - 1/R^2)}$$

$$\times \begin{pmatrix} \bar{\omega} + 1/\lambda - \frac{1}{2}(\bar{\lambda} + 1/\lambda) & 1 \\ 1 & \omega + \lambda - \frac{1}{2}(\lambda + 1/\bar{\lambda}) \end{pmatrix} d\omega$$

$$= \frac{1}{(2\pi)^2} \lim_{\varepsilon \to 0^+} \int \frac{f(\omega, \phi, \varepsilon)}{|\omega|^2 - 1 \pm (e^{-i\phi}\omega - e^{i\phi}\bar{\omega})\,\varepsilon + o(\varepsilon)}\, d\omega,$$

where $d\omega = d\omega_1\, d\omega_2$, and

$$f(\omega, \phi, \varepsilon) = e^{i\{x_1(\omega_1 + (\lambda + 1/\lambda - \bar{\lambda} - 1/\bar{\lambda})/4) + x_2(\omega_2 + (\lambda - 1/\lambda + \bar{\lambda} - 1/\bar{\lambda})/4i)\}}$$
$$\times \begin{pmatrix} \bar{\omega} - \frac{1}{2}(\bar{\lambda} - 1/\lambda) & 1 \\ 1 & \omega + \frac{1}{2}(\lambda - 1/\bar{\lambda}) \end{pmatrix}.$$

By Lemma 1, in order to complete the proof, we only need to show that

$$\lim_{\varepsilon \to 0^+} \left(\int_{N_\delta} \frac{f(\omega, \phi, \varepsilon)}{|\omega|^2 - 1 \pm (e^{-i\phi}\omega - e^{i\phi}\bar{\omega})\,\varepsilon}\, d\omega - \int_{N_\delta} \frac{f(\omega, \phi, \varepsilon)}{|\omega|^2 - 1 \pm (e^{-i\phi}\omega - e^{i\phi}\bar{\omega})\,\varepsilon + o(\varepsilon)}\, d\omega \right) = 0,$$

and

$$\lim_{\varepsilon \to 0^+} \int_{N_\delta} \frac{f(\omega, \phi, \varepsilon) - e^{ix\cdot\omega} \begin{pmatrix} \bar{\omega} & 1 \\ 1 & \omega \end{pmatrix}}{|\omega|^2 - 1 \pm (e^{-i\phi}\omega - e^{i\phi}\bar{\omega})\,\varepsilon}\, d\omega = 0,$$

where $N_\delta = \{\omega = re^{i\theta} : |\sin(\theta - \phi)| \geqslant \delta,\ r > 0\}$, for all $\delta > 0$. However, this follows from Lemma 1 again. (See [11] for details.) Thus the proof is complete. ▮

Now let us define $\Psi_1(x, \mu, \lambda)$, $|\mu| = |\lambda| = 1$, which satisfies

$$\Psi_1(x, \mu, \lambda) = e^{ix \cdot \lambda} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + \int G_1(x-y, \mu)\, Q(y)\, \Psi_1(y, \mu, \lambda)\, dy, \tag{3.1}$$

with

$$G_1(x, \mu) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int \frac{e^{ix \cdot \xi}}{|\xi|^2 - 1 + (\bar{\mu}\xi - \mu\bar{\xi})\,\varepsilon} \begin{pmatrix} \bar{\xi} & 1 \\ 1 & \xi \end{pmatrix} d\xi. \tag{3.2}$$

By Lemma 6 and an argument similar to the proof of Theorem 1, Ψ_1 exists. Moreover we define $f_-(x, \lambda) = \lim_{R \to 1^-} f(x, R\lambda)$, and $f_+(x, \lambda) = \lim_{R \to 1^+} f(x, R\lambda)$ for any function $f \in L^\infty(\mathbf{C} \times \{\mathbf{C} \backslash S^1\}) \cap C^1(\mathbf{C} \times \{\mathbf{C} \backslash S^1\})$, $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Note that

$$\begin{aligned} \Psi_-(x, \lambda) &= \Psi_1(x, \lambda, \lambda), \\ \Psi_+(x, \lambda) &= \Psi_1(x, -\lambda, \lambda). \end{aligned} \tag{3.3}$$

Lemma 7. *With G_1 defined as in* (3.2),

$$\frac{\partial}{\partial \phi} G_1(x, e^{i\phi}) = -\frac{i}{4\pi} \left(e^{ix \cdot e^{i\phi}} \begin{pmatrix} e^{-i\phi} & 1 \\ 1 & e^{i\phi} \end{pmatrix} - e^{-ix \cdot e^{i\phi}} \begin{pmatrix} -e^{-i\phi} & 1 \\ 1 & -e^{i\phi} \end{pmatrix} \right).$$

Proof. Use the polar coordinate and set

$$f(x, \xi(r, \theta)) = e^{ix \cdot \xi(r, \theta)} \begin{pmatrix} \bar{\xi}(r, \theta) & 1 \\ 1 & \xi(r, \theta) \end{pmatrix} r.$$

Then by the dominate convergence theorem, we know that the limit

$$\lim_{\varepsilon \to 0^+} \int \frac{1}{r^2 - 1 + 2ir\varepsilon \sin(\theta - \phi)} \{f(x, \xi(r, \theta)) - f(x, \xi(1, \theta))\}\, dr\, d\theta$$

is independent of ϕ. So

$$\frac{\partial}{\partial \phi} \left\{ \lim_{\varepsilon \to 0^+} \int \frac{1}{r^2 - 1 + 2ir\varepsilon \sin(\theta - \phi)} \{f(x, \xi(r, \theta)) - f(x, \xi(1, \theta))\}\, dr\, d\theta \right\} = 0.$$

Furthermore for fixed θ, ϕ, let $r_\pm$ be the roots of $r^2 - 1 + 2ir\varepsilon \sin(\theta - \phi) = 0$, that is, $r_\pm = -i\varepsilon \sin(\theta - \phi) \pm \sqrt{1 - \varepsilon^2 \sin^2(\theta - \phi)}$. So

$$\begin{aligned}
\frac{\partial}{\partial\phi} G_1(x, e^{i\phi}) &= \frac{1}{(2\pi)^2} \frac{\partial}{\partial\phi} \lim_{\varepsilon\to 0^+} \int_0^{2\pi}\int_0^\infty \frac{f(x, \xi(r,\theta))}{r^2 - 1 + 2ir\varepsilon \sin(\theta-\phi)}\, dr\, d\theta \\
&= \frac{1}{(2\pi)^2} \frac{\partial}{\partial\phi} \lim_{\varepsilon\to 0^+} \int_0^{2\pi} f(x, \xi(1,\theta)) \\
&\quad \times \int_0^\infty \frac{1}{r^2 - 1 + 2ir\varepsilon \sin(\theta-\phi)}\, dr\, d\theta \\
&= \frac{1}{(2\pi)^2} \frac{\partial}{\partial\phi} \lim_{\varepsilon\to 0^+} \int_0^{2\pi} f(x, \xi(1,\theta)) \frac{1}{r_+ - r_-} \log \frac{r - r_+}{r - r_-}\bigg|_0^\infty d\theta \\
&= \frac{1}{(2\pi)^2} \frac{\partial}{\partial\phi} \lim_{\varepsilon\to 0^+} \int_0^{2\pi} f(x, \xi(1,\theta)) \frac{1}{2\sqrt{1-\varepsilon^2 \sin^2(\theta-\phi)}} \\
&\quad \times \log \frac{r - \sqrt{1-\varepsilon^2\sin^2(\theta-\phi)} + i\varepsilon\sin(\theta-\phi)}{r + \sqrt{1-\varepsilon^2\sin^2(\theta-\phi)} + i\varepsilon\sin(\theta-\phi)}\bigg|_0^\infty d\theta \\
&= -\frac{1}{8\pi^2}\frac{\partial}{\partial\phi}\int_0^{2\pi} f(x, \xi(1,\theta))\, i\pi(\operatorname{sgn}\{\sin(\theta-\phi)\})\, d\theta \\
&= -\frac{i}{4\pi}\left(e^{ix\cdot e^{i\phi}}\begin{pmatrix} e^{-i\phi} & 1 \\ 1 & e^{i\phi}\end{pmatrix} - e^{-ix\cdot e^{i\phi}}\begin{pmatrix} -e^{-i\phi} & 1 \\ 1 & -e^{i\phi}\end{pmatrix}\right). \quad \blacksquare
\end{aligned}$$

Now we are ready to find the discontinuous $\bar{\partial}$-data.

THEOREM 3. *If Q is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon} \cap L^1$, then we can find a function $R: S^1 \times S^1 \to \mathbf{C}$, such that*

$$\Psi_-(x, \lambda) - \Psi_+(x, \lambda) = \int_0^{2\pi} R(e^{i\phi}, \lambda)\, \Psi_+(x, e^{i\phi})\, d\phi,$$

$$m_-(x, \lambda) - m_+(x, \lambda) = \int_0^{2\pi} R(e^{i\phi}, \lambda)\, e^{ix\cdot(-\lambda + e^{i\phi})} m_+(x, e^{i\phi})\, d\phi,$$

and $\sup_{S^1\times S^1} |R| \leqslant C\,\|Q\|_1$ *for some positive constant C.*

Proof. We first write $\Psi_1 = \binom{\Psi_{1,1}}{\Psi_{1,2}}$. By (3.1), Lemma 7, and an argument similar to the proof of Theorem 2, we have

$$\begin{aligned}
&\frac{\partial}{\partial\phi}\Psi_1(x, e^{i\phi}, \lambda) \\
&\quad = -\frac{i}{4\pi}\{[e^{-i\phi}\widehat{q_1\Psi_{1,2}}(e^{i\phi}, e^{i\phi}, \lambda) + \widehat{q_2\Psi_{1,1}}(e^{i\phi}, e^{i\phi}, \lambda)]\, \Psi_1(x, e^{i\phi}, e^{i\phi}) \\
&\qquad - [-e^{-i\phi}\widehat{q_1\Psi_{1,2}}(-e^{i\phi}, e^{i\phi}, \lambda) \\
&\qquad + \widehat{q_2\Psi_{1,1}}(-e^{i\phi}, e^{i\phi}, \lambda)]\, \Psi_1(x, e^{i\phi}, -e^{i\phi})\} \\
&\quad = W_1(e^{i\phi}, \lambda)\, \Psi_1(x, e^{i\phi}, e^{i\phi}) - W_2(e^{i\phi}, \lambda)\, \Psi_1(x, e^{i\phi}, -e^{i\phi}),
\end{aligned}$$

or

$$\frac{\partial}{\partial\phi}\Psi_1(x, e^{i\phi}, \lambda) = W_1(e^{i\phi}, \lambda)\,\Psi_1(x, e^{i\phi}, e^{i\phi}) - W_2(e^{i\phi}, \lambda)\,\Psi_1(x, e^{i\phi}, -e^{i\phi}),$$

where

$$W_1(e^{i\phi}, \lambda) = -\frac{i}{4\pi}\left(e^{-i\phi}\widehat{q_1\Psi_{1,2}}(e^{i\phi}, e^{i\phi}, \lambda) + \widehat{q_2\Psi_{1,1}}(e^{i\phi}, e^{i\phi}, \lambda)\right),$$

$$W_2(e^{i\phi}, \lambda) = -\frac{i}{4\pi}\left(-e^{-i\phi}\widehat{q_1\Psi_{1,2}}(-e^{i\phi}, e^{i\phi}, \lambda) + \widehat{q_2\Psi_{1,1}}(-e^{i\phi}, e^{i\phi}, \lambda)\right).$$

Equation (3.3) then implies that

$$\Psi_-(x, \lambda) + \int_0^{2\pi} \tilde{W}_1(e^{i\phi}, \lambda)\,\Psi_-(x, e^{i\phi})\,d\phi$$
$$= \Psi_+(x, \lambda) + \int_0^{2\pi} \tilde{W}_2(e^{i\phi}, \lambda)\,\Psi_+(x, e^{i\phi})\,d\phi,$$

where

$$\tilde{W}_1(e^{i\phi}, \lambda) = \chi_{[\alpha, \alpha+\pi]}(\phi)\,W_1(e^{i\phi}, \lambda),$$
$$\tilde{W}_2(e^{i\phi}, \lambda) = \chi_{[\alpha-\pi, \alpha]}(\phi)\,W_2(-e^{i\phi}, \lambda), \qquad \alpha = \arg\lambda,$$

and χ_A is the characteristic function for the set A. Consequently, $\sup_{\lambda, \lambda'} |\tilde{W}_i(\lambda', \lambda)| \leqslant C\,\|Q\|_1$. Thus if Q is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon} \cap L^1$, we can conclude that

$$\Psi_-(x, \lambda) - \Psi_+(x, \lambda) = \left\{\sum_{k=1}^{\infty} (-1)^k \left(\int_0^{2\pi} \tilde{W}_1(e^{i\phi}, \lambda)\cdot\right)^k + \sum_{k=0}^{\infty} (-1)^k \left(\int_0^{2\pi} \tilde{W}_1(e^{i\phi}, \lambda)\cdot\right)^k \times \int_0^{2\pi} \tilde{W}_2(-e^{i\phi}, \lambda)\cdot\right\}\Psi_+\,d\phi.$$

By setting $R(e^{i\phi}, \lambda)$ to be the kernel of the above integral operator the theorem follows. ∎

For convenience, we renormalize $m(x, \lambda)$ by defining

$$\Lambda(x, \lambda) = \frac{1}{\lambda}\left(m(x, \lambda) - \begin{pmatrix}1\\ \lambda\end{pmatrix}\right).$$

COROLLARY 1. *If* $(\partial^k/\partial x_j^k)\, Q \in L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, $k=0, 1$, *and* Q *is small in* $L^{2+\varepsilon} \cap L^{4/3-\varepsilon} \cap L^1$, *then* $\Lambda(x, \lambda)$ *is a uniformly bounded function in the* λ *variable which satisfies,*

if $|\lambda| \neq 1$,

$$\partial_{\bar\lambda} \Lambda(x, \lambda) = -\frac{\operatorname{sgn}(|\lambda|-1)}{4\pi\,|\lambda|^2}\, e^{ix\cdot\xi_\lambda}(-\bar\lambda \widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda) \times \left(\frac{-\Lambda(x, -1/\bar\lambda)}{\bar\lambda} + \begin{pmatrix} 1 \\ -1/\bar\lambda \end{pmatrix}\right);$$

if $|\lambda| = 1$,

$$\Lambda_-(x, \lambda) - \Lambda_+(x, \lambda) = \frac{1}{\lambda}\int_0^{2\pi} R(e^{i\phi}, \lambda)\, e^{ix\cdot(-\lambda + e^{i\phi})} \times \left(e^{i\phi}\Lambda_+(x, e^{i\phi}) + \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}\right) d\phi,$$

where $R(\lambda', \lambda)$ *is as in Theorem* 3;

if $|\lambda| \to \infty$, $\Lambda(x, \lambda) \to \binom{0}{0}$.

Proof. This is obvious from the definition of Λ, Theorem 2, Theorem 3, and a similar argument as in Lemma 3. ∎

Our scattering transform is defined from the above corollary, that is, from the $\bar\partial$-data of Λ.

DEFINITION 1. If $(\partial^k/\partial x_j^k)\, Q \in L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, $k=0, 1$, and Q is small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon} \cap L^1$, then we define the scattering data (or the $\bar\partial$-data) S_c, S_d as

$$S_c(\lambda) = -\frac{1}{4\pi}\frac{\operatorname{sgn}(|\lambda|-1)}{|\lambda|^2}(-\bar\lambda\widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda),$$

when $|\lambda| \neq 0, 1$;

$$S_d(\lambda', \lambda) = R(\lambda', \lambda),$$

when $|\lambda| = |\lambda'| = 1$, R is as in Theorem 3; and the scattering transform $T\colon L^\infty(\mathbf{C} \times \{\mathbf{C}\backslash S^1\}) \cap C^1(\mathbf{C} \times \{\mathbf{C}\backslash S^1\}) \to L^\infty(\mathbf{C} \times \{\mathbf{C}\backslash S^1\}) \cap C^1(\mathbf{C} \times \{\mathbf{C}\backslash S^1\})$ as

$$Tf = \begin{cases} -\dfrac{e^{ix\cdot\xi_\lambda}}{\bar\lambda}\, S_c(\lambda)\, f\left(x, -\dfrac{1}{\bar\lambda}\right), & \text{for } |\lambda| \neq 1; \\[2ex] \dfrac{1}{\lambda}\displaystyle\int_0^{2\pi} e^{i\{\phi + x\cdot(-\lambda + e^{i\phi})\}} S_d(e^{i\phi}, \lambda)\, f_+(x, e^{i\phi})\, d\phi, & \text{otherwise.} \end{cases}$$

4. The Inverse Problem

We now turn to the problem of inverting the scattering transform: given the $\bar\partial$-data S_c, S_d, or equivalently, the scattering transform, try to find the corresponding Q.

Lemma 8. *Suppose* $f \in L^{2-\varepsilon}(\mathbf{C}\backslash S^1) \cap L^{2+\varepsilon}(\mathbf{C}\backslash S^1)$, $g \in C^\alpha(S^1)$, $0<\alpha<1$. *Then the following are equivalent.*

$$\begin{cases} \partial_{\bar\lambda} A(\lambda) = f(\lambda), & \text{if } \ |\lambda| \neq 1; \\ A_-(\lambda) - A_+(\lambda) = g(\lambda), & \text{if } \ |\lambda| = 1; \\ A \to C, & \text{as } \ |\lambda| \to \infty; \\ A \in L^\infty(\mathbf{C}\backslash S^1) \cap C^1(\mathbf{C}\backslash S^1). \end{cases} \tag{a}$$

$$A(\lambda) = -\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{f(\zeta)}{\zeta-\lambda}\, d\bar\zeta \wedge d\zeta + \frac{1}{2\pi i}\oint_{S^1} \frac{g(\zeta)}{\zeta-\lambda}\, d\zeta + C. \tag{b}$$

Proof. By a direct application of the Cauchy integral formula [10], we can prove that (a) implies (b). Conversely, suppose (b) holds. Since $1/\lambda * \cdot$ maps $L^{2-\varepsilon} \cap L^{2+\varepsilon}$ to C_0, A tends to C as $|\lambda| \to \infty$. Moreover if $|\lambda| \neq 1$, $\partial_{\bar\lambda} A(\lambda) = f(\lambda)$ is followed by an argument similar to the proof of Lemma 4. Finally, we can prove $A_-(\lambda) - A_+(\lambda) = g(\lambda)$ by Sokhotski formula [8]. ∎

Lemma 9. *Suppose* $(|\lambda| + 1/|\lambda|^2)\, S_c(\lambda)$ *is sufficiently small in* $L^{2-\varepsilon}(\mathbf{C}\backslash S^1) \cap L^{2+\varepsilon}(\mathbf{C}\backslash S^1)$ *and* $\sup_{S^1 \times S^1} |S_d(\lambda', \lambda)| \ll 1$. *Then there exists a unique function* $A \in L^\infty(\mathbf{C} \times \mathbf{C}) \cap C^1(\mathbf{C} \times \{\mathbf{C}\backslash S^1\})$ *such that,*

if $|\lambda| \neq 1$,

$$\partial_{\bar\lambda} A(x,\lambda) = e^{ix\cdot\xi_\lambda} S_c(\lambda) \left(-\frac{1}{\bar\lambda} A\left(x, -\frac{1}{\bar\lambda}\right) + \begin{pmatrix} 1 \\ -1/\bar\lambda \end{pmatrix} \right);$$

if $|\lambda| = 1$,

$$\begin{aligned} &A_-(x,\lambda) - A_+(x,\lambda) \\ &\quad = \frac{1}{\lambda}\int_0^{2\pi} S_d(e^{i\phi}, \lambda)\, e^{ix\cdot(-\lambda + e^{i\phi})} \left(e^{i\phi} A_+(x, e^{i\phi}) + \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \right) d\phi; \end{aligned}$$

if $|\lambda| \to \infty$, $A(x,\lambda) \to \binom{0}{0}$.

Proof. Let us first consider the integral equation

$$\begin{aligned}\Lambda(x,\lambda) = &-\frac{1}{2\pi i}\int_{\mathbf{C}}\frac{1}{\zeta-\lambda}e^{ix\cdot\xi_\zeta}S_c(\zeta)\left(-\frac{1}{\bar{\zeta}}\Lambda\left(x,-\frac{1}{\bar{\zeta}}\right)+\begin{pmatrix}1\\-1/\bar{\zeta}\end{pmatrix}\right)d\bar{\zeta}\wedge d\zeta\\ &+\frac{1}{2\pi i}\oint_{S^1}\frac{1}{\zeta-\lambda}\frac{1}{\zeta}\int_0^{2\pi}S_d(e^{i\phi},\zeta)\,e^{ix\cdot(-\zeta+e^{i\phi})}\\ &\times\left(e^{i\phi}\Lambda_+(x,e^{i\phi})+\begin{pmatrix}1\\e^{i\phi}\end{pmatrix}\right)d\phi,\end{aligned} \tag{4.1}$$

or

$$(1-CT\cdot)\,\Lambda = CT\begin{pmatrix}1/\lambda\\1\end{pmatrix}, \tag{4.2}$$

where

$$Cf(x,\lambda) = -\frac{1}{2\pi i}\int_{\mathbf{C}}\frac{f(x,\zeta)}{\zeta-\lambda}d\bar{\zeta}\wedge d\zeta+\frac{1}{2\pi i}\oint_{S^1}\frac{f(x,\zeta)}{\zeta-\lambda}d\zeta, \tag{4.3}$$

$$Tf(x,\lambda) = \begin{cases}-\dfrac{1}{\bar{\lambda}}e^{ix\cdot\xi_\lambda}S_c(\lambda)\,f\left(x,-\dfrac{1}{\bar{\lambda}}\right), & \text{if } |\lambda|\neq 1,\\ \dfrac{1}{\lambda}\displaystyle\int_0^{2\pi}S_d(e^{i\phi},\lambda)\,e^{ix\cdot(-\lambda+e^{i\phi})}e^{i\phi}f_+(x,e^{i\phi})\,d\phi, & \text{if } |\lambda|=1.\end{cases} \tag{4.4}$$

By an argument similar to the proof of Lemma 8, we can prove CT maps $L^\infty(\mathbf{C}\times\mathbf{C})\cap C(\mathbf{C}\times\{\mathbf{C}\backslash S^1\})$ to itself with norm less than some positive constant C_0 depending only on $\|S_c/\lambda\|_{L^{2-\varepsilon}\cap L^{2+\varepsilon}}+\|S_d\|_{L^\infty}$. So $(1-CT\cdot)$ is invertible and there exists $\Lambda(x,\lambda)\in L^\infty(\mathbf{C}\times\mathbf{C})\cap C(\mathbf{C}\times\{\mathbf{C}\backslash S^1\})$,

$$\Lambda(x,\lambda) = (1-CT\cdot)^{-1}\,CT\begin{pmatrix}1/\lambda\\1\end{pmatrix}$$

satisfying (4.1). By Lemma 8 again this lemma follows. ∎

Recall that

$$P_\lambda(D) = \begin{pmatrix}(2/i)\,\partial_{\bar{x}}+\lambda & -1\\ -1 & (2/i)\,\partial_x+1/\lambda\end{pmatrix}.$$

LEMMA 10. *If T is defined as in* (4.4), *then*

$$P_\lambda(D)\,Tf = TP_\lambda(D)\,f.$$

Proof. This follows directly from the definition of $P_\lambda(D)$ and T. ∎

THEOREM 4. *If* $(|\lambda| + 1/|\lambda|^2)\, S_c$ *is small in* $L^{2-\varepsilon}(\mathbf{C}) \cap L^{2+\varepsilon}(\mathbf{C})$, *and* $\sup_{S^1\times S^1} |S_d| \ll 1$, *then*

$$P_\lambda(D)\, m = Q(x)\, m,$$

with

$$m = \lambda \Lambda(x, \lambda) + \begin{pmatrix} 1 \\ \lambda \end{pmatrix},$$

$$Q(x) = \begin{pmatrix} 0 & -C_* T(\Lambda_1 + 1/\zeta) \\ C_*(1/\zeta)\, T(\Lambda_2 + 1) & 0 \end{pmatrix},$$

where

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix},$$

T defined as in (4.2), (4.3), (4.4),

$$C_* f(x) = -\frac{1}{2\pi i} \int_{\mathbf{C}} f(x, \zeta)\, d\bar{\zeta} \wedge d\zeta + \frac{1}{2\pi i} \oint_{S^1} f(x, \zeta)\, d\zeta. \tag{4.5}$$

Proof. By (4.2) and Lemma 10, we have

$$\begin{aligned}
P_\lambda(D)\, \Lambda &= P_\lambda(D)(1 - CT)^{-1}\, CT \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \\
&= P_\lambda(D)(1 - CT)^{-1} \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \\
&= [P_\lambda(D), (1 - CT)^{-1}] \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \\
&= (1 - CT)^{-1}\, [P_\lambda(D), CT](1 - CT)^{-1} \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \\
&= (1 - CT)^{-1}\, [P_\lambda(D), C] T (1 - CT)^{-1} \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \left\{ -C_* T\left(\Lambda_1 + \frac{1}{\zeta}\right) \right\} (1 - CT\cdot)^{-1}\, 1 \\ \left\{ C_* \frac{1}{\zeta} T(\Lambda_2 + 1) \right\} (1 - CT\cdot)^{-1} \frac{1}{\lambda} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -C_* T\left(\Lambda_1 + \frac{1}{\zeta}\right) \\ C_* \frac{1}{\zeta} T(\Lambda_2 + 1) & 0 \end{pmatrix} \left(\Lambda + \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \right).
\end{aligned}$$

And the theorem follows from the definition of $m(x, \lambda)$. ∎

5. An Associated Hierachy of Nonlinear Evolutions

In this section, we find a corresponding hierachy of evolution equations from the scattering side.

LEMMA 11. *Suppose Λ is defined by* (4.1). *Then if* $|\lambda| = 1$, *we have*

$$\Lambda_+(x, \lambda) = \left(1 + \frac{T_d}{2}\right)^{-1} \left(\Lambda - \frac{T_d}{2}\begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix}\right)(x, \lambda),$$

where

$$T_d f(x, \lambda) = \frac{1}{\lambda} \int_0^{2\pi} S_d(e^{i\phi}, \lambda)\, e^{ix \cdot (-\lambda + e^{i\phi})} e^{i\phi} f(x, e^{i\phi})\, d\phi, \qquad \textit{for} \quad |\lambda| = 1.$$

Proof. The lemma is directly followed by applying Sokhotski formula [8] to (4.1). ∎

As a result (4.1) can be rewritten as

$$(1 - CH)\Lambda = CH\begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix}, \tag{5.1}$$

where

$$Hf(x, \lambda) = \begin{cases} -(1/\bar{\lambda})\, e^{ix \cdot \xi_\lambda} S_c(\lambda)\, f(x, -1/\bar{\lambda}), & \text{if} \quad |\lambda| \neq 1, \\ T_d(1 + T_d/2)^{-1} f(x, \lambda), & \text{if} \quad |\lambda| = 1. \end{cases} \tag{5.2}$$

For simplicity, we define a paring $\langle\ ,\ \rangle_*$

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_* = \begin{pmatrix} C_*(f_1 g_1) \\ C_*(f_2 g_2) \end{pmatrix}, \tag{5.3}$$

where C_* is defined as in (4.5), for $f = \binom{f_1}{f_2}$, $g = \binom{g_1}{g_2}$, $f_i, g_i \in L^\infty(\mathbf{C} \times \mathbf{C}) \cap C^1(\mathbf{C} \times \{\mathbf{C} \backslash S^1\})$, and an operator $\mathscr{T}$ given by

$$\mathscr{T}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & -f_1 \\ f_2 & 0 \end{pmatrix}.$$

Consequently, since $P_\lambda(D)$ commutes with H, by an argument similar to the proof in Theorem 4, (5.1) then implies

$$Q = \mathscr{T}\left\langle H\left(\Lambda + \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix}\right), \begin{pmatrix} 1 \\ 1/\lambda \end{pmatrix}\right\rangle_*, \tag{5.4}$$

where Q is defined as in Theorem 4.

LEMMA 12. *If H, $\langle\ ,\ \rangle_*$ is defined as* (5.2), (5.3), *then there exists $\tilde{H}$ such that*

$$\langle f, Hg\rangle_* = \langle \tilde{H}f, g\rangle_*,$$

for $f=\binom{f_1}{f_2}$, $g=\binom{g_1}{g_2}$, $f_i, g_i \in L^\infty(\mathbf{C}\times\mathbf{C})\cap C^1(\mathbf{C}\times\{\mathbf{C}\backslash S^1\})$. *As a matter of fact,*

$$\tilde{H}f(x,\lambda)=\begin{cases}(1/\bar{\lambda}\,|\lambda|^2)\,e^{-ix\cdot\xi_\lambda}S_c(-1/\bar{\lambda})\,f(x,-1/\bar{\lambda}), & \text{if } |\lambda|\neq 1,\\ \tilde{T}_d(1+\tilde{T}_d/2)^{-1}f(x,\lambda), & \text{if } |\lambda|=1,\end{cases}$$

where

$$\tilde{T}_d f(x,\lambda)=\int_0^{2\pi} S_d(\lambda, e^{i\phi})\,e^{ix\cdot(-e^{i\phi}+\lambda)}f(x,e^{i\phi})\,d\phi, \qquad \text{for } |\lambda|=1.$$

Proof. This follows directly from the definition of H, $\langle\ ,\ \rangle_*$ and a change of variables. ∎

LEMMA 13. *With Q, A as in* (5.4) *and* (5.1), *then*

$$\frac{\partial Q}{\partial t}=\mathcal{T}\left\langle\left(\frac{\partial H}{\partial t}\right)\left(A+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right),\tilde{A}+\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*,$$

where

$$\tilde{A}=-(1+C\tilde{H})^{-1}C\tilde{H}\begin{pmatrix}1\\1/\lambda\end{pmatrix}. \tag{5.5}$$

Proof. Note that $\tilde{C}=-C$. Then (5.1), (5.4) imply

$$\begin{aligned}\frac{\partial Q}{\partial t}&=\mathcal{T}\left\langle\left(\frac{\partial H}{\partial t}\right)\left(A+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right)+H\left(\frac{\partial A}{\partial t}\right),\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*,\\&=\mathcal{T}\left\langle(1-HC)^{-1}\left(\frac{\partial H}{\partial t}\right)\left(A+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right),\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*,\\&=\mathcal{T}\left\langle\left(\frac{\partial H}{\partial t}\right)\left(A+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right),(1+C\tilde{H})^{-1}\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*,\\&=\mathcal{T}\left\langle\left(\frac{\partial H}{\partial t}\right)\left(A+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right),\tilde{A}+\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*.\end{aligned}$$ ∎

LEMMA 14. *Suppose* $(1+|\lambda|^4)\,S_c(\lambda)$ *is sufficiently small in* $L^{2-\varepsilon}\cap L^{2+\varepsilon}(|\lambda|^{-4}\,d\lambda_1\,d\lambda_2)$, $\sup_{S^1\times S^1}|S_d|\ll 1$, *and* $\tilde{\Lambda}$ *is defined as in* (5.5). *Then* $\tilde{\Lambda}\in L^\infty(\mathbf{C}\times\mathbf{C})\cap C^1(\mathbf{C}\times\{\mathbf{C}\backslash S^1\})$, *and* $\tilde{\Lambda}\to\binom{0}{0}$, *as* $|\lambda|\to\infty$. *Moreover if we let*

$$\tilde{P}_\lambda(D)=\begin{pmatrix}-(2/i)\,\partial_x+1/\lambda & -1\\ -1 & -(2/i)\,\partial_{\bar{x}}+\lambda\end{pmatrix},$$

then

$$\tilde{P}_\lambda(D)\,\tilde{\Lambda}=Q(x)\left(\tilde{\Lambda}+\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right),$$

where Q *is as in* (5.4).

Proof. First of all, note that $\tilde{\mathbf{P}}_\lambda(D)$ commutes with $\tilde{H}$. Thus by an argument similar to the proof in Theorem 4, we can show

$$\tilde{P}_\lambda(D)\,\tilde{\Lambda}=\tilde{Q}(x)\left(\tilde{\Lambda}+\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right),$$

where

$$\tilde{Q}(x)=\begin{pmatrix}0 & -C_*(1/\zeta)\,\tilde{H}(\tilde{\Lambda}_1+1)\\ C_*\tilde{H}(\tilde{\Lambda}_2+1/\zeta) & 0\end{pmatrix}.$$

By (5.4), Lemma 12, the above expression for $\tilde{Q}$ then implies

$$\begin{aligned}\tilde{Q}(x)&=\mathscr{T}\left\langle\begin{pmatrix}1/\lambda\\1\end{pmatrix},\tilde{H}\left(\tilde{\Lambda}+\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right)\right\rangle_*\\&=\mathscr{T}\left\langle\begin{pmatrix}1/\lambda\\1\end{pmatrix},\tilde{H}(1+C\tilde{H})^{-1}\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*\\&=\mathscr{T}\left\langle H(1-CH)^{-1}\begin{pmatrix}1/\lambda\\1\end{pmatrix},\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*\\&=\mathscr{T}\left\langle H\left(\Lambda+\begin{pmatrix}1/\lambda\\1\end{pmatrix}\right),\begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_*\\&=Q(x).\ \blacksquare\end{aligned}$$

For simplicity, we write $fg=\binom{f_1g_1}{f_2g_2}$, for $f=\binom{f_1}{f_2}$, $g=\binom{g_1}{g_2}$, $f_i, g_i\in L^\infty(\mathbf{C}\times\mathbf{C})\cap C^1(\mathbf{C}\times\{\mathbf{C}\backslash S^1\})$ for the rest of the paper.

LEMMA 15. *Suppose that at* $t=0$, $(|\lambda|^4+1/|\lambda|^2)\,S_c$ *is sufficiently small in*

$L^{2+\varepsilon} \cap L^{2-\varepsilon}((1+1/|\lambda|^4)\, d\lambda_1\, d\lambda_2)$, *and* S_d *is sufficiently small in* $L^\infty(S^1 \times S^1)$. *Moreover for* $t \ll 1$, $\partial H/\partial t = [\phi_a, H]$, *where* $\phi_a = 1/(\lambda - a) + 1/\bar{a}(\bar{a}\lambda + 1)$, Λ, $\tilde{\Lambda}$, *and* Q *are defined as in* (5.1), (5.5), (5.4). *Then*

$$\frac{\partial Q}{\partial t} = \mathscr{T}\left\{\Lambda\tilde{\Lambda}(a) + \frac{1}{\bar{a}^2}\Lambda\tilde{\Lambda}\left(-\frac{1}{\bar{a}}\right) + \Lambda(a)\begin{pmatrix}1\\1/a\end{pmatrix} + \tilde{\Lambda}(a)\begin{pmatrix}1/a\\1\end{pmatrix}\right.$$
$$\left. + \Lambda\left(-\frac{1}{\bar{a}}\right)\begin{pmatrix}1/\bar{a}^2\\-1/\bar{a}\end{pmatrix} + \tilde{\Lambda}\left(-\frac{1}{\bar{a}}\right)\begin{pmatrix}-1/\bar{a}\\1/\bar{a}^2\end{pmatrix} + \phi_a(0)\begin{pmatrix}\tilde{\Lambda}_1(0)\\\Lambda_2(0)\end{pmatrix}\right\}.$$

Proof. $\partial H/\partial t = [\phi_a, H]$ implies that $S_c(\lambda, t) = S_c(\lambda, 0)\, e^{t\{\phi_a(\lambda) - \phi_a(-1/\bar{\lambda})\}}$, and $S_d(e^{i\alpha}, \lambda, t) = S_d(e^{i\alpha}, \lambda, 0)\, e^{t\{\phi_a(\lambda) - \phi_a(e^{i\alpha})\}}$. Note that $\phi_a(\lambda) - \phi_a(-1/\bar{\lambda})$ is purely imaginary. Then by the assumption on $S_c(\lambda, 0)$, $S_d(e^{i\alpha}, \lambda, 0)$ and Lemmas 9 and 14, Λ, $\tilde{\Lambda}$ are well defined for $t \ll 1$.

Second, if we take $\partial_{\bar{\lambda}}$ in the distribution sense, then by (5.1), (5.5), we have $\partial_{\bar{\lambda}}\Lambda = H\{\Lambda + \binom{1/\lambda}{1}\}$, $\partial_{\bar{\lambda}}\tilde{\Lambda} = -\tilde{H}\{\tilde{\Lambda} + \binom{1}{1/\lambda}\}$, and $\partial_{\bar{\lambda}}(\Lambda\tilde{\Lambda}) = (\partial_{\bar{\lambda}}\Lambda)\,\tilde{\Lambda} + \Lambda(\partial_{\bar{\lambda}}\tilde{\Lambda})$. Lemma 13 and the Cauchy integral formula then imply

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \mathscr{T}\left\langle \phi_a H\left\{\Lambda + \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right\}, \tilde{\Lambda} + \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_* \\
&\quad - \mathscr{T}\left\langle H\left\{\phi_a\left[\Lambda + \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right]\right\}, \tilde{\Lambda} + \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_* \\
&= \mathscr{T}\left\langle \phi_a H\left\{\Lambda + \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right\}, \tilde{\Lambda} + \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_* \\
&\quad + \mathscr{T}\left\langle \phi_a\left[\Lambda + \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right], -\tilde{H}\left\{\tilde{\Lambda} + \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\}\right\rangle_* \\
&= \mathscr{T}\left\langle \phi_a\, \partial_{\bar{\lambda}}\Lambda, \tilde{\Lambda} + \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_* + \mathscr{T}\left\langle \phi_a\left[\Lambda + \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right], \partial_{\bar{\lambda}}\tilde{\Lambda}\right\rangle_* \\
&= \mathscr{T}\langle \phi_a, \partial_{\bar{\lambda}}(\Lambda\tilde{\Lambda})\rangle_* + \mathscr{T}\left\langle (\partial_{\bar{\lambda}}\Lambda)\,\phi_a, \begin{pmatrix}1\\1/\lambda\end{pmatrix}\right\rangle_* \\
&\quad + \mathscr{T}\left\langle (\partial_{\bar{\lambda}}\tilde{\Lambda})\,\phi_a, \begin{pmatrix}1/\lambda\\1\end{pmatrix}\right\rangle_* \\
&= \mathscr{T}\left\{\Lambda\tilde{\Lambda}(a) + \frac{1}{\bar{a}^2}\Lambda\tilde{\Lambda}\left(-\frac{1}{\bar{a}}\right) + \Lambda(a)\begin{pmatrix}1\\1/a\end{pmatrix} + \tilde{\Lambda}(a)\begin{pmatrix}1/a\\1\end{pmatrix}\right. \\
&\quad \left. + \Lambda\left(-\frac{1}{\bar{a}}\right)\begin{pmatrix}1/\bar{a}^2\\-1/\bar{a}\end{pmatrix} + \tilde{\Lambda}\left(-\frac{1}{\bar{a}}\right)\begin{pmatrix}-1/\bar{a}\\1/\bar{a}^2\end{pmatrix} + \phi_a(0)\begin{pmatrix}\tilde{\Lambda}_1(0)\\\Lambda_2(0)\end{pmatrix}\right\}. \quad \blacksquare
\end{aligned}$$

For the rest of the paper, we say S_c, S_d are sufficiently small if S_c, S_d satisfy the assumption of Lemma 15.

LEMMA 16. *If* $|a| \to \infty$ *and* Λ *is defined as in* (5.1), *then* $\Lambda(x, a)$, $\Lambda(x, -1/\bar{a})$ *can be expanded in*

$$\Lambda(x, a) = \begin{pmatrix} 1/a \\ 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} B_k(x) = \begin{pmatrix} 1/a \\ 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \begin{pmatrix} B_{k,1}(x) \\ B_{k,2}(x) \end{pmatrix},$$

$$\Lambda\left(x, -\frac{1}{\bar{a}}\right) = \begin{pmatrix} -\bar{a} \\ 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{\bar{a}^k} A_{k+1}(x) = \begin{pmatrix} -\bar{a} \\ 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{\bar{a}^k} \begin{pmatrix} A_{k+1,1}(x) \\ A_{k+1,2}(x) \end{pmatrix},$$

with

$$B_{0,1} = q_1,$$

$$\frac{2}{i} \partial_x B_{0,2} = q_1 + q_2 + q_1 q_2,$$

$$B_{k,1} = (q_1 + 1) B_{k-1,2} - \frac{2}{i} \partial_{\bar{x}} B_{k-1,1}, \qquad k > 0,$$

$$\frac{2}{i} \partial_x B_{k-1,2} = (q_2 + 1) B_{k-1,1} - B_{k-2,2}, \qquad k > 1,$$

and

$$\frac{2}{i} \partial_{\bar{x}} A_{1,1} = q_1 + q_2 + q_1 q_2,$$

$$A_{1,2} = q_2,$$

$$\frac{2}{i} \partial_{\bar{x}} A_{k+1,1} = (q_1 + 1) A_{k+1,2} + A_{k,1}, \qquad k > 0,$$

$$A_{k+2,2} = (q_2 + 1) A_{k+1,1} - \frac{2}{i} \partial_x A_{k+1,2}, \qquad k \geqslant 0.$$

Thus the coefficients B_i, A_i *can be expressed in terms of* q_i.

Proof. By expanding (5.1), we get

$$B_k(x) = -\frac{1}{2\pi i} \int_{\mathbf{C}} \lambda^k H \left\{ \Lambda + \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \right\} d\bar{\lambda} \wedge d\lambda + \frac{1}{2\pi i} \oint_{S^1} \lambda^k H \left\{ \Lambda + \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \right\} d\lambda,$$

$$A_{k+1}(x) = -\frac{1}{2\pi i} \int_{\mathbf{C}} \frac{1}{(-\lambda)^{k+1}} H\Lambda \, d\bar{\lambda} \wedge d\lambda + \frac{1}{2\pi i} \oint_{S^1} \frac{1}{(-\lambda)^{k+1}} H\Lambda \, d\lambda.$$

Second the properties of the coefficients B_k, A_k can be derived by plugging the series of $\Lambda(x, a)$, $\Lambda(x, -1/\bar{a})$ in both sides of

$$P_\lambda(D)\,\Lambda = Q(x)\left(\Lambda + \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix}\right). \quad \blacksquare$$

LEMMA 17. *If* $|a| \to \infty$ *and* $\tilde{\Lambda}$ *is defined as in* (5.5), *then* $\tilde{\Lambda}(x, a)$, $\tilde{\Lambda}(x, -1/\bar{a})$ *can be expanded in*

$$\tilde{\Lambda}(x, a) = \begin{pmatrix} 1 \\ 1/a \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \tilde{B}_k(x) = \begin{pmatrix} 1 \\ 1/a \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \begin{pmatrix} \tilde{B}_{k,1}(x) \\ \tilde{B}_{k,2}(x) \end{pmatrix},$$

$$\tilde{\Lambda}\left(x, -\frac{1}{\bar{a}}\right) = \begin{pmatrix} 1 \\ -\bar{a} \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{\bar{a}^k} \tilde{A}_{k+1}(x) = \begin{pmatrix} 1 \\ -\bar{a} \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{\bar{a}^k} \begin{pmatrix} \tilde{A}_{k+1,1}(x) \\ \tilde{A}_{k+1,2}(x) \end{pmatrix},$$

with

$$-\frac{2}{i}\partial_x \tilde{B}_{0,1} = q_1 + q_2 + q_1 q_2,$$

$$\tilde{B}_{0,2} = q_2,$$

$$-\frac{2}{i}\partial_x \tilde{B}_{k-1,1} = (q_1 + 1)\,\tilde{B}_{k-1,2} - \tilde{B}_{k-2,1}, \qquad k > 1,$$

$$\tilde{B}_{k,2} = (q_2 + 1)\,\tilde{B}_{k-1,1} + \frac{2}{i}\partial_{\bar{x}} \tilde{B}_{k-1,2}, \qquad k > 0,$$

and

$$\tilde{A}_{1,1} = q_1,$$

$$-\frac{2}{i}\partial_{\bar{x}} \tilde{A}_{1,2} = q_1 + q_2 + q_1 q_2,$$

$$-\tilde{A}_{k+2,1} = (q_1 + 1)\,\tilde{A}_{k+1,2} + \frac{2}{i}\partial_x \tilde{A}_{k+1,1}, \qquad k \geqslant 0.$$

$$-\frac{2}{i}\partial_{\bar{x}} \tilde{A}_{k+1,2} = (q_2 + 1)\,\tilde{A}_{k+1,1} + \tilde{A}_{k,2}, \qquad k > 0,$$

Thus the coefficients $\tilde{B}_i$, $\tilde{A}_i$ *can be expressed in terms of* q_i.

Proof. This lemma can be proved by Lemma 14 and a similar argument as in the proof of the previous lemma. $\blacksquare$

THEOREM 5. *Suppose* $|a| \to \infty$, $\partial H/\partial t = [\phi_a, H]$, *where* ϕ_a *is as in Lemma* 15 *and* Q *is defined as in* (5.4). *Then the hierachy of nonlinear evolution equation of* Q *is*

$$\frac{\partial Q}{\partial t} = \mathscr{T}\left\{\sum_{i\geqslant 2}\frac{1}{a^i}\left\{\sum_{0\leqslant j\leqslant i-2} B_j\tilde{B}_{i-j-2} + \begin{pmatrix} B_{i-1,1} \\ B_{i-2,2}\end{pmatrix} + \begin{pmatrix}\tilde{B}_{i-2,1} \\ \tilde{B}_{i-1,2}\end{pmatrix}\right\}\right.$$
$$\left. + \sum_{i\geqslant 2}\frac{1}{\bar{a}^i}\left\{\sum_{1\leqslant j\leqslant i-1} A_j\tilde{A}_{i-j} + \begin{pmatrix} A_{i-1,1} \\ -A_{i,2}\end{pmatrix} + \begin{pmatrix} -\tilde{A}_{i,1} \\ \tilde{A}_{i-1,2}\end{pmatrix}\right\}\right\},$$

where B_i, $\tilde{B}_i$, A_i, $\tilde{A}_i$ *are defined as in Lemmas* 16 *and* 17.

Proof. This follows directly from Lemmas 15, 16, and 17. ∎

COROLLARY 2. *Suppose* S_c *and* S_d *are sufficiently small at* $t=0$, *and for* $t \ll 1$, $k \geqslant 2$, $\partial H/\partial t = [\phi_{a,k}, H]$, *where* $\phi_{a,k}(\lambda) = -\lambda^k/a^{k+1} + (-1)^{k+1}(\bar{a}^{k+1}\lambda^k)$, *and* Q *is defined as in* (5.4). *Then the nonlinear evolution equation of* Q *is*

$$\frac{\partial Q}{\partial t} = \mathscr{I}\left\{\frac{1}{a^{k+1}}\left\{\sum_{0\leqslant j\leqslant k-1} B_j\tilde{B}_{k-j-1} + \begin{pmatrix} B_{k,1} \\ B_{k-1,2}\end{pmatrix} + \begin{pmatrix}\tilde{B}_{k-1,1} \\ \tilde{B}_{k,2}\end{pmatrix}\right\}\right.$$
$$\left. + \frac{1}{\bar{a}^{k+1}}\left\{\sum_{1\leqslant j\leqslant k} A_j\tilde{A}_{k-j+1} + \begin{pmatrix} A_{k,1} \\ -A_{k+1,2}\end{pmatrix} + \begin{pmatrix} -\tilde{A}_{k+1,1} \\ \tilde{A}_{k,2}\end{pmatrix}\right\}\right\},$$

where B_i, $\tilde{B}_i$, A_i, $\tilde{A}_i$ *are defined as in Lemmas* 16 *and* 17.

Proof. If $|a| \to \infty$, the Laurent expansion of

$$\begin{aligned}\phi_a(\lambda) - \phi_a\left(-\frac{1}{\bar{\lambda}}\right) &= \sum_{k=1}^{\infty}\left\{-\frac{\lambda^k}{a^{k+1}} + \frac{(-1)^{k+1}}{\bar{a}^{k+1}\lambda^k}\right\} + \sum_{k=1}^{\infty}\left\{-\frac{(-1)^{k+1}}{a^{k+1}\bar{\lambda}^k} + \frac{\bar{\lambda}^k}{\bar{a}^{k+1}}\right\} \\ &= \sum_{k=1}^{\infty}\left\{\left[-\frac{\lambda^k}{a^{k+1}} + \frac{\bar{\lambda}^k}{\bar{a}^{k+1}}\right] + \left[\frac{(-1)^{k+1}}{\bar{a}^{k+1}\lambda^k} - \frac{(-1)^{k+1}}{a^{k+1}\bar{\lambda}^k}\right]\right\} \\ &= \sum_{k=1}^{\infty}\left\{\phi_{a,k}(\lambda) - \phi_{a,k}\left(-\frac{1}{\bar{\lambda}}\right)\right\}.\end{aligned}$$

Then the corollary follows from Theorem 5 and the assumption. ∎

In particular we have the following corollary which is closely related to the Davey–Stewartson system.

COROLLARY 3. *Suppose S_c, S_d are sufficiently small at $t=0$, and for $t \ll 1$, $\partial H/\partial t = [i(\lambda^2 - 1/\lambda^2), H]$, Q is defined as in (5.4). Then*

$$\frac{\partial Q}{\partial t} = \frac{1}{i} \begin{pmatrix} 0 & \begin{array}{l} \omega(1+q_1) \\ \quad +2(\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2)\, q_1 \end{array} \\ \begin{array}{l} -\omega(1+q_2) \\ \quad -2(\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2)\, q_2 \end{array} & 0 \end{pmatrix},$$

where

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \omega = 4 \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) (q_1 + q_2 + q_1 q_2).$$

Proof. By setting $a = e^{i(\pi/6)}$ in Corollary 2 and noting that

$$\tilde{B}_{1,1} + B_{1,2} = B_{0,2}^2 - \frac{2}{i}\, \partial_{\bar{x}} B_{0,2},$$

$$A_{2,1} + \tilde{A}_{2,2} = -A_{1,1}^2 + \frac{2}{i}\, \partial_x A_{1,1},$$

we get

$$\frac{\partial Q}{\partial t} = \frac{1}{i} \begin{pmatrix} 0 & \begin{array}{l} \omega_0(1+q_1) \\ \quad -\{((2/i)\,\partial_{\bar{x}})^2 + ((2/i)\,\partial_x)^2\}\, q_1 \end{array} \\ \begin{array}{l} -\omega_0(1+q_2) \\ \quad +\{((2/i)\,\partial_{\bar{x}})^2 + ((2/i)\,\partial_x)^2\}\, q_2 \end{array} & 0 \end{pmatrix},$$

where

$$\omega_0 = \frac{4}{i}\, \partial_{\bar{x}} B_{0,2} + \frac{4}{i}\, \partial_x A_{1,1}.$$

Therefore the corollary follows by setting $\omega = \omega_0$. ∎

COROLLARY 4. *If $Q_0 = \left(\begin{smallmatrix} 0 & \amalg_1 \\ \amalg_2 & 0 \end{smallmatrix}\right)$, $\partial^k \amalg_i/\partial x_j^k$ is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, for $k \leqslant 6$. Then the initial value problem of*

$$\begin{aligned} \frac{\partial Q}{\partial t} = \mathscr{T} \Bigg\{ & \frac{1}{a^{k+1}} \Bigg\{ \sum_{0 \leqslant j \leqslant k-1} B_j \tilde{B}_{k-j-1} + \begin{pmatrix} B_{k,1} \\ B_{k-1,2} \end{pmatrix} + \begin{pmatrix} \tilde{B}_{k-1,1} \\ \tilde{B}_{k,2} \end{pmatrix} \Bigg\} \\ & + \frac{1}{\bar{a}^{k+1}} \Bigg\{ \sum_{1 \leqslant j \leqslant k} A_j \tilde{A}_{k-j+1} + \begin{pmatrix} A_{k,1} \\ -A_{k+1,2} \end{pmatrix} + \begin{pmatrix} -\tilde{A}_{k+1,1} \\ \tilde{A}_{k,2} \end{pmatrix} \Bigg\} \Bigg\} \end{aligned} \tag{5.6}$$

$$Q(x,0) = Q_0(x)$$

can be locally solved, where B_i, $\tilde{B}_i$, A_i, $\tilde{A}_i$ are defined as in Lemmas 16 *and* 17.

Proof. Since for $k \leqslant 6$, $\partial^k \text{U}_i/\partial x_j^k$ is sufficiently small in $L^{2+\varepsilon} \cap L^{4/3-\varepsilon}$, we can define the scattering data $S_c(\lambda, 0)$ and $S_d(\lambda', \lambda, 0)$ of Q_0 by Definition 1. Moreover $S_c(\lambda, 0)$, and $S_d(\lambda', \lambda, 0)$ are sufficiently small. Thus we can define $S_c(\lambda, t)$, and $S_d(\lambda', \lambda, t)$, H by $\partial H/\partial t = [\phi_{a,k}, H]$ and (5.2), $\Lambda(x, \lambda, t)$ by (5.1), and Q'' by (5.4) successively. Note that $Q''(x, 0) = Q_0(x)$.

Corollary 2 then implies that for $t \ll 1$, Q'' satisfies (5.6). So Q can be solved by setting $Q(x, t) = Q''(x, t) = \mathscr{T} \langle H(\Lambda + (\begin{smallmatrix} 1/\lambda \\ 1 \end{smallmatrix})), (\begin{smallmatrix} 1 \\ 1/\lambda \end{smallmatrix}) \rangle_*$ for $t \ll 1$. ∎

For the rest of the paper, we say Q_0 is sufficiently small and smooth if Q_0 satisfies the assumption of Corollary 4.

COROLLARY 5. *If Q_0 is sufficiently small and smooth. Then the initial value problem of*

$$\frac{\partial Q}{\partial t} = \frac{1}{i} \begin{pmatrix} 0 & \begin{matrix} \omega(1+q_1) \\ +2(\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2)\, q_1 \end{matrix} \\ \begin{matrix} -\omega(1+q_2) \\ -2(\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2)\, q_2 \end{matrix} & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \omega = 4 \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) (q_1 + q_2 + q_1 q_2), \tag{5.7}$$

$$Q(x, 0) = Q_0(x)$$

can be locally solved.

Proof. The corollary can be proved by the same method as in the proof of Corollary 4, but with $\phi_{a,k}$ replaced by $i(\lambda^2 - 1/\lambda^2)$, and with Corollary 2 replaced by Corollary 3. ∎

6. AN APPLICATION: SOLVABILITY OF THE DAVEY–STEWARTSON EQUATIONS

We start with the consideration of the following two boundary value problems: given $\lambda \in \mathbf{C}$, $|\lambda| \neq 0, 1$,

$$P_\lambda(D)\, m(x, \lambda) = Q(x)\, m(x, \lambda), \qquad m \to (\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}), \quad \text{as} \quad |x| \to \infty; \tag{6.1}$$

$$P_\lambda(D)\, g(x, \lambda) = Q(x)\, g(x, \lambda), \qquad g \to (\begin{smallmatrix} 1/\lambda \\ 1 \end{smallmatrix}), \quad \text{as} \quad |x| \to \infty. \tag{6.2}$$

From the discussion in previous sections, we know that if Q is sufficiently small and smooth, the scattering problems of (6.1), (6.2) can be solved by defining

$$\Lambda = \frac{1}{\lambda}\left(m - \begin{pmatrix} 1 \\ \lambda \end{pmatrix}\right), \tag{6.3}$$

$$\Lambda' = g - \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix}, \tag{6.4}$$

$$S_c(\lambda) = -\frac{1}{4\pi}\frac{\operatorname{sgn}(|\lambda|-1)}{|\lambda|^2}(-\bar{\lambda}\widehat{q_1 m_2} + \widehat{q_2 m_1})(\xi_\lambda, \lambda),$$

$$\text{when} \quad |\lambda| \neq 1; \tag{6.5}$$

$$S_d(\lambda', \lambda) = R(\lambda', \lambda), \qquad \text{when} \quad |\lambda| = |\lambda'| = 1,$$

$$R \text{ is as in Theorem 3.} \tag{6.6}$$

$$S'_c(\lambda) = \frac{1}{4\pi}\frac{\operatorname{sgn}(|\lambda|-1)}{\bar{\lambda}^2}(-\bar{\lambda}\widehat{q_1 g_2} + \widehat{q_2 g_1})(\xi_\lambda, \lambda),$$

$$\text{when} \quad |\lambda| \neq 1; \tag{6.7}$$

$$S'_d(\lambda', \lambda) = R'(\lambda', \lambda), \qquad \text{when} \quad |\lambda| = |\lambda'| = 1,$$

$$R' \text{ is as in Theorem 3} \tag{6.8}$$

except that W_1, W_2 are replaced by

$$W'_1(e^{i\phi}, \lambda) = -\frac{i}{4\pi}(\widehat{q_1\Psi_{2,2}} + e^{i\phi}\widehat{q_2\Psi_{2,1}})(e^{i\phi}, e^{i\phi}, \lambda),$$

$$W'_2(e^{i\phi}, \lambda) = -\frac{i}{4\pi}(\widehat{q_1\Psi_{2,2}} - e^{i\phi}\widehat{q_2\Psi_{2,1}})(-e^{i\phi}, e^{i\phi}, \lambda),$$

where $\Psi_2 = \binom{\Psi_{2,1}}{\Psi_{2,2}}$ satisfies

$$\Psi_2(x, \mu, \lambda) = e^{ix\cdot\lambda}\begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} + \int G_1(x-y, \mu)\, Q(y)\, \Psi_2(y, \mu, \lambda)\, dy.$$

Besides

$$\partial_{\bar{\lambda}}\Lambda(x, \lambda) = e^{ix\cdot\xi_\lambda}S_c(\lambda)\left(-\frac{1}{\bar{\lambda}}\Lambda\left(x, -\frac{1}{\bar{\lambda}}\right) + \begin{pmatrix} 1 \\ -1/\bar{\lambda} \end{pmatrix}\right),$$

$$\text{if} \quad |\lambda| \neq 1; \tag{6.9}$$

$$(\Lambda_- - \Lambda_+)(x, e^{i\alpha}) = \int_0^{2\pi} e^{-i\alpha} S_d(e^{i\phi}, e^{i\alpha})\, e^{ix\cdot(-e^{i\alpha}+e^{i\phi})} \times \left(e^{i\phi}\Lambda_+(x, e^{i\phi}) + \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}\right) d\phi \tag{6.10}$$

$$\Lambda(x, \lambda) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{if} \quad |\lambda| \to \infty; \tag{6.11}$$

$$\partial_{\bar{\lambda}} \Lambda'(x, \lambda) = e^{ix\cdot\xi_\lambda} S'_c(\lambda) \left(\Lambda'\left(x, -\frac{1}{\bar{\lambda}}\right) + \begin{pmatrix} -\bar{\lambda} \\ 1 \end{pmatrix}\right), \quad \text{if} \quad |\lambda| \neq 1, \tag{6.12}$$

$$(\Lambda'_- - \Lambda'_+)(x, e^{i\alpha}) = \int_0^{2\pi} S'_d(e^{i\phi}, e^{i\alpha})\, e^{ix\cdot(-e^{i\alpha}+e^{i\phi})} \times \left(\Lambda'_+(x, e^{i\phi}) + \begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}\right) d\phi; \tag{6.13}$$

$$\Lambda'(x, \lambda) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{if} \quad |\lambda| \to \infty. \tag{6.14}$$

Moreover

$$(1 - CH)^{-1}\, \Lambda = CH \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix},$$

$$(1 - CH')^{-1}\, \Lambda' = CH' \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix},$$

where

$$Hh(x, \lambda) = \begin{cases} -\dfrac{1}{\bar{\lambda}} e^{ix\cdot\xi_\lambda} S_c(\lambda)\, h\left(x, -\dfrac{1}{\bar{\lambda}}\right), & \text{if} \quad |\lambda| \neq 1, \\ T_d\left(1 + \dfrac{T_d}{2}\right)^{-1} h(x, \lambda), & \text{if} \quad |\lambda| = 1. \end{cases} \tag{6.15}$$

$$H'h(x, \lambda) = \begin{cases} e^{ix\cdot\xi_\lambda} S'_c(\lambda)\, h\left(x, -\dfrac{1}{\bar{\lambda}}\right), & \text{if} \quad |\lambda| \neq 1, \\ T'_d\left(1 + \dfrac{T'_d}{2}\right)^{-1} h(x, \lambda), & \text{if} \quad |\lambda| = 1, \end{cases} \tag{6.16}$$

and

$$T_d h(x, \lambda) = \frac{1}{\lambda} \int_0^{2\pi} S_d(e^{i\phi}, \lambda)\, e^{ix\cdot(-\lambda + e^{i\phi})} e^{i\phi} h_+(x, e^{i\phi})\, d\phi, \tag{6.17}$$

$$T'_d h(x, \lambda) = \int_0^{2\pi} S'_d(e^{i\phi}, \lambda)\, e^{ix\cdot(-\lambda + e^{i\phi})} h_+(x, e^{i\phi})\, d\phi, \tag{6.18}$$

for $|\lambda| = 1$. Thus by a similar method as in the proof of Theorem 4, there exist Q'' and Q' such that

$$P_\lambda(D)\, m = Q'' m, \tag{6.19}$$

$$P_\lambda(D)\, g = Q' g. \tag{6.20}$$

Now let us define $A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and an operator U given by $U\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(x, \lambda) = \overline{\begin{pmatrix} -h_2(x, -1/\bar{\lambda}) \\ h_1(x, -1/\bar{\lambda}) \end{pmatrix}}$, for $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, $h_i \in L^\infty(\mathbf{C} \times \mathbf{C}) \cap C^1(\mathbf{C} \times \{\mathbf{C} \backslash S^1\})$.

LEMMA 18. *Suppose Q is sufficiently small and smooth, and* $m(x, \lambda) = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$, $g(x, \lambda) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ *satisfy* (6.1), (6.2). *Then*

$$\frac{1}{\lambda} m = g.$$

Moreover $Q = Q^*$ *if and only if*

$$Um = g.$$

Proof. $(1/\lambda)\, m = g$ directly follows by the uniqueness property. On the other hand, since

$$U(P_\lambda(D)\, m) = -P_\lambda(D)\, U(m),$$

$$U(Qm) = -Q^* U(m),$$

so

$$P_\lambda(D)\, U(m) = Q^* U(m),$$

and the lemma is completed by the uniqueness property. ∎

Remark. The "if" part of Lemma 18 still remains valid without the smallness and smoothness assumption on Q.

LEMMA 19. *If Q is the solution of* (5.7) *given by Corollary* 5. *Then* $Q(x, 0) = Q^*(x, 0)$ *implies* $Q(x, t) = Q^*(x, t)$, *for* $t \ll 1$.

Proof. We first consider the systems (6.1) and (6.2) with $Q(x)$ replaced by $Q(x,0)$. Since $Q(x,0)$ is sufficiently small and smooth, there exist $m(x,\lambda,0)$, $g(x,\lambda,0)$ satisfy (6.1), and (6.2). By Lemma 18, we have

$$\frac{1}{\lambda}\,m(x,\lambda,0)=g(x,\lambda,0), \tag{6.21}$$

$$U(m)(x,\lambda,0)=g(x,\lambda,0). \tag{6.22}$$

Now let us define A, A', S_c, S'_c, S_d, S'_d, H, H' at $t=0$ as in (6.3) to (6.8), (6.15), and (6.16). Moreover we define

$$S_c(\lambda,t)=S_c(\lambda,0)\,e^{t\{\psi(\lambda)-\psi(-1/\bar{\lambda})\}}, \tag{6.23}$$

$$S_d(e^{i\phi},\lambda,t)=S_d(e^{i\phi},\lambda,0)\,e^{t\{\psi(\lambda)-\psi(e^{i\phi})\}}, \tag{6.24}$$

$$S'_c(\lambda,t)=S'_c(\lambda,0)\,e^{t\{\psi(\lambda)-\psi(-1/\bar{\lambda})\}}, \tag{6.25}$$

$$S'_d(e^{i\phi},\lambda,t)=S'_d(e^{i\phi},\lambda,0)\,e^{t\{\psi(\lambda)-\psi(e^{i\phi})\}}, \tag{6.26}$$

where $\psi(\lambda)=i(\lambda^2-1/\lambda^2)$.

Successively define $A(x,\lambda,t)$, $A'(x,\lambda,t)$ by (6.9)–(6.11), (6.12)–(6.14), $m(x,\lambda,t)$, $g(x,\lambda,t)$ by (6.3), (6.4), and H, H' by (6.15), (6.16), $Q''(x,t)$ and $Q'(x,t)$ by (6.19), (6.20).

$Q''=Q$ follows from the assumption. Thus by (6.19), (6.20) Lemma 18 and the remark following it, it is sufficient to show that

$$\frac{1}{\lambda}\,m(x,t)=g(x,t),$$

$$U(m)(x,t)=g(x,t).$$

By (6.21), (6.9), (6.12), we have $S_c(\lambda,0)=(1/\lambda)\,S'_c(\lambda,0)$. And (6.21), (6.10), (6.13) imply $S_d(e^{i\phi},\lambda,0)=S'_d(e^{i\phi},\lambda,0)$. Therefore by (6.23)–(6.26)

$$S_c(\lambda,t)=\frac{1}{\lambda}\,S'_c(\lambda,t),$$

$$S_d(e^{i\phi},\lambda,t)=S'_d(e^{i\phi},\lambda,t).$$

Thus

$$\frac{1}{\lambda}\,m(x,t)=g(x,t)$$

follows from (6.3), (6.4), and (6.9)–(6.14). On the other hand, by (6.22), (6.9), (6.12), we have $-(1/|\lambda|^2\bar{\lambda})\,\overline{S_c(-1/\bar{\lambda}, 0)} = S_c'(\lambda, 0)$. And (6.22), (6.10), (6.13) imply $\overline{S_d(-e^{i\phi}, -\lambda, 0)} = S_d'(e^{i\phi}, \lambda, 0)$. Thus (6.23)–(6.26) result in

$$-\frac{1}{|\lambda|^2\bar{\lambda}}\,\overline{S_c\left(-\frac{1}{\bar{\lambda}}, t\right)} = S_c'(\lambda, t),$$

$$\overline{S_d(-e^{i\phi}, -\lambda, t)} = S_d'(e^{i\phi}, \lambda, t).$$

So (6.3), (6.4), (6.9)–(6.14) imply

$$U(m)(x, t) = g(x, t). \quad \blacksquare$$

THEOREM 6. *If $v_0 - 1$ is sufficiently small and smooth, then the initial value problem of the Davey–Stewartson system*

$$i\frac{\partial v}{\partial t} + \omega v + 2\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)v = 0;$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\omega = 4\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)|v|^2,$$

$$v(x, 0) = v_0$$

can be locally solved.

Proof. Set $q_1 = v - 1$, $q_2 = \bar{v} - 1$ in $Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}$ in Corollary 5. Then the theorem follows directly from Corollary 5 and Lemma 19. $\blacksquare$

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