JOURNAL OF APPROXIMATION THEORY 3, 114–116 (1970)

Letter to the Editor

Convergence of Cubic Spline Interpolants

The purpose of this note is to communicate the proof of a result ascribed to us by Meir and Sharma in their recent paper [1].

Let C denote the space of all continuous functions f on $[0, 1]$ which are periodic: $f(0) = f(1)$. This space is assigned the supremum norm. Let a "point group" be prescribed:

$$\begin{array}{ccccc} & & x_0^{(0)} & & \\ & x_0^{(1)} & & x_1^{(1)} & \\ x_0^{(2)} & & x_1^{(2)} & & x_2^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We assume that for each n, $0 = x_0^{(n)} < \cdots < x_n^{(n)} = 1$. For each n, let S_n denote the linear subspace of C consisting of all periodic cubic splines with "joints" $x_0^{(n)},\ldots, x_n^{(n)}$. Thus, $s \in S_n$ if, and only if, $s \in C$, $s' \in C$, $s'' \in C$, and on each of the intervals $[x_0^{(n)}, x_1^{(n)}],\ldots,[x_{n-1}^{(n)}, x_n^{(n)}]$, s reduces to a cubic polynomial. For each n and for each $f \in C$, let $L_n f$ denote the unique element of S_n which interpolates to f at the joints $x_0^{(n)},\ldots, x_n^{(n)}$. The operator L_n is a linear projection of C onto S_n.

An important problem which is still unsolved in the theory of spline approximation is to determine a simple necessary and sufficient condition on the point group in order that $L_n f$ converge uniformly to f for all $f \in C$.

$$\text{Define} \quad h_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)} \quad \text{and} \quad h^{(n)} = \max_{1 \leqslant i \leqslant n} h_i^{(n)}.$$

The condition $h^{(n)} \to 0$ implies that $\bigcup_{n=1}^{\infty} S_n$ is *dense* in C. This implication follows, for example, from Theorem 2 of [2]. By Banach's Theorem, one obtains then the following result:

THEOREM. *If* $h^{(n)} \to 0$, *then the following conditions are equivalent:*

(1) $L_n f \to f$ *(uniformly) for all* $f \in C$.

(2) $\limsup \| L_n \| < \infty$.

The effect of this theorem is to focus attention on the problem of estimating $\| L_n \|$. Several such estimates were given in [2]. In [1], Meir and Sharma prove that if the number $m_n = \max_{|i-j|=1} h_i^{(n)}/h_j^{(n)}$ is less than $\sqrt{2}$, then

$$\| f - L_n f \| \leqslant [1 + \tfrac{3}{4} m_n (2 - m_n^2)^{-1}]\, \omega(f; h^{(n)}).$$

This result can be strengthened as follows.

THEOREM. *If* $m_n < 2$, *then* $\| L_n \| < 6(2 - m_n)^{-1}$ *and* $\| f - L_n f \| < 30(2 - m_n)^{-1} \omega(f; h^{(n)})$. *Consequently, the condition* $\limsup m_n < 2$ *is sufficient to guarantee* $L_n f \to f$ *for all* $f \in C$.

Proof. One starts with Eq. (1), p. 96, of [2], written in the form

$$q_i h_i \lambda_{i-1} + 2h_i \lambda_i + p_i h_i \lambda_{i+1} = 3p_i h_i h_{i+1}^{-1}(f_{i+1} - f_i) + 3q_i(f_i - f_{i-1}). \tag{A}$$

Denote the right member of this equation by r_i, and set $\mu_i = h_i \lambda_i$. In the notation, the dependence upon n is suppressed. Equation (A) becomes

$$q_i h_i h_{i-1}^{-1} \mu_{i-1} + 2\mu_i + p_i h_i h_{i+1}^{-1} \mu_{i+1} = r_i . \tag{B}$$

Now refer to Eq. (2), p. 97, of [2]. Let $\|f\| \leqslant 1$ and $s = Lf$. Then on the interval $[x_{i-1}, x_i]$, we have

$$\begin{aligned}
|s(x)| &\leqslant |A_i(x)| + |B_i(x)| + |\lambda_{i-1}|\,|C_i(x)| + |\lambda_i|\,|D_i(x)| \\
&= 1 + |\mu_{i-1}| \frac{h_i}{h_{i-1}} \frac{C_i(x)}{h_i} - |\mu_i| \frac{D_i(x)}{h_i} \\
&\leqslant 1 + \max\{m\,|\mu_{i-1}|, |\mu_i|\} \frac{C_i(x) - D_i(x)}{h_i} \\
&\leqslant 1 + \frac{1}{4} \max\{m\,|\mu_{i-1}|, |\mu_i|\} \\
&\leqslant 1 + \frac{m}{4} \max_{1 \leqslant i \leqslant n} |\mu_i|.
\end{aligned}$$

Let $|\mu_j| = \max |\mu_i|$. From Eq. (B) we have

$$2\,|\mu_j| \leqslant |r_j| + q_j m\,|\mu_j| + p_j m\,|\mu_j| = |r_j| + m\,|\mu_j|.$$

Hence,

$$\max_{1 \leqslant i \leqslant n} |\mu_i| \leqslant |r_j|\,(2 - m)^{-1}.$$

Since $\|f\| \leqslant 1$, we have $|r_j| \leqslant 6p_j m + 6q_j \leqslant 6m$. Thus, $\max |\mu_i| \leqslant 6m(2 - m)^{-1}$. From an inequality above we, therefore, obtain

$$|s(x)| \leqslant 1 + \tfrac{3}{2} m^2 (2 - m)^{-1} < 6(2 - m)^{-1}.$$

This establishes the asserted bound on $\| L_n \|$. Now let f be any element of C and let s be its best approximation in S_n. Meir and Sharma, improving upon results in [2], have shown in [1] that $\|f - s\| \leqslant 5\omega(f; h)$. Consequently,

$$\begin{aligned}
\|f - Lf\| &= \|(f - s) - L(f - s)\| \\
&= \|(I - L)(f - s)\| \\
&\leqslant \|I - L\|\,\|f - s\| \\
&\leqslant (1 + \|L\|)\,\|f - s\| \\
&\leqslant [2 + \tfrac{3}{2} m^2 (2 - m)^{-1}]\, 5\omega(f; h) \\
&\leqslant 30(2 - m)^{-1}\, \omega(f; h).
\end{aligned}$$

These results emphasize what was pointed out in [2], namely, that the convergence depends not on the boundedness of $K_n = \max_i h_i^{(n)} / \min_i h_i^{(n)}$ but upon the magnitude of $m_n = \max_{|i-j|=1} h_i^{(n)} / h_j^{(n)}$.

CONJECTURE. In order that $L_n f \to f$ for all $f \in C$, it is necessary that $\limsup m_n < 2$.

References

1. A. Meir and A. Sharma, On uniform approximation by cubic splines, *J. Approximation Theory* **2** (1969), 270–274.
2. E. W. Cheney and F. Schurer, A note on the operators arising in spline approximation, *J. Approximation theory* **1** (1968), 94–102.

E. W. Cheney*

Department of Mathematics,
Michigan State University,
East Lansing, Michigan 48823

F. Schurer

Technological University Eindhoven,
Eindhoven, The Netherlands

Received October 3, 1969

* The first author was supported by the Air Force Office of Scientific Research.

Printed in Belgium