Notes on cycles through a vertex or an arc in regular 3-partite tournaments

1. Introduction

We shall assume that the reader is familiar with standard terminology on directed graphs (see, e.g., Bang-Jensen and Gutin [1]). In this note, if we speak of a cycle, then we mean a directed cycle. If $xy$ is an arc of a digraph $D$, then we write $x \rightarrow y$ and say $x$ dominates $y$. If $X$ and $Y$ are two disjoint vertex sets of a digraph $D$ such that every vertex of $X$ dominates every vertex of $Y$, then we say that $X$ dominates $Y$, denoted by $X \rightarrow Y$; otherwise denoted by $X \nrightarrow Y$. If $D'$ is a vertex set or a subdigraph of a digraph $D$, then we define $N_D^+(x)$ as the set of vertices of $D'$ which are dominated by $x$ and $N_D^-(x)$ as the set of vertices of $D'$ which dominate $x$. The numbers $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are called the out-degree and in-degree of $x$ in $D'$, respectively. When $D' = D$, $N_D^+(x)$, $d_D^+(x)$ and $d_D^-(x)$ are also denoted by $N^+(x)$, $N^-(x)$, $d^+(x)$ and $d^-(x)$, respectively. For two vertex sets $X$, $Y$ of a digraph $D$, we define $X - Y = \{x|x \in X, x \not\in Y\}$. A $c$-partite tournament is an orientation of a complete $c$-partite graph. A digraph $D$ is regular, if $d^+(x) = d^-(y) = d^+(y)$ for all $x, y \in V(D)$.

For cycles in regular 3-partite tournaments, Volkmann [2] obtained the following four results.

**Theorem 1.1.** If $D$ is a regular 3-partite tournament, then every arc of $D$ is contained in a 3- or 4-cycle.

**Theorem 1.2.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every arc of $D$ is contained in a $4r$, $5r$, or $6r$-cycle.

**Theorem 1.3.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every arc of $D$ is contained in a $5r$, $6r$, or $7r$-cycle.

**Theorem 1.4.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then $D$ contains a 6-cycle.

In 2007, Stella and Volkmann [3] obtained the following result.

**Theorem 1.5.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 3$, then $D$ contains a 9-cycle.

In 2007, Volkmann [4] gave the following problem and conjecture.

**Problem 1.6.** Let $D$ be an $r$-regular 3-partite tournament with $r \geq 3$. Is every vertex of $D$ contained in an $m$-cycle for each $m \in \{6, 9, \ldots, |V(D)|\}$?

**Conjecture 1.7.** If $D$ is a regular 3-partite tournament, then every arc of $D$ is contained in an $m$, $(m + 1)$- or $(m + 2)$- cycle for each $m \in \{3, 4, \ldots, |V(D)| - 2\}$. Volkmann [2] gave examples showing that there exists an infinite family of regular 3-partite tournaments $D$ that even has vertices which are not contained in a 3-cycle. Let $D$ be a 3-partite tournament with partite sets $V_1$, $V_2$, $V_3$ such that $|V_1| = |V_2| = |V_3| = r$ and $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1$. Obviously, $D$ is $r$-regular and $D$ has only cycles of lengths 3, 6, 9, $\ldots$, $|V(D)| = 3r$. This shows that Problem 1.6 would be best possible.

Also Volkmann [2] gave examples showing that for each integer $t$ with $3 \leq t \leq |V(D)|$, there exists an infinite family of regular 3-partite tournaments $D$ such that there are at least three arcs in $D$ which are not contained in a cycle of length $t$. This gives support to Conjecture 1.7.

In this note, we prove that if $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every vertex of $D$ is contained in a 6-cycle and every arc of $D$ is contained in a $5r$- or $6r$-cycle.

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2. Main results

The next lemma is well-known and easy to prove.

Lemma 2.1. If $D$ is an $r$-regular 3-partite tournament with partite sets $V_1, V_2, V_3$ and $v$ is a vertex of $D$, then $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$.

Lemma 2.2. If $D$ is an $r$-regular 3-partite tournament with partite sets $U, V, W$ and $u$ is a vertex of $U$, then $d^+_V(u) = d^-_W(u)$ and $d^-_W(u) = d^-_V(u)$.

Proof. Obviously, $d^+(u) = d^+_V(u) + d^+_W(u)$ and $d^-(u) = d^-_W(u) + d^-_V(u)$. By Lemma 2.1, $d^+(u) = |W| = r$, so $d^+_V(u) = d^-_W(u)$. Similarly, $d^-_W(u) = d^-_V(u)$. □

Theorem 2.3. If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every vertex of $D$ is contained in a 6-cycle and every arc of $D$ is contained in a 5- or 6-cycle.

Proof. Let $V_1$, $V_2$, $V_3$ be the partite sets of $D$, and let $a$ be an arbitrary vertex of $D$ and $ab$ be an arbitrary arc of $D$ containing $a$. Obviously, it is enough to prove that $a$ is in a 6-cycle and $ab$ is in a 5- or 6-cycle. Without loss of generality, we suppose that $a \in V_1$ and $b \in V_2$. By Lemma 2.1, $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$, for any vertex $v$ of $D$. We now distinguish two cases.

Case 1. $V_3 \to a \to V_2$.

By Lemma 2.2, $d^+_V(x) \geq 1$ and $d^-_V(y) \geq 1$ for any $x \in V_2$, $y \in V_3$. So there are at least $r$ arcs from $V_2$ to $V_3$.

Subcase 1.1. There are exactly $r$ arcs from $V_2$ to $V_3$.

We array the vertices of $V_2$ into $b_1 = b, b_2, \ldots, b_r$ and the vertices of $V_3$ into $c_1, c_2, \ldots, c_r$ such that $b_i \to c_i$ and $V_3 - \{c_1\} \to b_1$ and $a \to b_1$. As $d^-(b) = r, b \to V_1 - \{a\}$. Similarly, $V_1 - \{a\} \to c_1$.

Let $a_2 \in V_1 - \{a\}$ be arbitrary. Then $aba_2c_1b_2c_2a$ is a 6-cycle through $ab$.

Subcase 1.2. There are more than $r$ arcs from $V_2$ to $V_3$.

First, we prove that $ab$ is in a 5- or 6-cycle.

Suppose that $V_2 \not\to N^+_V(b)$. Let $x$ be a vertex in $N^+_V(b)$ such that $d^+_V(x) \geq 1$. Let $y \in N^+_V(x)$ be arbitrary. By Lemma 2.2, there is an arc $yz$, where $z \in V_3 - \{x\}$. Obviously, $abxyz$ is a 5-cycle through $ab$.

Suppose that $V_2 \to N^+_V(b)$ and $b \to V_3$. Obviously, $V_1 \to V_2 \to V_3$. So $abxyza$ is a 6-cycle through $ab$, where $y \in V_1 - \{a\}, z \in V_2 - \{b\}, x, u \in V_3$ and $x \neq u$.

Suppose that $V_2 \to N^+_V(b)$ and $b \not\to V_3$. Let $x \in N^+_V(b)$ be arbitrary. As $V_2 \to x$, by Lemma 2.2, $x \to V_1$. Let $z \in V_3 - N^+_V(b)$ be arbitrary. As $z \to b$, there is an arc $yz$ where $y \in V_1 - \{a\}$ by Lemma 2.2. So $abxyz$ is a 5-cycle through $ab$.

Second, we prove that $ab$ is in a 6-cycle. Obviously, there is a vertex $x \in V_1$ such that $d^+_V(x) \geq 2$, so by Lemma 2.2, there is an arc $xy$ where $y \in V_1 - \{a\}$. By Lemma 2.2, there is an arc $yz$ where $z \in V_2$. As $a \to z$, there is an arc $zu$ where $u \in V_3 - \{x\}$ by Lemma 2.2. As $d^+_V(x) \geq 2$, there is an arc $ux$ where $u \in V_2 - \{z\}$. Obviously, $abxyzu$ is a 6-cycle containing $a$.

Case 2. $V_3 \to a \to V_2$.

Obviously, $V_2$ can be divided into two nonempty parts $V'_2, V''_2$ such that $V''_2 \to a \to V'_2$. Similarly, $V_3$ can be divided into two nonempty parts $V'_3, V''_3$ such that $V''_3 \to a \to V'_3$. Let $V' = V'_2 \cup V'_3$ and $V'' = V''_2 \cup V''_3$. Obviously, $N^+(a) = V', N^-(a) = V'$ and $|V'| = |V''| = r$.

First, we prove that $ab$ is in a 5- or 6-cycle.

If $V'_3 \not\to b$, then there is an arc $bx, x \in V'_2$. By Lemma 2.2, there is an arc $xy, y \in V_1 - \{a\}$. As $x \in V'$ and $|V''| = r$, there is an arc $yz$ where $z \in V''$ by Lemma 2.1. Otherwise $V'' \to y, d^+(y) \geq r + 1$, a contradiction. So $abxyz$ is a 5-cycle through $ab$.

If $V'_3 \to b$, then there is an arc $bx, x \in V_1 - \{a\}$. Suppose that $x \to V''$. Let $yz$ be an arc in $V''$. Obviously, $abxyz$ is a 5-cycle through $ab$. Suppose that $x \not\to V''$. There is an arc $xy, y \in V' - \{b\}$. If $V'_1 \to y$, then there is an arc $yz$ where $z \in V''$. Clearly, $abxyz$ is a 5-cycle through $ab$. If $V'_1 \to y$, then there is an arc $zy$, where $z \in V_1 - \{a, x\}$. As $y \in V'$ and $|V''| = r$, there is an arc $zu$ where $u \in V'$ by Lemma 2.2. So $abxyzu$ is a 6-cycle through $ab$.

Second, we prove that $ab$ is in a 6-cycle. Obviously, there are arcs between $V'_2$ and $V'_3$. Without loss of generality, assume that there is an arc $xy$ from $V'_2$ to $V'_3$. By Lemma 2.2, there is an arc $yz$ where $z \in V_1 - \{a\}$.

If $z \to V''$, then let $uv$ be an arbitrary arc in $V''$. Clearly, $abxyzuv$ is a 6-cycle containing $a$. If $z \not\to V''$, then there is an arc $zu$ where $u \in V' - \{y\}$.

Suppose that $V_1 \not\to u$. There is an arc $uv$, where $u \in V_1 - \{a, z\}$. As $u \in V'$ and $|V''| = r$, there is an arc $vw$ where $w \in V''$ by Lemma 2.1. So $abxyzuvw$ is a 6-cycle containing $a$.

Suppose that $V_1 \to u$ and $u \neq x$. Obviously, $abxyzuv$ is a 6-cycle containing $a$. If $u \in V'_2$, then $u \to V_2$. Let $v$ be a vertex of $V''_2$. Obviously, $abxyzuv$ is a 6-cycle containing $a$.

Suppose that $V_1 \to u$ and $u = x$. Obviously, $abxyzuv$ is a 6-cycle containing $a$. If $v \to V_2 - \{y\}$, then there is an arc $vw, w \in V_3 - \{y\}$. So $abxyzuvw$ is a 6-cycle containing $a$. If $v \to V_3 - \{y\}$, then $z \to v$ by the fact $v \to a$ and Lemma 2.1. Let $w$ be a vertex of $V''_3$. Clearly, $abxyzuvw$ is a 6-cycle containing $a$. This completes the proof. □
Obviously, Theorems 1.2–1.4 are corollaries of the theorem above. Considering Theorem 1.1, we give the following conjectures.

**Conjecture 2.4.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every arc of $D$ is contained a $3k - 1$- or $3k$-cycle for $k = 2, 3, \ldots, r$.

**Conjecture 2.5.** If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$, then every arc of $D$ is contained a $3k$- or $3k + 1$-cycle for $k = 1, 2, \ldots, r - 1$.

**References**


