Primitive 2-structures with the \((n - 2)\)-property

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Abstract


A fundamental notion in the theory of 2-structures is that of a primitive 2-structure. In (Ehrenfeucht, Rozenberg 1990), it is proved that primitivity is hereditary in the sense that each primitive 2-structure on \(n\) elements, where \(n \geq 3\), contains a primitive substructure on either \(n - 1\) or \(n - 2\) elements. In this paper we determine the class of primitive 2-structures on \(n\) elements that do not contain primitive substructures on \(n - 1\) elements: these 2-structures are said to satisfy the \((n - 2)\)-property. We show that for each \(n \geq 3\), there is a restricted number of primitive 2-structures on \(n\) elements satisfying this property. In fact, for each \(n > 4\), there are four different reversible 2-structures up to isomorphism, satisfying the \((n - 2)\)-property, while for \(n\) odd, there are five different 2-structures with this property.

1. Introduction

The theory of 2-structures, introduced by Ehrenfeucht and Rozenberg [4], is a convenient framework to investigate graphs as well as other mathematical structures in Computer Science [2]. A key notion in this theory which is related to the main result about 2-structures, a strong decomposition theorem [5], is that of a primitive 2-structure. In fact, primitive 2-structures are one of the basic components from which 2-structures are decomposed or constructed [5]. It is shown that understanding primitivity is crucial for the comprehension of 2-structures [3, 4, 6]. The study of this notion is also relevant in graph theory and some related areas where primitivity has
been introduced under different names, in relation to a decomposition of graphs [9,11]. This decomposition is used to obtain fast algorithms for problems on graphs and partial orders [10]. Since graphs are represented by 2-structures, as is shown in [4], the study of primitive 2-structures easily leads to understanding primitive graphs.

Some steps in the direction of the understanding of primitivity have been taken in [3,6] and [7], where this problem is analyzed by expressing the primitivity of a 2-structure in terms of its substructures. In [3] it is proved that primitivity is hereditary in the sense that a primitive 2-structure on \( n \) elements contains a primitive substructure on either \( n-1 \) or \( n-2 \) elements. This means that the primitivity of a 2-structure is preserved by the removal of one or two elements from the 2-structure. This result leaves open the problem of characterizing the class of the primitive 2-structures on \( n \) elements which are minimal in the sense that the removal of any element of the 2-structure results in a substructure which is nonprimitive.

In this paper we solve this problem by describing this class of primitive 2-structures. We prove that for each \( n > 3 \), there is a limited number of 2-structures on \( n \) elements with the \((n-2)\)-property. The characterization of these 2-structures is based on the notion of a chain of clans of a 2-structure, which is a sequence of elements of the 2-structure, \( c = \langle x_1, \ldots, x_n \rangle \), such that when an element \( x_i \in c \), for \( 1 < i < n \), is removed from the 2-structure, a unique nontrivial clan consisting of the adjacent elements of \( x_i \) in \( c \), i.e. \( x_{i-1}, x_{i+1} \), is obtained. In fact, the \((n-2)\)-property is related to the existence in a primitive 2-structure of a maximal chain of clans whose elements form the whole domain of the 2-structure, except for at most one element.

We prove that the \((n-2)\)-property is hereditary in the sense that given a 2-structure \( g \) with the \((n-2)\)-property, all primitive substructures of \( g \) on \( n > 3 \) elements have this property.

### 2. Preliminaries

We now recall some of the basic notions of the theory of 2-structures. Most of the notation here is the same as in the work [4], to where the reader is referred for more details on the theory of 2-structures.

For a finite set \( D \), \(|D|\) denotes its cardinality and \( \emptyset \) denotes the empty set. A 2-edge over \( D \) is an ordered pair \((x, y)\) such that \( x, y \in D \) and \( x \neq y \); \( E_2(D) \) denotes the set of all 2-edges over \( D \). For a 2-edge \((x, y)\), its reverse, denoted \((x, y)^{-1}\), is the 2-edge \((y, x)\). In this paper we consider finite sets only.

**Definition 2.1.** A 2-structure, is an ordered pair \((D, R)\) such that \( D \) is a nonempty finite set, and \( R \) is an equivalence relation on \( E_2(D) \).

The term “2-structure” is abbreviated as \( 2s \). For a \( 2s \) \( g = (D, R) \), \( D \) is referred to as the domain of \( g \), and \( R \) as the equivalence relation of \( g \). We use \( \text{dom}(g) \) and \( \text{rel}(g) \) to denote \( D \) and \( R \), respectively. We say that \( e, e' \in E_2(D) \) are \( g \)-equivalent (or simply equivalent) iff \( e \sim R e' \).
Definition 2.2. Let $g=(D, R)$ be a 2s, and let $X$ be a nonempty subset of $D$. The substructure of $g$ induced by $X$, denoted $\text{sub}_g(X)$, is the 2s $h=(X, R')$, such that $R'=R\cap(E_2(X) \times E_2(X))$. A 2s $h$ is a substructure of $g$ iff there exists $X \subseteq D$ such that $h=\text{sub}_g(X)$.

Given a 2s $g=(D, R)$ and an element $x \in D$, to simplify the notation, we use $g_{-x}$ rather than $\text{sub}_g(D-\{x\})$ to denote the substructure of $g$ induced by $D-\{x\}$.

A pictorial representation of the 2s $g$ is given by an edge-labeled graph, where equivalent edges get the same edge label, with edges in different equivalence classes getting different edge labels. This is illustrated in Fig. 1. A double arrowed edge with one label denotes an edge that is equivalent to its reverse.

Example 2.1. Let $g=(D, R)$ be a 2s, where $R$ induces on $E_2(D)$ the following partition $\mathcal{P} = \{P_1, P_2, P_3\}$ (see Fig. 1(a)):

- $P_1 = \{(1,3), (3,1), (1,2), (3,2)\}$
- $P_2 = \{(2,1), (2,4), (4,2), (4,1), (4,3), (2,3)\}$
- $P_3 = \{(1,4), (3,4)\}$

Let $X = \{1,3,4\}$, then $\text{sub}_g(X) = (X, R')$, where $R'$ induces on $E_2(X)$ the partition $\mathcal{P}' = \{P'_1, P'_2, P'_3\}$ (see Fig. 1(b))

- $P'_1 = X \cap P_1 = \{(1,3), (3,1)\}$
- $P'_2 = X \cap P_2 = \{(4,3), (4,1)\}$
- $P'_3 = X \cap P_3 = \{(3,4), (1,4)\}$

The following is the central notion of the theory of 2-structures.

Definition 2.3. Let $g=(D, R)$ be a 2s. A clan of $g$ is a subset $X$ of $D$, such that for all $z \in D-X$, and all $x, y \in X$, $(z, x) R (z, y)$ and $(x, z) R (y, z)$.

Hence, a subset $X$ of the domain of a 2s $g$ is a clan iff all elements of $X$ are connected by equivalent edges in the same way to each element from outside of $X$, while each element from outside of $X$ is “connected” in the same way to all elements of $X$. We use
\(\mathcal{C}(g)\) to denote the set of all clans of \(g\). Obviously, \(\emptyset, D \in \mathcal{C}(g)\) and for each \(x \in D\), \(\{x\} \in \mathcal{C}(g)\). These clans are called trivial. We use \(\mathcal{C}_2(g)\) to denote the set of nontrivial clans of a 2s \(g\). Obviously for each \(X \in \mathcal{C}_2(g)\), \(|X| \geq 2\), and \(D \notin \mathcal{C}_2(g)\). To simplify the notation we write \(\mathcal{C}_2(g) = \{X\}\).

We recall some basic properties of clans of a 2s. The next proposition describes the relationship between clans of a 2s and clans of its substructures.

**Proposition 2.1** ([4]). Let \(g = (D, R)\) be a 2s and let \(X\) be a nonempty subset of \(D\) and let 
\(h = \text{sub}_X(D)\). If \(Y \in \mathcal{C}(h)\), then \(Y \cap X \in \mathcal{C}(h)\).

**Proposition 2.2** ([4]). Let \(g = (D, R)\) be a 2s and let \(X, Y \in \mathcal{C}(g)\). Then
1. \(X \cap Y \in \mathcal{C}(g)\),
2. if \(X \cap Y \neq \emptyset\), then \(X \cup Y \in \mathcal{C}(g)\) and
3. if \(Y - X \neq \emptyset\), then \(X - Y \in \mathcal{C}(g)\).

**Example 2.2.** Let \(g = (D, R)\) be the 2s of Fig. 1(a). Then
\[\mathcal{C}(g) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}, \{1, 3\}\} \quad \text{and} \quad \mathcal{C}_2(g) = \{1, 3\}\).

A subclass of the class of 2-structures that plays an important role in proving properties of 2-structures is that of the reversible 2-structures.

**Definition 2.4.** A 2s \(g = (D, R)\) is a reversible 2-structure, abbreviated r2s, iff for all \(e_1, e_2 \in E_2(D)\), \(e_1 R e_2\) implies \(e_1^{-1} R e_2^{-1}\).

**Example 2.3.** Let \(g = (D, R)\) be a 2s, where \(R\) induces on \(E_2(D)\) the following partition
\[\mathcal{P} = \{P_1, P_2, P_3, P_4\}\] see Fig. 2
\[P_1 = \{(1, 5), (5, 1), (3, 5), (5, 3), (2, 4), (4, 2), (1, 3), (3, 1)\},\]
\[P_2 = \{(2, 1), (2, 3), (4, 3), (4, 1)\},\]
\[P_3 = \{(3, 4), (1, 4), (3, 2), (1, 2)\},\]
\[P_4 = \{(5, 4), (4, 5), (5, 2), (2, 5)\}.

Clearly, \(g\) is a reversible 2s.

Given an arbitrary 2-structure \(g\), there is an easy construction that allows one to consider a reversible version of \(g\) which is defined as follows.

**Definition 2.5.** Let \(g = (D, R)\) be a 2s. The reversible version of \(g\), denoted \(\text{rver}(g)\), is the 2s \(g' = (D, R')\), where \(R'\) is the equivalence relation on \(E_2(D)\) defined by \(e_1 R e_2\) iff \(e_1 R e_2\) and \(e_1^{-1} R e_2^{-1}\).

In [4] some basic properties of the reversible version of a 2s are proved. These explain the importance of reversible 2-structures in proving properties of 2-structures.
Proposition 2.3. Let \( g \) be a 2s.

1. \( \text{rver}(g) \) is reversible.
2. A 2s \( h \) is a substructure of \( g \) iff \( \text{rver}(h) \) is a substructure of \( \text{rver}(g) \).
3. \( \mathcal{C}(g) = \mathcal{C}(\text{rver}(g)) \).

By this result a 2s and its reversible version, which is a reversible 2s, have the same set of clans. Moreover, if \( h \) is a substructure of \( g \), then \( h \) has the same set of clans as \( \text{rver}(h) \), which is the substructure of \( \text{rver}(g) \) induced by the same subset of the domain that induces \( h \) in \( g \). By this fact, all properties concerning clans of a 2s \( g \) and its substructures, can be proved using \( \text{rver}(g) \) instead of \( g \) itself. It can be easily seen that the condition to be satisfied by a set in order for it to be a clan is simpler for an r2s. In fact, given an r2s \( g = (D, R) \), a subset \( X \) of \( D \) is a clan of \( g \) when for all \( z \in D - X \), and all \( x, y \in X \), \( (z, x) R (z, y) \). Hence it is convenient to prove properties about clans using reversible 2-structures instead of arbitrary 2-structures. In this sense \( \text{rver}(g) \) can be considered as a "normal form" for a 2s.

Let us recall in the following some natural notions about 2-structures that are particularly useful in the theory of reversible 2-structures. Some of these notions are extensively illustrated in [6].

Definition 2.6. Let \( g = (D, R) \) be a 2s. A 2-edge \( e \in E_2(D) \) is symmetric iff \( e R e^{-1} \), otherwise \( e \) is asymmetric.

A 2s \( g \) is called symmetric if it consists of only symmetric edges; \( g \) is antisymmetric iff \( g \) consists of only asymmetric edges.

Given a 2s \( g = (D, R) \), the equivalence relation \( R \) induces a partition \( \mathcal{P} \) of all 2-edges over \( D \). Thus, depending on the context, \( g \) can also be conveniently represented in the form \( g = (D, \mathcal{P}) \). The partition \( \mathcal{P} \) is denoted \( \text{part}(g) \) when not named explicitly. In the paper we will use the notation \( g = (D, \mathcal{P}) \), when we mean \( \mathcal{P} \) to be a partition, otherwise we will denote \( g \) in the form \( (D, R) \), when \( R \) is meant to be the equivalence relation of \( g \).
For $P \in \mathcal{P}$, $P^{-1}$ denotes the set $\{ e^{-1} : e \in P \}$. If $g$ is an r2s, then by Definition 2.4, $P^{-1} \in \mathcal{P}$. Clearly for each $P \in \mathcal{P}$, either $P = P^{-1}$ and all edges in $P$ are symmetric edges and $P$ is called symmetric, or $P \cap P^{-1} = \emptyset$ and all edges in $P$ are asymmetric and $P$ is called antisymmetric. Given $P \in \mathcal{P}$, the symmetric closure of $P$, denoted by $\text{sym}(P)$, is the set $P \cup P^{-1}$.

For an r2s $g$, and each $P \in \mathcal{P}$, a feature of $g$ is defined as the set $\{ P, P^{-1} \}$ if $P$ is antisymmetric, or as the singleton $\{ P \}$ if $P$ is symmetric. In particular a feature is called symmetric if it consists of a singleton, otherwise it is called antisymmetric.

The 2s $g$ can be described through the set of its features, and hence $g$ is represented in the form $g = (D, \mathcal{F})$, where $\mathcal{F}$ is the set of all its features.

**Example 2.4.** Let $g$ be the r2s of Example 2.3. $P_1, P_4$ are symmetric, and $P_2, P_3$ are antisymmetric; hence $g$ is neither symmetric nor antisymmetric. Then $g$ can be described through the set of its features, i.e. $g = (D, \mathcal{F})$, where $\mathcal{F} = \{ F_1, F_2, F_3 \}$, $F_1 = \{ P_1 \}$, $F_2 = \{ P_4 \}$ and $F_3 = \{ P_2, P_3 \}$. $F_1$, $F_2$ are symmetric, while $F_3$ is antisymmetric.

The following subclass of the class of 2-structures is important in the theory of 2-structures.

**Definition 2.7.** Let $g = (D, R)$ be a 2s. Then $g$ is primitive iff $g$ contains only trivial clans.

It is proved in [5] that primitive 2-structures are one of the basic types from which each 2s can be constructed or decomposed.

Next lemma follows directly from the definition of a clan and of a primitive 2s.

**Lemma 2.1.** Let $g = (D, R)$ be a primitive 2s and let $x_0 \in D$ such that $X \in \mathcal{C}_2 (g_{x_0})$. Then for each $z \in D$, $z \neq x_0$, $X \notin \mathcal{C}(g_{-z})$.

An important notion that arises in the theory of 2-structures, when we assume an algebraic perspective, is that of isomorphism between different 2-structures [4]. This notion is naturally defined as follows:

**Definition 2.8.** Let $g_1 = (D_1, R_1)$ and $g_2 = (D_2, R_2)$, be 2-structures. A function $\phi : D_1 \rightarrow D_2$, is an isomorphism from $g_1$ onto $g_2$, iff $\phi$ is a bijection such that for all $x, y$, $u, v \in D$, $(\phi(x), \phi(y))$ and $(\phi(u), \phi(v))$ are $g_2$-equivalent iff $(x, y), (u, v)$ are $g_1$-equivalent.

We will use this notion in order to enumerate the objects in the class of primitive 2-structures that satisfy the $(n - 2)$-property.

3. The $(n - 2)$-property

In [3] it has been proved that primitivity is hereditary in the following sense.

**Proposition 3.1.** Let $g$ be a primitive 2s such that $|\text{dom}(g)| \geq 3$. Then there exists a primitive substructure $h$ of $g$ such that either $|\text{dom}(h)| = |\text{dom}(g)| - 1$ or $|\text{dom}(h)| = |\text{dom}(g)| - 2$. 
In this paper we investigate primitive 2-structures on \( n > 4 \) elements which are minimal in the sense that they do not contain primitive substructures on \( n-1 \) elements. This condition for a primitive 2-structure is referred to as the \((n-2)\)-property.

**Definition 3.1.** Let \( g = (D, R) \) be a primitive 2-structure. Then \( g \) satisfies the \((n-2)\)-property iff for every \( x \in D \), each substructure \( g_{-x} \) of \( g \) induced by \( D - \{x\} \) is not primitive.

Obviously by Proposition 3.1, a primitive 2s satisfying the \((n-2)\)-property contains a primitive substructure induced by \( D - \{x, y\} \) for some \( x, y \in D \).

### 4. The local and global elements of a 2s

In the next sections, we investigate the existence of 2-structures with the \((n-2)\)-property. Since this property concerns clans of substructures, and the notion of clan is simpler for reversible 2-structures, as seen in Section 2, it will be easier to analyze the \((n-2)\)-property for the class of reversible 2-structures. Furthermore, the results obtained in the paper about this class of 2-structures can be easily extended to the case of arbitrary 2-structures (see Proposition 2.3). Hence, in the rest of the paper we will consider only reversible 2-structures, except in the last section, where we determine the class of arbitrary 2-structures with the \((n-2)\)-property.

A fundamental step to analyze primitive 2-structures consists in investigating reductions for these 2-structures, that is how primitivity is changed by the removal of elements from the domain.

In this section we investigate methods of “destroying” the primitivity of an r2s \( g \). Given a substructure \( h \) of \( g \), there are two methods of “destroying” the primitivity of \( h \) by extending it by one element of \( g \), these are referred as “local” and “global”.

**Lemma 4.1 ([3]).** Let \( g = (D, R) \) be an r2s, let \( D_0 \subseteq D \) be such that \( |D_0| \geq 3 \) and \( \text{sub}_g(D_0) \) is primitive and let \( x \in D - D_0 \) be such that \( \text{sub}_g(D_0 \cup \{x\}) \) is not primitive. Then \( \text{sub}_g(D_0 \cup \{x\}) \) has a unique nontrivial clan, and moreover either

1. \( D_0 = \mathcal{E}_2(\text{sub}_g(D_0 \cup \{x\})) \), or
2. \( \{x, y\} = \mathcal{E}_2(\text{sub}_g(D_0 \cup \{x\})) \), for some \( y \in D_0 \).

Let \( g = (D, R) \) be an r2s and \( D_0 \subseteq D \), such that \( |D_0| \geq 2 \) and \( \text{sub}_g(D_0) \) is primitive. We call \( x \in D - D_0 \) local for \( D_0 \) in \( g \), iff \( \text{sub}_g(D_0 \cup \{x\}) \) has a unique 2-element clan \( \{x, k\} \), and in this case we denote the element \( k \in D_0 \) by \( \text{uni}_g(D_0, x) \). We call \( x \) global for \( D_0 \) in \( g \) iff \( D_0 \) is the only nontrivial clan of \( \text{sub}_g(D_0 \cup \{x\}) \).

In Lemma 4.4 we shall prove that an r2s \( g = (D, R) \), satisfying \((n-2)\)-property has two elements \( x, y \in D - \{x, y\} \), which are both local for \( D - \{x, y\} \) in \( g \).

Before introducing this result, we give some technical lemmas which will be used to prove it.
Lemma 4.2. Let \( g = (D, R) \) be an \( r_2s \) and let \( \{ x, a \} = \mathcal{C}_2(g) \) for \( a, x \in D \). Then \( \text{sub}_g(D - \{ x \}) \) is a primitive substructure of \( g \).

Proof. If \( |D| = 3 \), the lemma is immediate, so assume that \( |D| > 3 \). Let \( h = \text{sub}_g(D - \{ x \}) \). Assume to the contrary that \( h \) is nonprimitive. Hence \( h \) contains a nontrivial clan; let \( X \) be such a clan.

(i) Suppose that \( a \notin X \). Then for all \( t_1, t_2 \in X \subset D - \{ x \} \), \( (a, t_1) R (a, t_2) \). Since \( \{ x, a \} = \mathcal{C}_2(g) \), \( (x, t) R (a, t) \) for all \( t \in X \), thus implying that \( (x, t_1) R (x, t_2) \). This implies that \( X \) is a nontrivial clan of \( g \), where \( X \neq \{ x, a \} \), thus contradicting the fact that there is a unique nontrivial clan in \( g \).

(ii) Suppose that \( a \in X \). Then for each \( d \in D - \{ x \} \), \( d \notin X \), \( (d, a) R (d, t) \) for all \( t \in X \). Since \( \{ x, a \} = \mathcal{C}_2(g) \), \( (d, a) R (d, x) \), consequently \( (d, x) R (d, t) \). This implies that \( X \cup \{ x \} \neq \{ x, a \} \) is a clan of \( g \) that is nontrivial, since \( X \subset D - \{ x \} \), but this contradicts the fact that there is only one nontrivial clan in \( g \).

Then (i) and (ii) imply that the assumption that \( h \) is not primitive leads to a contradiction. Thus the lemma holds. \( \Box \)

Corollary 4.1. Let \( g = (D, R) \) be an \( r_2s \) such that \( |D| \geq 3 \). The following statements are equivalent:

1. \( \{ x, a \} = \mathcal{C}_2(g) \), with \( x, a \in D \),
2. \( x \) is local for \( D - \{ x \} \) in \( g \), where \( \text{uni}_g(D - \{ x \}, x) = a \).

Proof. Follows from Lemma 4.2 and the definition of a local element. \( \Box \)

We now define a new notion of locality and globality of an element in an \( r_2s \): that of an element \( x \) that is local or global for the domain of the primitive \( r_2s \) obtained by removing \( x \) from \( g \). In this case, \( x \) is called \( r \)-local or \( r \)-global (we use the prefix \( r \)-). These notions will simplify the comprehension of the lemmas in the rest of the paper.

Definition 4.1. Let \( g = (D, R) \) be an \( r_2s \) and let \( x \) be an element of \( D \). Then \( x \) is \( r \)-local in \( g \) iff \( x \) is local for \( D - \{ x \} \) in \( g \), and \( x \) is \( r \)-global in \( g \) iff \( x \) is global for \( D - \{ x \} \) in \( g \).

Let us now investigate the local and global elements in a primitive \( r_2s \) with the \((n-2)\)-property.

Lemma 4.3. Let \( g = (D, R) \) be a primitive \( r_2s \) satisfying the \((n-2)\)-property and let \( |D| = 4 \). Then each element \( x \in D \) is either local or global for \( D - \{ x, y \} \) in \( g \), for any \( y \in D - \{ x \} \).

Proof. Since \( g \) satisfies the \((n-2)\)-property, \( g_{-d} \) and \( g_{-d'} \) are not primitive for all \( d, d' \in D \). Suppose that \( X_1 \in \mathcal{C}_2(g_{-d}) \) and \( X_2 \in \mathcal{C}_2(g_{-d'}) \). Then \( |X_1| = |X_2| = 2 \), and \( X_1 \neq X_2 \), otherwise by Lemma 2.1, we contradict the fact that \( g \) is primitive. Moreover the following condition (a) is verified: if \( X_1 = \{ d', k \} \), \( X_2 = \{ d, k' \} \), for \( k, k' \in D - \{ d, d' \} \), then \( k \neq k' \). In fact, if \( k = k' \), then it follows that \( \{ d, d', k \} \) is a clan of \( g \), which contradicts the primitivity of \( g \).
Let us determine the nontrivial clans of $g_{-y}$ for an arbitrary element $y \in D$. Clearly, any 2-element subset of $D - \{y\}$ can be a nontrivial clan of $g_{-y}$. Assume that $g_{-y}$ has at least two nontrivial clans $X_1$ and $X_2$. Obviously, $|X_1 \cap X_2| = 1$, hence let $X_1 = \{a, b\}$ and $X_2 = \{x, a\}$ for $\{a, b, x\} = D - \{y\}$. By Lemma 2.1 and condition (a), we have that $\{y, b\}$ is the unique nontrivial clan of $g_{-x}$. Hence, by Lemma 2.1 and condition (a), it follows that $g_{-y}$ has no nontrivial clans, which is not possible. Thus, $g_{-y}$ must have a unique nontrivial clan. This proves that $x$ is either local or global for $D - \{x, y\}$ for any $y \in D - \{x\}$.

The next result will be important for investigating the $(n-2)$-property of primitive 2-structures.

**Lemma 4.4.** Let $g = (D, R)$ be a primitive r2s that satisfies the $(n-2)$-property. Then there exist two elements $x, y \in D$ such that $\text{sub}_g(D_0), D_0 = D - \{x, y\}$, is primitive and $x, y$ are both local for $D_0$ in $g$, where $\text{uni}_g(D_0, x) \neq \text{uni}_g(D_0, y)$.

**Proof.** Since $g$ satisfies the $(n-2)$-property, there exists $D_0 \subseteq D$ such that $\text{sub}_g(D_0)$ is primitive and $D_0 = D - \{x, y\}$ for $x, y \in D$. Since $g_{-x}$ and $g_{-y}$ are not primitive, by Lemma 4.1 (or Lemma 4.3 if $|D| = 4$), each of $x, y$, is local or global for $D_0$ in $g$. We have three cases to consider.

(i) $x, y$ are both local for $D_0$ in $g$.

(ii) $x$ is local and $y$ is global for $D_0$ in $g$. Since $x$ is local for $D_0$ in $g$, then $\{x, a\} = \mathcal{S}_2(g_{-y})$ for some $a \in D_0$ and by Corollary 4.1, $a$ is local for $D - \{a, y\}$ in $g$. Let $D' = D - \{a, y\}$. Thus $\text{sub}_g(D')$ is primitive. Since $g_{-x}$ is not primitive, by Lemma 4.1 (or Lemma 4.3 if $|D| = 4$), $y$ is local or global for $D'$ in $g$. Let us assume that $y$ is global. This implies that $(y, x) R (y, d)$, for $x, d \in D - \{a, y\}$. Since $y$ is also global for $D_0$ in $g$, $(y, a) R (y, d)$, which implies that $(y, x) R (y, a)$. Consequently, since $\{x, a\} = \mathcal{S}_2(g_{-y})$, it follows that $\{x, y\}$ is a clan of $g$, and this contradicts the fact that $g$ is primitive. Thus $y$ must be local for $D'$ in $g$. Hence, $a$ and $y$ are both local for $D - \{y, a\}$ in $g$.

(iii) $x$ is global and $y$ is local for $D_0$ in $g$. This case is analogous to the previous one.

(iv) $x, y$ are global for $D_0$ in $g$. This implies that $D_0$ is a trivial clan of $g$, which leads to a contradiction.

By cases (i), (ii), (iii) and (iv), it follows that there exist two elements $x_1, x_2 \in D$ that are local for $D - \{x_1, x_2\} = D'$ in $g$. Let us assume that $\text{uni}_g(D', x_1) = \text{uni}_g(D', x_2) = a$. Thus, by the definition of a local element, we have that $\{x_1, x_2, a\}$ is a nontrivial clan of $g$, which leads to a contradiction. Hence, $\text{uni}_g(D', x_1) \neq \text{uni}_g(D', x_2)$. Thus the lemma holds.

The following lemmas illustrate properties of local elements in an r2s.

**Lemma 4.5.** Let $g = (D, R)$ be an r2s and let $D_0 \subseteq D$ be such that $\text{sub}_g(D_0)$ is primitive and $x, y \in D - D_0$ are local for $D_0$ in $g$ and let $\text{uni}_g(D_0, x) = x_1$ and $\text{uni}_g(D_0, y) = y_1$. If $x_1 \neq y_1$, then $(x, y) R (x_1, y_1)$.

**Proof.** Since $x$ is local for $D_0$ in $g$, then $(y, x) R (y_1, x_1)$; similarly since $y$ is local for $D_0$ in $g$, $(x_1, y) R (x_1, y_1)$ and consequently $(x, y) R (x_1, y)$. □
Fig. 3 illustrates Lemma 4.5.

**Lemma 4.6.** Let \( g = (D, R) \) be an \( r2s \) and let \( x, y \in D \) be such that \( \text{sub}_g(D_0), D_0 = D - \{x, y\}, \) is primitive and both \( x, y \) are local for \( D_0 \) in \( g \), where \( x_1 = \text{uni}_g(D_0, x) \neq \text{uni}_g(D_0, y) = y_1 \). If \( g \) is not primitive, then \( \mathcal{C}_2(g) = \{ \{x, x_1\}, \{y, y_1\}\} \).

**Proof.** If \( g \) is not primitive, \( g \) contains a nontrivial clan \( X \). Since \( x, y \) are both local for \( D_0 \) in \( g \), \( \{x, x_1\} = \mathcal{C}_2(g - x) \) and \( \{y, y_1\} = \mathcal{C}_2(g - y) \). By Proposition 2.1, \( X - \{x\} \in \mathcal{C}(g - x) \) and \( X - \{y\} \in \mathcal{C}(g - y) \). By the uniqueness of the nontrivial clan in \( g - x \) and \( g - y \), we have \( X = \{x, x_1\} \), or \( X = \{y, y_1\} \) or \( X = \{x, y\} \). Since \( x \) is local, if \( X = \{x, y\} \), then \( X \cup \{x_1\} \) is clan of \( g \), but this is a contradiction. Now we prove that \( \{x, x_1\} \in \mathcal{C}_2(g) \) iff \( \{y, y_1\} \in \mathcal{C}_2(g) \). In fact, let \( \{d, \text{uni}_g(D_0, d)\} \) be a clan of \( g \), for \( d \in \{x, y\} \) and \( d' = \{x, y\} - d \). Thus \( (d', d) R (d', \text{uni}_g(D_0, d)) \), by Lemma 4.5 \( (d', \text{uni}_g(D_0, d)) R (\text{uni}_g(D_0, d'), d) \), which implies that \( (d', d) R (\text{uni}_g(D_0, d'), d) \) and since \( d' \) is local, it follows that \( \{d', \text{uni}_g(D_0, d')\} \) is a clan of \( g \). Thus the lemma holds.

Given a primitive reversible 2-structure \( g = (D, R) \) satisfying the \((n - 2)\)-property, there are two local elements \( x, y \in D \) for \( D - \{x, y\} \) in \( g \). Now we analyze how the primitivity of the 2-structure \( g \) is destroyed by removing \( \text{uni}_g(D_0, x) \) from \( g \) (see Corollary 4.2). Similarly to the case of Lemma 4.1, there are two methods of "destroying" the primitivity of \( g \) by removing \( \text{uni}_g(D_0, x) \). These methods are analogously referred to as "local" and "global".

**Lemma 4.7 (Global–local rule).** Let \( g = (D, R) \) be an \( r2s \) such that \( |D| > 4 \). Let \( x, y \in D \) be such that \( \{x, a\} = \mathcal{C}_2(g - x), \{y, b\} = \mathcal{C}_2(g - y), \) with \( a, b \in D \). Let \( g - a \) be nonprimitive. Then either

1. \( y \) is \( r \)-global in \( g - a \), or
2. \( y \) is \( r \)-local in \( g - a \), where \( \{y, k\} = \mathcal{C}_2(g - a) \), for \( k \in D - \{x, y\} \), such that \( k \neq b \) iff \( g \) is primitive.
Proof. By Corollary 4.1, \( x, y \) are local for \( D - \{x, y\} \) in \( g \). By Lemma 4.2, since \( \{x, a\} = \mathcal{C}_2(g_{-a}) \), \( \text{sub}_g(D_0) \) is primitive, \( D_0 = D - \{a, y\} \). Since \( g_{-a} = \text{sub}_g(D_0 \cup \{y\}) \) is not primitive, by Lemma 4.1, \( y \) is \( r \)-local or \( r \)-global in \( g_{-a} \). Assume that \( y \) is \( r \)-local where \( \{y, k\} = \mathcal{C}_2(g_{-a}) \). Assume first that \( k = x \). Since \( y \) is local for \( D - \{x, y\} \), it follows that \( \{y, x, b\} \) is a nontrivial clan of \( g_{-a}(|\text{dom}(g_{-a})| \geq 4) \), which contradicts the fact that \( \{y, k\} \) is the unique nontrivial clan of \( g_{-a} \). Thus \( k \in D - \{x, y\} \).

Let us now prove that \( k \neq b \) iff \( g \) is primitive. Let \( g \) be primitive. Since \( \{y, b\} = \mathcal{C}_2(g_{-a}) \), \( (x, y), (x, b) \) are not equivalent. Thus \( \{y, b\} \) cannot be a clan of \( g_{-a} \). Hence \( k \neq b \). On the other end assume that \( k \neq b \). Let \( g \) be nonprimitive. By Lemma 4.6, \( \mathcal{C}_2(g) = \{\{y, b\}, \{x, a\}\} \), which implies by Proposition 2.1 that \( \{y, b\} \in \mathcal{C}_2(g_{-a}) \). Since \( \{y, k\} \) is the unique nontrivial clan of \( g_{-a} \), \( k = b \), thus obtaining a contradiction. Hence if \( k \neq b \), \( g \) must be primitive. \( \square \)

Lemma 4.8. Let \( g = (D, R) \) be a primitive 2-s such that \( |D| = 4 \). Let \( x, y \in D \) be local for \( D_0 \) in \( g \), where \( D_0 = D - \{x, y\} \) and \( \text{uni}_g(D_0, x) = a \) for \( a \in D \). Let \( g_{-a} \) be nonprimitive. Then \( y \) is \( r \)-global in \( g_{-a} \).

Proof. Let \( X \) be a nontrivial clan of \( g_{-a} \). Since \( \{y, b\} = \mathcal{C}_2(g_{-x}) \) and \( g \) is primitive, \( X \neq \{y, b\} \). Thus \( X = \{x, t\} \) for \( t \in \{y, b\} \). If \( X = \{x, y\} \), then since \( \{x, a\} = \mathcal{C}_2(g_{-x}) \), it follows that \( \{x, y, a\} \) is a nontrivial clan of \( g \), which contradicts the assumption that \( g \) is primitive. Hence \( \{x, b\} = D - \{y, a\} \) is the unique nontrivial clan of \( g_{-a} \). \( \square \)

Corollary 4.2 follows directly from the definition of a local element, the global–local rule (4.7) and Lemma 4.8.

Corollary 4.2. Let \( g = (D, R) \) be a primitive 2-s and let \( x, y \in D \) be such that \( \text{sub}_g(D_0) \), \( D_0 = D - \{x, y\} \), is a primitive substructure of \( g \) and \( x, y \) are local for \( D_0 \) in \( g \), with \( \text{uni}_g(D_0, x) = a \), for \( a \in D_0 \). Let \( g_{-a} \) be nonprimitive. Then \( y \) is \( r \)-global in \( g_{-a} \), or

1. \( y \) is \( r \)-global in \( g_{-a} \), or
2. \( y \) is \( r \)-local in \( g_{-a} \), where \( \{y, k\} = \mathcal{C}_2(g_{-a}) \), for \( k \in D_0 \), \( k \neq \text{uni}_g(D_0, y) \).

The global–local rule (4.7) is useful in proving the main results of the paper about the \((n-2)\)-property (Theorems 6.1, 6.2) since it describes the nontrivial clans of nonprimitive substructures on \( n - 1 \) elements contained in primitive reversible 2-structures on \( n \) elements. In fact, given a 2-structure that satisfies the \((n-2)\)-property, there are two elements \( x, y \) that are local for \( D_0 = D - \{x, y\} \), where \( \text{uni}_g(D_0, x) = a \) and \( \text{uni}_g(D_0, y) = b \). By the global–local rule (4.7) we can determine the clans of \( g_{-a} \) and \( g_{-b} \). If \( y \) is \( r \)-local, then since \( \{y, k\} = \mathcal{C}_2(g_{-a}) \) and \( \{x, a\} = \mathcal{C}_2(g_{-x}) \), we can apply the global–local rule (4.7) to compute the clans of \( g_{-k} \) and repeat this step each time an \( r \)-local element is obtained. Thus the global–local rule (4.7) gives a general rule for determining how primitivity is violated when single elements are removed from an r2s with the \((n-2)\)-property. The way in which primitivity is destroyed is described by the notion of a chain of clans of an r2s, illustrated in the next section.
A chain of clans of an r2s $g$ consists of a sequence $c$ of different elements of $g$ such that when each element $x$ of $c$ is removed, $g_{-x}$ contains a unique nontrivial clan consisting of the two adjacent elements to $x$ in the sequence. Now we observe that the sequence $x, y, a, k$ obtained by the global–local rule (4.7) is a chain of clans, and by applying again this rule a longer chain can be computed.

5. Chains of clans

In this section we define the notion of a chain of clans of an r2s and analyze properties of chains of clans of an r2s. This notion plays an important role in studying primitive 2-structures satisfying the $(n-2)$-property.

By $(x_1, \ldots, x_m)$, we denote the sequence of elements $x_1, \ldots, x_m$.

**Definition 5.1.** Let $g = (D, R)$ be an r2s. A chain of clans of $g$ is a sequence $c = (x_1, \ldots, x_m)$ of different elements of $D$ such that $m \geq 3$ and for all $i \in \{1, \ldots, m-2\}$, 

$$\{x_i, x_{i+2}\} = \mathcal{C}_2(g_{-x_i}).$$

Clearly, Definition 5.1 also holds for arbitrary 2-structures.

**Example 5.1.** Let $g = (D, R)$ be the r2s of Fig. 2. The sequence $c = (1, 2, 3, 4)$ is a chain of clans of $g$. The r2s $g$ is not primitive since $\mathcal{C}_2(g) = \{\{1, 3\}, \{2, 4\}\}$.

By the global–local rule (4.7), it follows that an r2s containing a chain of clans of length $m > 4$ is a primitive r2s, as is proved in next lemma.

**Lemma 5.1.** Let $g = (D, R)$ be an r2s and let $c = (x_1, \ldots, x_m)$ be a chain of clans of $g$ such that $m > 4$. Then $g$ is primitive.

**Proof.** By the definition of a chain of clans $\mathcal{C}_2(g_{-x_i}) = \{x_{i-1}, x_{i+1}\}$, $\mathcal{C}_2(g_{-x_{i+1}}) = \{x_i, x_{i+2}\}$, and $\mathcal{C}_2(g_{-x_{i+2}}) = \{x_{i+1}, x_{i+3}\}$. Since $x_{i+3} \neq x_{i-1}$, the global–local rule (4.7) implies that $g$ is primitive. $\square$

**Example 5.2.** Let $g$ be the r2s of Fig. 4. The sequence $c = (1, 2, 3, 4)$ is a chain of clans. Then $g$ is primitive but does not satisfy the $(n-2)$-property.

We give some lemmas concerning properties of chains of clans of an r2s. Obviously, by Lemma 5.1 these properties refer to primitive 2-structures whenever the chains of clans have length $m > 4$.

**Lemma 5.2.** Let $g = (D, R)$ be an r2s and let $c = (x_1, \ldots, x_m)$ be a chain of clans of $g$. Then

$$(z, x_j) R (z, x_{j+2k}),$$

where $z \in D$, $z \notin \{x_j, \ldots, x_{j+2k}\}$ and $1 \leq j \leq m$ and $1 \leq j + 2k \leq m$. 


Proof. Since \( \mathcal{C}_2(g - x_{j+1}) = \{x_j, x_{j+2}\} \), then \((z, x_j) R (z, x_{j+2})\), for \( z \notin \{x_j, x_{j+1}, x_{j+2}\}\).

By transitivity of \( R \) the lemma follows. \( \square \)

The next lemmas describe what clans are obtained when the extreme elements of a chain of clans are removed and primitivity is violated.

**Lemma 5.3.** Let \( g = (D, R) \) be an \( r2s \) such that \(|D| > 4\), and let \( c = \langle x_1, \ldots, x_m \rangle \) be a chain of clans of \( g \). Then

1. if \( g - x_m \) is not primitive, \( x_{m-1} \) is \( r \)-local or \( r \)-global in \( g - x_m \).
2. if \( g - x_1 \) is not primitive, \( x_2 \) is \( r \)-local or \( r \)-global in \( g - x_1 \).

Proof. By the definition of a chain of clans, \( \mathcal{C}_2(g - x_m) = \{x_{m-1}, x_m\} \). By Lemma 4.2, \( \mathcal{C}_2(g - x_m) \) is a primitive \( r2s \). Hence, if \( g - x_m \) is not primitive, by Lemma 4.1, \( x_{m-1} \) is \( r \)-local or \( r \)-global in \( g - x_m \). Since \( c' = \langle x_m, \ldots, x_1 \rangle \) is a chain of clans, from statement (1) of the lemma, statement (2) follows. \( \square \)

**Remark 5.1.** Let \( g = (D, R) \) be a primitive \( r2s \) such that \( c = \langle x_1, \ldots, x_4 \rangle \) is a chain of clans of \( g \), and \(|D| = 4\). Then \( x_3, x_2 \) are \( r \)-local in \( g - x_4 \) and \( g - x_1 \), respectively. It is easy to verify that this follows from Lemma 4.8.

The following lemma describes what happens when \( x_m \) is removed from \( g \), and \( x_{m-1} \) is \( r \)-local in \( g - x_m \), where \( x_m, x_{m-1} \) are elements of a chain of clans of \( g \).

**Lemma 5.4.** Let \( g = (D, R) \) be an \( r2s \) such that \(|D| > 4\), and let \( c = \langle x_1, \ldots, x_m \rangle \) be a chain of clans of \( g \) such that \( x_{m-1} \) is \( r \)-local in \( g - x_m \), where \( \{x_{m-1}, k\} = \mathcal{C}_2(g - x_m) \). Then \( k \neq x_i \), for \( i \in \{2, \ldots, m\} \).

Proof. Let us assume that \( k = x_j \) for some \( x_j \) in the chain of clans of \( g \). Obviously \( k \neq x_m \). Hence \( \mathcal{C}_2(g - x_m) = \{x_{m-1}, x_1\} \) and by Lemma 4.2, \( \mathcal{C}_2(g - x_m) \) is a primitive \( r2s \). Then by Lemma 4.1, if \( g - x_j \) is not primitive, \( x_m \) is \( r \)-local or \( r \)-global in \( g - x_j \). If
Let \( j \in \{2, \ldots, m-1\} \), then by the definition of a chain of clans \( \mathscr{C}_2(g-x_j) = \{x_{j-1}, x_{j+1}\} \). Hence \( x_m \) must be \( r \)-local, thus implying that \( \mathscr{C}_2(g-x_j) = \{x_{m}, t\} \), where \( t \in D - \{x_{m}, x_j\} \). Consequently \( x_{j+1} \) must be equal to \( x_m \). (Obviously \( x_m \neq x_{j-1} \) because \( x_m \) is the last element in the chain of clans). This implies that \( x_j = x_{m-1} \), and consequently the previous assumption \( \mathscr{C}_2(g-x_m) = \{x_{m-1}, x_j\} \) is not verified when \( j \in \{2, \ldots, m\} \). Thus the lemma holds.

6. The main theorem

In this section we prove the main result of this paper stating that there exists a restricted number of primitive 2-structures satisfying the \((n-2)\)-property.

The proof of this result consists of the following steps. Let \( g \) be a primitive 2s satisfying the \((n-2)\)-property. We show first that there exists a chain of clans of \( g \) which has length at least four. We let then \( c \) be a maximal chain of clans, and show that \( c \) contains all the elements of the domain excepting at most one element. This result gives a characterization of 2-structures with the \((n-2)\)-property, in terms of chains of clans. Finally, in Propositions 6.1, 6.2 and 6.3, we completely describe all 2-structures that have this characterization.

**Lemma 6.1.** Let \( g=(D, R) \) be a primitive 2s satisfying the \((n-2)\)-property. Then \( g \) contains a chain of clans \( c=\langle x_1, x_2, \ldots, x_m \rangle \) such that \( 4 \leq m \leq |D| \).

**Proof.** By Lemma 4.4, there exists \( D_0 = D - \{x, y\} \), such that \( \text{sub}_g(D_0) \) is primitive and \( x, y \) are both local for \( D_0 \) in \( g \). Let \( a = \text{uni}_g(D_0, x) \) and \( b = \text{uni}_g(D_0, y) \). Hence \( \mathscr{C}_2(g-x) = \{x, a\} \) and \( \mathscr{C}_2(g-x) = \{y, b\} \), and thus the sequence \( a, y, x, b \) is a chain of clans of \( g \) as required. \( \square \)

Next Theorem 6.1 shows the characterization of the 2-structures satisfying the \((n-2)\)-property in terms of their chain of clans; such a result is obtained by proving that if more than one element of the domain of an 2s \( g \) is not contained in the maximal chain of clans \( c \) of \( g \), then \( g \) must not be primitive. Moreover, the chain \( c \) must be either cyclic or bordered as specified in Definition 6.1.

**Definition 6.1.** Let \( c=\langle x_1, \ldots, x_m \rangle \) be a chain of clans of an 2s \( g \). Then \( c \) is cyclic iff \( x_2, x_{m-1} \) are \( r \)-local in \( g-x_1, g-x_m \), respectively, with \( \mathscr{C}_2(g-x_1) = \{x_2, x_m\} \) and \( \mathscr{C}_2(g-x_m) = \{x_{m-1}, x_1\} \). The chain \( c \) is bordered iff \( x_2, x_{m-1} \) are \( r \)-global in \( g-x_1, g-x_m \), respectively.

In the above definition, cyclic means that \( c \) has only local elements, and hence for each \( i, 1 \leq i \leq m \), also \( \langle x_i, \ldots, x_m, x_1, \ldots, x_{i-1} \rangle \) is a chain of clans. If the extreme elements of \( c \) are global, then \( c \) cannot be cycled around and \( c \) is called bordered.

**Lemma 6.2.** Let \( g=(D, R) \) be a 2s such that \(|D| \geq 4\), and let \( c=\langle x_1, \ldots, x_m \rangle \) be a chain of clans of \( g \). If \( c \) is bordered or \( c \) is a cyclic chain with \( m = |D| \), then \( g \) is primitive.
Proof. By Lemma 5.1, if \( m > 4 \), then \( g \) is primitive. If \( c \) is cyclic, then by Remark 5.1, \( |D| > 4 \). This implies that \( m > 4 \), and hence \( g \) is primitive. So assume that \( m = 4 \) and \( c \) is bordered. By the definition of a chain of clans, \( x_2 \) and \( x_3 \) are local for \( D - \{ x_2, x_3 \} \). By Lemma 4.6, if \( g \) is nonprimitive \( \mathcal{Q}_2(g) = \{ \{ x_2, x_4 \}, \{ x_1, x_3 \} \} \). By Proposition 2.1, it follows that \( \{ x_2, x_4 \} \in \mathcal{Q}_2(g_{-x_1}) \), which leads to a contradiction since \( D - \{ x_1, x_2 \} \) is the unique nontrivial clan of \( g_{-x_1} \). Thus \( g \) must be a primitive \( r_2s \). □

Theorem 6.1. Let \( g = (D, R) \) be an \( r_2s \) such that \( |D| \geq 4 \). Then \( g \) satisfies the \((n-2)\)-property if \( g \) contains a chain of clans \( c = \langle x_1, \ldots, x_m \rangle \) of maximal length such that one of the following statements holds:

1. If \( m \) is odd, then \( c \) is cyclic or bordered, \( D = \{ x_1, \ldots, x_m \} \).
2. If \( m \) is even, then \( c \) is bordered, \( D = \{ x_1, \ldots, x_m \} \) or \( D = \{ x_1, \ldots, x_m, z \} \), where \( g_{-z} \) is nonprimitive.

Proof. We first assume that \( g \) satisfies the \((n-2)\)-property and prove that either statement (1) or (2) holds.

Let \( c = \langle x_1, \ldots, x_m \rangle \) be a maximal chain of clans of \( g \) (by Lemma 6.1 such a chain exists and \( m \geq 4 \)). Then by Lemma 5.3, \( x_2 \) is r-local or r-global in \( g_{x_1} \), and \( x_{m-1} \) is r-local or r-global in \( g_{-x_m} \). These two cases have to be considered.

Case 1. Let us assume that \( x_{m-1} \) is r-local in \( g_{-x_m} \).

We now show that \( c \) is cyclic and \( D = \{ x_1, \ldots, x_m \} \), where \( m \) is odd. In this case, since \( x_{m-1} \) is r-local, \( \mathcal{Q}_2(g_{-x_m}) = \{ x_{m-1}, z \} \), for \( z \in D - \{ x_{m-1}, x_m \} \), and moreover by Remark 5.1, \( |D| > 4 \). It follows that \( z \not\in c \), otherwise \( x_1, \ldots, x_m, z \) would be a chain of clans, contradicting the assumption that \( c \) has maximal length. Furthermore, by Lemma 5.4, we must have that \( z = x_1 \), and hence, \( \mathcal{Q}_2(g_{-x_m}) = \{ x_{m-1}, x_1 \} \). Consequently, \( x_2, \ldots, x_m, x_1 \) is also a chain of clans, and it is easy to verify that it is maximal. Moreover, \( x_m \) is r-local in \( g_{-x_1} \). In fact, by Lemma 5.3, \( x_m \) is r-global or r-local in \( g_{-x_1} \). On the other hand, \( x_2 \) is r-local or r-global in \( g_{-x_1} \). This implies that \( \mathcal{Q}_2(g_{-x_1}) = \{ x_2, x_m \} \). Hence \( c \) is cyclic.

Now we consider separately these two cases: \( m \) is odd or even.

(i) If \( m \) is odd, we obtain that \( \{ x_1, \ldots, x_m \} \) is a clan of \( g \). Hence, by primitivity of \( g \), it must be \( D = \{ x_1, \ldots, x_m \} \). In fact, assume that \( z \) is an arbitrary element in \( D - \{ x_1, \ldots, x_m \} \). Since \( m \) is odd, by Lemma 5.2, \( (z, x_m) R (z, x_i) \) for every \( i \) such that \( i \equiv 1 \pmod{2}, 1 \leq i < m \). By Lemma 5.2, \( (z, x_{m-1}) R (z, x_j) \) for each \( j \) such that \( j \equiv 0 \pmod{2}, 1 < j < m - 1 \). Since \( \{ x_1, x_{m-1} \} \) is a clan of \( g_{-x_m} \), \( (z, x_1) R (z, x_{m-1}) \). It follows by transitivity of \( R \) that \( (z, x_k) R (z, x_{k'}) \), for every \( x_k, x_{k'} \in c \). Thus the set \( \{ x_1, \ldots, x_m \} \) is a nontrivial clan of \( g \).

(ii) If \( m \) is even, we obtain that \( \{ x_1, x_{m-1} \} \) is a nontrivial clan of \( g \). This case violates primitivity of \( g \), hence it cannot occur. In fact, by Lemma 5.2, \( (x_m, x_{m-1}) R (x_m, x_1) \). Since \( X = \{ x_1, x_{m-1} \} \) is a clan of \( g_{-x_m} \), it follows that \( X \) is a clan of \( g \).

Case 2. Assume that \( x_{m-1} \) is r-global in \( g_{-x_m} \).

By Case 1, also \( x_2 \) is r-global in \( g_{-x_1} \), and hence \( c \) is bordered. We show that: (i) if \( m \) is odd, then \( D = \{ x_1, \ldots, x_m \} \), (ii) if \( m \) is even, then either \( D = \{ x_1, \ldots, x_m, z \} \), for \( z \not\in c \), or \( D = \{ x_1, \ldots, x_m \} \).
(i) Assume \( m \) is odd. If \( |D| > m \), then there exists \( z \in D \), such that \( z \notin c \). Since \( c \) is bordered, i.e. \( x_2, x_{m-1} \) are \( r \)-global, by Lemma 5.2, for any element \( x_i \in c \), \( 1 < i < m \), \((x_i, x_2) R (z, x_2) \) and \((x_i, x_{m-1}) R (z, x_{m-1}) \), where \((z, x_2)\) is equivalent to \((z, x_{m-1})\); it follows by transitivity of \( R \) that \((x_i, x_{m-1}) R (x_i, x_2) \). Consequently, by applying Lemma 5.2, it follows that \((x_j, x_{j+1})\) and \((x_j, x_{j-1})\) are equivalent for \( j \) odd, \( 1 < j < m \). Hence \( \{x_{j-1}, x_{j+1}\} \) is a nontrivial clan of \( g \), as this is a clan of \( g_{-x_j} \), which leads to a contradiction. Hence, it must be \( |D| = m \).

(ii) Assume \( m \) is even. If \( |D| > m + 1 \), we show that \( g \) is nonprimitive. By Lemma 5.2, given an element \( x_i \in c \), either \((z, x_i) R (z, x_{m-1}) \) or \((z, x_i) R (z, x_2) \), for all \( z \in D - \{x_1, \ldots, x_m\} \). Since \( x_2 \) and \( x_{m-1} \) are \( r \)-global, it follows that \((z_1, x_i) R (z_2, x_i)\), for all \( z_1, z_2 \in D - \{x_1, \ldots, x_m\} \) and \( x_i \in c \). Thus the set \( D - \{x_1, \ldots, x_m\} \) is a nontrivial clan of \( g \), which contradicts the fact that \( g \) is primitive. Hence, it must be \( |D| = m \) or \( |D| = m + 1 \).

Thus Cases (1) and (2) prove the two statements (1) or (2) of the theorem and this shows one direction of the theorem.

Let us assume now that \( g \) satisfies statement (1) or (2) of the theorem. By Lemma 6.2, \( g \) is primitive, moreover for any \( d \in c \), \( g_{-d} \) is nonprimitive. Hence \( g \) satisfies the \((n-2)\)-property. The propositions that follow completely describe the \( r2s \) \( g \) satisfying statement (1) or (2).

The following proposition describes the \( r2s \) \( g \) containing a chain \( c \) which is cyclic and verifies statement (1) of Theorem 6.1. In this case \( g \) has only one feature and it is antisymmetric, i.e. \( \mathcal{F} = \{P, P^{-1}\} \). The set \( P \), can be seen as a set of asymmetric edges of a graph with set of vertices \( D \), while the edges in \( P^{-1} \) are the nonedges of such graph. Then it turns out that \( g \) represents a particular graph (see Fig. 9), a tournament, which is a graph on vertex set \( D \), such that for any pair of elements \( x, y \in D \), there exists exactly one asymmetric edge that connects \( x \) and \( y \) [8].

Hence, we give the following definition.

**Definition 6.2.** An \( r2s \) \( g = (D, \mathcal{F}) \) is a tournament iff \( \mathcal{F} = \{P, P^{-1}\} \).

**Proposition 6.1.** Let \( g = (D, \mathcal{F}) \) be a primitive \( r2s \) such that \( |D| = m \), and \( m > 3 \) is odd. Let \( c = \langle x_1, \ldots, x_m \rangle \) be a chain of clans of \( g \) such that \( c \) is cyclic. Then \( g \) is a tournament, where \( \mathcal{F} = \{P, P^{-1}\} \) and

\[ P = \{ (x_i, x_j) : 1 \leq i < j \leq m, i \equiv 0 \pmod{2}, j \equiv 1 \pmod{2} \ \text{or vice versa} \} \]

\[ \cup \{ (x_i, x_j) : 1 \leq j < i \leq m, i, j \equiv 0 \pmod{2}, \text{or } i, j \equiv 1 \pmod{2} \} . \]

**Proof.** Assume that \( rel(g) = R \). We show that \( P \) is a set of equivalent edges which are asymmetric. Let \( e = (x_i, x_j) \) be an arbitrary edge in \( P \) such that \( 1 \leq i < j \leq m \). Since \( c \) is cyclic, \( \langle x_i, \ldots, x_k, x_{k+1} \rangle \) is a chain of clans. By Lemma 5.2, \( e \) is equivalent to an edge \( e' = (x_k, x_{k+l}) \), where \( l > k \), and \( l, k \) are both even or odd. Now, given an arbitrary edge \( e'' = (x_k, x_{k+l}) \), such that \( z \geq z' \) and \( e'' \in P \), we show in the following that \( e'' \) is equivalent to \((x_1, x_{m-1})\). Hence, \( e' R (x_k, x_{k-1}) \). Since \( e R e' \), consequently all edges in \( P \) are equivalent to \((x_1, x_{m-1}) \).
In fact, if $z$, $z'$ are even, since $\langle x_{z-1}, \ldots, x_1, x_m, x_{m-1}, \ldots, x_z \rangle$ is a chain of clans, then by Lemma 5.2, $(x_{z}, x_{z'}) \mathcal{R} (x_1, x_{z'})$, and $(x_1, x_{m-1}) \mathcal{R} (x_1, x_{m-1})$. Assume that $z$, $z'$ are odd, with $z \neq m$. If $z = m$, then by Lemma 5.2, $(x_{z}, x_{z'}) \mathcal{R} (x_1, x_{z'})$, if $z' = m - 2$, or $(x_{z}, x_{z'}) \mathcal{R} (x_{m-2}, x_{m-4})$, if $z' = m - 2$. Hence, $(x_{z}, x_{z'})$ is equivalent to $(x_k, x_k')$, with $k \neq m$, and hence we pose $z = k$, $z' = k'$. Since $\langle x_z, \ldots, x_1, x_m, \ldots, x_{z+1} \rangle$ is a chain of clans, then, by Lemma 5.2, $(x_z, x_{z'}) \mathcal{R} (x_1, x_{m-1})$. This proves that $e''$ is equivalent to $(x_1, x_{m-1})$.

Clearly, $P \cup P^{-1} = E_2(D)$. Observe that $P$ must be an antisymmetric set of edges, otherwise all 2-edges from $E_3(D)$ are equivalent, which is not possible. In fact, for $1 < i < m$, by the definition of a chain of clans, $\{x_{i-1}, x_{i+1} \}$ is a nontrivial clan of $g - x_i$; since $g$ is primitive, $(x_i, x_{i-1})$ and $(x_i, x_{i+1})$ are not equivalent edges. Thus the proposition holds. □

**Proposition 6.2.** Let $g = (D, \mathcal{F})$ be a primitive r2s such that $|D| = m + 1$, $m$ is even and $c = \langle x_1, \ldots, x_m \rangle$ is a maximal chain of clans of $g$ which is bordered. Let $g_{-z}$ be a nonprimitive substructure, for $z \notin c, z \in D$. Then $g$ is a tournament, where $\mathcal{F} = \{P, P^{-1}\}$ and

$$P = \{(x_i, x_j): 1 < i < j \leq m\} \cup \{(x_i, z): i \equiv 0 \pmod{2}, 1 < i \leq m\} \cup \{(z, x_i): i \equiv 1 \pmod{2}, 1 < i \leq m\}.$$

**Proof.** Assume $rel(g) = R$. Since $g_{-z}$ is nonprimitive, let $Z$ be a nontrivial clan of $g_{-z}$. By Lemma 5.2, it is easily verified that any set $Z = \{x_i, \ldots, x_k\}$, where $1 \leq i < k < m$ or $1 < l < k \leq m$ is a nontrivial clan of $g_{-z}$.

We now show that each edge in $P$ is equivalent to $(y, x_{m-1})$, for some $y \neq x_m$.

Assume $e = (x_i, x_j)$, where $e \in P$ and $i, j \neq m - 1$. If $i$ or $j$ is equal to $m - 1$, then by Lemma 5.2, $e$ is equivalent to an edge $e' = (x_l, x_k)$ such that $l, k \neq m - 1$, thus we pose $i = l$ and $j = k$. Since $Z = \{x_l, \ldots, x_{m-1}\}$ is a clan of $g_{-z}$, then $(x_l, x_j) \mathcal{R} (x_l, x_{m-1})$. Now assume that $e = (x_i, z)$, or $e = (z, x_i)$. By Lemma 5.2, since $i \equiv 0 \pmod{2}$ (or $i \equiv 1 \pmod{2}$), $e$ is equivalent to $(x_z, z)$ (or to $(z, x_{m-1})$), where the latter is equivalent to $(x_z, x_{m-1})$, as $c$ is bordered.

These different cases prove that each edges $e \in P$ is equivalent to an edge $(y, x_{m-1})$, $y \neq x_m$ and since $c$ is bordered, $e$ is equivalent to $(x_1, x_{m-1})$. This proves that all edges in $P$ are equivalent.

Clearly, $P \cup P^{-1} = E_2(D)$. We have that $P$ is an antisymmetric set of edges. In fact, since for $1 < i < m, \{x_{i-1}, x_{i+1}\}$ is a nontrivial clan of $g_{-x_i}$, and $g$ is primitive, $(x_i, x_{i-1})$ and $(x_i, x_{i+1})$ are not equivalent edges, which implies that $P \cup P^{-1}$ is not a set of equivalent edges, i.e. $P$ must be antisymmetric. □

The r2s $g$ of previous proposition is illustrated in Fig. 10.

Finally, let us describe the 2-structures with the $(n-2)$-property containing a bordered chain of clans $c$; we show that such 2-structures are specified by the following Definitions 6.3 and 6.4.
**Definition 6.3.** Let \( g=(D, \overline{\mathcal{F}}) \) be an \( r \)-structure on \( n \) elements, \( n \) even, such that 
\[ D = \{x_1, \ldots, x_n\}. \]
The \( r \)-structure \( g \) is **even-bordered** iff it satisfies the following property: given 
\[ P_1, P_2 \subseteq \mathcal{E}_2(D), \] 
such that 
\[ P_1 = \{(x_i, x_j) : i \equiv 1 \pmod{2}, j \equiv 0 \pmod{2}, 1 \leq i < j \leq n \}, \]
\[ P_2 = \{(x_i, x_j) : 1 \leq i < j \leq n \} - P_1, \]
then \( P_1, P_2 \) are two sets of equivalent edges, where the edges in \( P_1 \) are not equivalent 
to the edges in \( P_2 \cup P_2^{-1} \).

Clearly, there are four different even-bordered 2-structures, which we denote by \( g_1n, g_2n, g_3n \) and \( g_4n \). These 2-structures are illustrated in Fig. 5 and 6. \( g_1n=(D, \{\{P_1, P_1^{-1}\}, \{P_2, P_2^{-1}\}\}) \) has 2 antisymmetric features, \( g_2n=(D, \{\{\text{sym}(P_1)\}, \{P_2, P_2^{-1}\}\}) \) and \( g_3n=(D, \{\{P_1, P_1^{-1}\}, \{\text{sym}(P_2)\}\}) \) have 1 symmetric and 1 antisymmetric feature, \( g_4n=(D, \{\{\text{sym}(P_1)\}, \{\text{sym}(P_2)\}\}) \) has 2 symmetric features.

Observe that, if \(|D|=4\), then the 2-structures \( g_2n \) and \( g_3n \) are isomorphic.

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**Fig. 5.**
Definition 6.4. Let $g = (D, \mathcal{F})$ be a $2$-structure on $n$ elements, $n$ odd, such that $D = \{x_1, \ldots, x_n\}$. The $2$-structure $g$ is odd-bordered if it satisfies the following property: given $P_1, P_2 \subseteq E_2(D)$, such that

$$P_1 = \{(x_i, x_j) : i \equiv 0 \pmod{2}, j \equiv 1 \pmod{2}, \text{ or vice versa, or } i \equiv 0 \pmod{2}, j \equiv 0 \pmod{2}, 1 \leq i < j \leq n\},$$

$$P_2 = \{(x_i, x_j) : 1 \leq i < j \leq n\} - P_1,$$

then $P_1, P_2$ are two sets of equivalent edges, where the edges in $P_1$ are not equivalent to the edges in $P_2$, and $P_1$ is antisymmetric.

Clearly, there are three different odd-bordered 2-structures, which we denote by $g_{1n}, g_{2n}$ and $g_{3n}$. These 2-structures are illustrated in Fig. 7 and 8.

$g_{1n} = (D, \{\{P_1, P_1^{-1}\}, \{P_2, P_2^{-1}\}\})$ has 2 antisymmetric features, $g_{2n} = (D, \{\{P_1, P_1^{-1}\}, \{\text{sym}(P_2)\}\})$ has 1 symmetric and 1 antisymmetric feature, $g_{3n} = (D, \{P_1 \cup P_2^{-1}, \{P_1 \cup P_2^{-1}\}^{-1}\})$ has 1 antisymmetric feature.
Proposition 6.3. Let $g=(D, \mathcal{F})$ be a primitive $r2s$ such that $|D|=m$ and $c=\langle x_1, \ldots, x_m \rangle$ is a chain of clans of $g$ which is bordered. If $m$ is even, then $g$ is even-bordered, while if $m$ is odd, then $g$ is odd-bordered.

Proof. Assume $rel(g)=R$. Consider $(x_i, x_j)\in E_2(D)$ and let us assume for simplicity that $i<j$. By applying Lemma 5.2, it is easy to prove that $(x_i, x_j)$ is equivalent to $(x_2, x_{m-1})$ or to $(x_1, x_m)$.

(i) Let $i, j$ be both even. Then by Lemma 5.2, $(x_i, x_j) \sim (x_2, x_j)$. Since $x_2$ is $r$-global in $g_{-x_i}$, $(x_i, x_j) \sim (x_2, x_{m-1})$.

(ii) Let $i, j$ be both odd. Then $(x_i, x_j) \sim (x_1, x_j)$. If $m$ is even, then $(x_i, x_j) \sim (x_1, x_{m-1})$. Since $x_{m-1}$ is $r$-global in $g_{-x_i}$, $(x_i, x_j) \sim (x_2, x_{m-1})$. Otherwise, if $m$ is odd, then $(x_i, x_j)$ is equivalent to $(x_1, x_m)$.

(iii) Let $i$ be even and $j$ odd. By Lemma 5.2, $(x_i, x_j) \sim (x_2, x_j)$, and since $x_2$ is $r$-global in $g_{-x_i}$, $(x_i, x_j) \sim (x_2, x_{m-1})$. 

Fig. 7.

$g^{1n}$

$g^{2n}$
(iv) Let \( i \) be odd and let \( j \) be even. If \( m \) is odd, then \((x_i, x_j) \mathbin{R} (x_i, x_{m-1})\), and since \( x_{m-1} \) is \( r \)-global in \( g_{-x_i} \), \((x_i, x_j) \mathbin{R} (x_2, x_{m-1})\). If \( m \) is even, then \((x_i, x_j) \mathbin{R} (x_1, x_m)\).

Let \( P_1 \) and \( P_2 \) be two sets of edges of \( g \) which are specified as in the first part of Definition 6.3, if \( m \) is even, or of Definition 6.4, if \( m \) is odd. By cases (i) to (iv), it follows that \( P_1 \) and \( P_2 \) are two sets of equivalent edges, in particular, if \( m \) is even, then all edges in \( P_1 \) are equivalent to \((x_1, x_m)\) and all edges in \( P_2 \) are equivalent to \((x_2, x_{m-1})\) (and vice versa if \( m \) is odd). We now show that \( P_1 \) and \( P_2 \) are completely defined as in Definitions 6.3 and 6.4. The following cases have to be considered.

(i) Let \( m \) be odd.

(i.1) Assume that \( P_1 \cup P_1^{-1} \) is a set of equivalent edges. This implies that \((x_2, x_{m-1}) \mathbin{R} (x_m, x_2)\). Since \( x_{m-1} \) is \( r \)-global in \( g_{-x_2} \), thus \((x_{m-1}, x_2) \mathbin{R} (x_1, x_{m-1})\). By Lemma 5.2, it follows that \((x_1, x_2) \mathbin{R} (x_2, x_1)\), and by transitivity of \( \mathbin{R} \), \((x_2, x_1) \mathbin{R} (x_{m-1}, x_2)\) is equivalent to \((x_2, x_1)\). Consequently, \( D - \{x_2\} \) is a clan of \( g \), since \( x_2 \) is \( r \)-global. This contradicts the primitivity of \( g \).

(i.2) Assume that \( P_1 \cup P_2 \) is a set of equivalent edges. This implies that \((x_1, x_m) \mathbin{R} (x_2, x_{m-1})\). Since \( x_{m-1} \) is \( r \)-global, then \((x_1, x_m), (x_1, x_{m-1}) \) are equivalent, which implies that \( D - \{x_1\} \) is a clan of \( g \), which contradicts the fact that \( g \) is primitive.

Hence, cases (i.1) and (i.2) imply that \( g \) is an odd-bordered \( r2s \), that is \( g \) can be one of the three 2-structures \( g_{1,m}^1, g_{2,m}^2 \) and \( g_{3,m}^3 \). It is easily verified that \( g \) is effectively one of such 2-structures.

(ii) Let \( m \) be even.

(ii.1) Assume that \( P_1 \cup P_2 \) is a set of equivalent edges. This case is the same as (i.2).

(ii.2) Assume that \( P_1 \cup P_2^{-1} \) is a set of equivalent edges. Then \((x_1, x_m) \mathbin{R} (x_{m-1}, x_2)\). This case leads to a contradiction. In fact, by Lemma 5.2, it is \((x_1, x_2) \mathbin{R} (x_{m-1}, x_2)\), and since \( x_2 \) is \( r \)-global in \( g_{-x_1} \), \( D - \{x_2\} \) is a clan of \( g \).
Thus, cases (ii.1) and (ii.2) imply that $g$ is even-bordered. Hence $g$ can be one of the four different 2-structures, $g_{1n}$, $g_{2n}$, $g_{3n}$ and $g_{4n}$. It is easily verified that $g$ is effectively one of such 2-structures. 

By previous propositions that describe the different 2-structures satisfying the $(n - 2)$-property, we state the following theorem.

**Theorem 6.2** (The main theorem). *For each $n > 4$, if $n$ is even, there are four 2-structures satisfying the $(n - 2)$-property, up to isomorphism, while if $n$ is odd there are five different 2-structures with this property.

In addition, if $n = 4$, there are three 2-structures that satisfy the $(n - 2)$-property up to isomorphism.*

**Proof.** Let $g$ be an $r_{2s}$ satisfying the $(n - 2)$-property. By Theorem 6.1 and Propositions 6.1, 6.2 and 6.3, $g$ can be one of four (three) different 2-structures if $|D| > 4$ is even ($|D| = 4$, respectively), and one of five if $|D|$ is odd. Clearly, by Definition 2.8, any $r_{2s}$ $g'$ isomorphic to $g$ has the $(n - 2)$-property.

Vice versa let $g_1 = (D_1, R_1)$ be an $r_{2s}$ satisfying the $(n - 2)$-property on $|D|$ elements. By Theorem 6.1, $g_1$ contains a chain of clans $c_1 = \langle y_1, \ldots, y_m \rangle$ and $g_1$ is described by one of the Propositions 6.1, 6.2 or 6.3. Let $c_1 = \langle x_1, \ldots, x_m \rangle$ be a chain of clans of $g$, where $g$ has the same characterization in terms of chain of clans of $g'$. Let $\varphi: D_1 \rightarrow D$, such that $\varphi(y_i) = x_i$ and if $D = \{ y_1, \ldots, y_m, k_1 \}$, $\varphi(k_1) = z$. Then $\varphi$ is a bijective function that preserves equivalence relation between edges. Thus $g_1$ is isomorphic to $g$. 

Figs. 5–10 illustrate the different 2-structures satisfying the $(n - 2)$-property (up to isomorphism) on $n$ elements, where $n > 3$. In order to simplify the pictures, we
Fig. 10.

represent the edges of an antisymmetric feature $F = \{ P, P^{-1} \}$ by drawing only edges in $P$.

Observe that in these figures the sequence $\langle x_1, \ldots, x_n \rangle$ is a chain of clans of the 2-structures illustrated.

All 2-structures described by Propositions 6.1, 6.2 and 6.3 have a nice characterization: they have at most 2 features.

**Corollary 6.1.** Each primitive 2rs satisfying the $(n-2)$-property has at most 2 features.

7. The $(n-2)$-property is hereditary

In this section we prove that the $(n-2)$-property is an hereditary property of primitive 2-structures in the sense that given a primitive 2rs $g$ satisfying the $(n-2)$-property, each primitive substructure of $g$ on $n > 3$ elements satisfies this property. This result is strictly related to the fact that each primitive 2rs $g$ on $n$ elements in this class is “built up” from a smaller 2rs $h$ of the same “type”, by adding to it two elements such that their connections to the elements in the domain of $h$ are described by a “repeated” pattern. In other words, we can describe recursively the construction of such 2-structures, by easy relations [1].

We now show that any primitive substructure $h$ of a primitive 2rs $g$ can be extended to a primitive substructure of $g$ obtained by adding two elements of $h$. The hereditary nature of the $(n-2)$-property will follow easily from this result.

**Lemma 7.1.** Let $g = (D, R)$ be a primitive 2rs. Then for any substructure $h$ of $g$, such that $|\text{dom}(h)| \leq |\text{dom}(g)| - 2$, there is a primitive substructure $h'$ of $g$ such that $|\text{dom}(h')| = |\text{dom}(h)| + 2$ and $\text{dom}(h) \subset \text{dom}(h')$.

**Proof.** Assume $D_0 = \text{dom}(h)$. If $|D_0| = |D| - 2$, then the lemma is trivially verified, hence let $|D_0| < |D| - 2$. By contradiction we show that there exist two elements $d, d' \in D - D_0$
such that $\text{sub}_g(D_0 \cup \{d, d'\})$ is primitive. Assume to the contrary that for any $z, z' \in D - D_0, \text{sub}_g(D_0 \cup \{z, z'\})$ is nonprimitive. By Lemma 4.1, an element $d \in D - D_0$ is either local or global for $D_0$ in $g$, or $\text{sub}_g(D_0 \cup \{d\})$ is primitive. Then the set $D - D_0$ can be partitioned into three sets $X_p, X_l, X_g$ such that $X_p$ contains all elements which are neither local nor global, $X_l$ contains the local elements for $D_0$ in $g$, while $X_g$ contains the global elements for $D_0$ in $g$. It is easy to verify that for $X \in \{X_l, X_p, X_g\}$, if $X \neq \emptyset$, then $h_X, h_X = \text{sub}_g(D_0 \cup X)$ is nonprimitive. In fact, if $X = X_p$, since by Lemma 4.6, $\{z, k\}, \{z', k'\}$ are nontrivial clans of $\text{sub}_g(D_0 \cup \{z, z'\})$, for $z, z' \in X_l$ and $\text{uni}_g(D_0, z) = k, \text{uni}_g(D_0, z') = k', k \neq k'$, we have that $\{z \in X_l: \text{uni}_g(D_0, z) = k\} \cup \{k\}$ is a nontrivial clan of $h_X$. If $X = X_g$, then $D_0$ is clan of $h_x$. For any $x, y \in X_p$, the nonprimitive substructure $\text{sub}_g(D_0 \cup \{x, y\})$ has the only nontrivial clan $\{x, y\}$. This implies that $X_0$ is a nontrivial clan of $h_x$, for $X = X_p$. Let $h_d$ be the substructure $\text{sub}_g(D_0 \cup \{d, d'\})$, for $d, d' \in \{x, y, z\}$, $d \neq d'$, and $x \in X_p, y \in X_l$ and $z \in X_g$, where $\text{uni}_g(D_0, y) = a$. Then $h_d$ is nonprimitive; if $X$ is a nontrivial clan of $h_d$, since $h$ is primitive and $X \cap D_0$ is a clan of $h$, it must be $X \cap D_0 = D_0$ or $|X \cap D_0| \leq 1$ (by Proposition 2.1). Hence we have the following properties: if $d = x, d' = y, then \{y, a\}$ is the unique nontrivial clan of $h_d$, if $d = x, d' = z$, then $D_0 \cup \{x\}$ is the unique nontrivial clan of $h_d$, if $d = y, d' = z$, then $D_0 \cup \{y\}$ and $\{y, a\}$ are nontrivial clans of $h_d$. Assume that $X_p \neq \emptyset, X_l \neq \emptyset$ and $X_g \neq \emptyset$. Clearly, by the above properties, it follows that $\{y, a\}$ and $D_0 \cup \{x, y\}$ are nontrivial clans of any substructure of $g'$ such that $g' = \text{sub}_g(D_0 \cup \{x, y, z\})$. This implies that $D - X_p$ and $\{y' \in X_l: \text{uni}_g(D_0, y') = a\} \cup \{a\}$ are nontrivial clans of $g$, which contradicts the primitivity of $g$. Similarly, if some set in $\{X_l, X_g, X_p\}$ is empty, it follows that $g$ is not primitive. Since, the initial assumption leads to a contradiction, there exist two elements $z, z' \in D - D_0$, such that the substructure $\text{sub}_g(D_0 \cup \{z, z'\})$ of $g$ is primitive. This proves the lemma. \[\square\]

Theorem 7.1 (The hereditary theorem). Let $g = (D, R)$ be a primitive $r2s$ with the $(n - 2)$-property, such that $|D| > 5$. Then each primitive substructure $h$ of $g$ such that $|\text{dom}(h)| > 3$, satisfies the $(n - 2)$-property.

Proof. Let $\text{dom}(h) = D_0$. We prove the theorem by induction on $|D - D_0|$. Assume $|D - D_0| = 2$. Let there exist an element $x \in D_0$ such that $h_x$ is primitive. By Lemma 7.1, $h_x$ is extended to a primitive substructure of $g$, $h'$, such that $|\text{dom}(g) - \text{dom}(h')| = 1$, which contradicts the fact that $g$ has the $(n - 2)$-property. Then for any $x \in D_0, h_x$ is nonprimitive, i.e. $h$ satisfies the $(n - 2)$-property. Let us assume now that $|D - D_0| = n$. By Lemma 7.1, $h$ is a substructure of a primitive substructure of $g$ such that $|D - \text{dom}(g')| = n - 2$. By induction $g'$ has the $(n - 2)$-property. As above, we can show that $h$ satisfies the $(n - 2)$-property. \[\square\]

8. Arbitrary 2-structures satisfying the $(n - 2)$-property

In the previous sections we have analyzed the reversible primitive 2-structures with the $(n - 2)$-property. The main result of the paper is Theorem 6.2, which states that
there is a fixed number of reversible primitive 2-structures satisfying the \((n-2)\)-property. We can easily extend the theorem to the general case of arbitrary 2-structures. The following considerations are important to obtain this result. Since the \((n-2)\)-property of a 2s is related to the clans of all its substructures, by Proposition 2.3, an arbitrary 2s satisfies this property iff its reversible version does. Secondarily we note that all reversible 2-structures satisfying the \((n-2)\)-property have at most two features. Hence, in order to translate the theorem into arbitrary 2-structures, we have to determine the arbitrary 2-structures such that their reversible version is a 2s with the \((n-2)\)-property. This step will be closely related to the construction of the reversible version of a 2s. The following lemma follows directly from Definition 2.4.

**Lemma 8.1 ([6]).** Let \(g=(D, \mathcal{P})\) be a 2s and let \(h=(D, \mathcal{P}')\) be the reversible version of \(g\). Let \(\text{pair}_g\) be the following mapping from \(\mathcal{P}'\) to \(\mathcal{P} \times \mathcal{P}:\) for each \(P' \in \mathcal{P}'\), \(\text{pair}_g(P')=(P_1, P_2)\), where \(P_1, P_2 \in \mathcal{P}\), \(P' \subseteq P_1\), and \(P'^{-1} \subseteq P_2\). Then the following conditions hold:

1. \(|\mathcal{P}'| \equiv |\mathcal{P}|\) and \(g\) is reversible iff \(|\mathcal{P}'|=|\mathcal{P}|\).
2. \(P' \in \mathcal{P}'\) iff \(\text{pair}_g(P')=(P_1, P_2)\), for some \(P_1, P_2 \in \mathcal{P}\).
3. \((P', P'^{-1})\) is a symmetric feature iff \(\text{pair}_g(P')=(P_1, P_1)\) for some \(P_1 \in \mathcal{P}\).

Given an r2s \(h\), we call \textit{reversible version set} of \(h\), the set of nonreversible 2-structures such that their reversible version is \(h\).

**Definition 8.1.** Let \(h=(D, \mathcal{P}')\) be a reversible 2s. Then the \textit{reversible version set} of \(h\), denoted by \(\text{rvers}(h)\) is the set:

\[
\text{rvers}(h) = \{g: g\text{ is a 2s, } rver(g)=h \text{ and } g \text{ is not reversible}\}
\]

We now describe the reversible version set of a reversible 2s with at most 2 features.

**Theorem 8.1.** Let \(h=(D, \mathcal{P})\) be a reversible 2s with \(|\mathcal{F}| \leq 2\).

1. If \(h\) has 2 antisymmetric features, then \(|\text{rvers}(h)|=4\), where for each \(g \in \text{rvers}(h)\), \(|\text{part}(g)|=3\).
2. If \(h\) has 1 symmetric and 1 antisymmetric feature, then \(\text{rvers}(h)=\{g, g'\}\), where \(|\text{part}(g)| = |\text{part}(g')| = 2\).

In all other cases, \(\text{rvers}(h)=\emptyset\).

**Proof.** Let \(g=(D, \mathcal{P})\) be a 2s such that \(rver(g)=h\), where \(h=(D, \mathcal{P}')\). We have the following cases to consider.

(i) \(h\) has 2 symmetric features. Then \(\mathcal{P}' = \{R_1, R_2\}\), where both \(R_1\) and \(R_2\) are symmetric. By Lemma 8.1, \(\text{pair}_g(R_1)=(P_1, P_1)\) and \(\text{pair}_g(R_2)=(P_2, P_2)\), where \(P_1, P_2\) are symmetric and \(\mathcal{P}=\{P_1, P_2\}\). But then \(g=h\), and \(g\) is reversible. Thus \(\text{rvers}(h)=\emptyset\).

(ii) \(h\) has 2 antisymmetric features. Then \(\mathcal{P}' = \{R_{11}, R_{12}, R_{21}, R_{22}\}\), where \(\{R_{11}, R_{12}\}\) and \(\{R_{21}, R_{22}\}\) are antisymmetric features. By Lemma 8.1, \(|\mathcal{P}'|>|\mathcal{P}|\).
that is \(|\mathcal{P}| \leq 3\). Since all classes of \(h\) are antisymmetric, then \(|\mathcal{P}| = 3\), where all classes of \(\mathcal{P}\) are antisymmetric. Clearly \(\mathcal{P} = \{P_1, P_2, P_1\}\), where \(P_1 \in \{R_{11}, R_{12}\}\), \(P_2 \in \{R_{21}, R_{22}\}\) and \(P_{12} = \mathcal{P} - \{P_1 \cup P_2\}\). Hence \(|\text{rver}_S(h)| = 4\).

(iii) \(h\) has 1 symmetric and 1 antisymmetric feature. Then \(\mathcal{P} = \{R_{11}, R_{12}, R_2\}\), where \(\{R_{11}, R_{12}\}\) is the antisymmetric feature of \(h\) and \(R_2\) is the symmetric feature of \(h\). By Lemma 8.1, \(|\mathcal{P}| < 3\). If \(|\mathcal{P}| = 1\), then \(g\) is reversible. Hence, \(\mathcal{P} = \{P_1, P_2\}\), where \(P_1 \in \{R_{11}, R_{12}\}\) and \(P_2 = R_2 \cup \{R_{11}, R_{12}\} - P_1\). Consequently \(\text{rver}_S(h) = \{g, g'\}\).

(iv) \(h\) has 1 symmetry feature. Then \(\mathcal{P}\) is symmetric and \(|\mathcal{P}| = 1\). Hence, \(|\mathcal{P}| = 1\), \(g = h\) and \(g\) is reversible. Thus \(\text{rver}_S(h) = \emptyset\).

(v) \(h\) has 1 antisymmetric feature. Then \(\{R_{11}, R_{12}\}\) is the unique feature of \(h\).

Since all classes in \(g\) are antisymmetric, \(|\mathcal{P}| = 2\), and by Lemma 8.1, \(g\) is reversible. Thus \(\text{rver}_S(h) = \emptyset\).

We observe that, by Proposition 2.3, a 2s \(g\) and its reversible version \(\text{rver}(g)\) have the same set of clans, then an arbitrary 2s \(g\) satisfying the \((n-2)\)-property contains the same chain of clans of \(\text{rver}(g)\).

**Lemma 8.2.** Let \(g = (D, \mathcal{P})\) be an arbitrary 2s such that satisfies the \((n-2)\)-property and let \(c = \langle x_1, \ldots, x_m \rangle\) be a chain of clans of \(g\), where \(|D| = |c|\), and \(c\) is bordered. Let \(g' = (D, \mathcal{P}')\) be such that \(\mathcal{P}' = \{P_i : P_i = P_i^{-1}\text{, for } P_i \in \mathcal{P}\}\). Then \(g\) is isomorphic to \(g'\).

**Proof.** Clearly, \(c\) is also a chain of clans of \(g'\). In order to distinguish the domain of \(g'\) from that of \(g\), we write \(\tilde{x}_i\), when \(x_i\) is an element in \(\text{dom}(g') - D_g\). Let \(\varphi : D_g \rightarrow D_{g'}\), such that \(\varphi(x_i) = \tilde{x}_{m-i+1}\), for all \(1 \leq i \leq m\). Obviously, \(\varphi\) is a bijective function, and \(c' = \langle \varphi(x_1), \ldots, \varphi(x_m) \rangle\) is a chain of clans of \(g'\). By Proposition 6.3, for each edge \(e = (x_i, x_j) \in E_2(D_g)\), \(e\) is equivalent to \((x_2, x_{m-1})\) or to \((x_1, x_m)\) or to \((x_{m-1}, x_2)\), \((x_m, x_1)\). Suppose that \((x_i, x_j) \in P_i\), where \((x_i, x_j) R (x_2, x_{m-1})\). Then \((\varphi(x_i), \varphi(x_j)) = (\tilde{x}_{m-i+1}, \tilde{x}_{m-j+1})\), and by cases (from (i) to (iv)) of Proposition 6.3, it follows that \((\tilde{x}_{m-i+1}, \tilde{x}_{m-j+1})\) is equivalent to \((\tilde{x}_{m-1}, \tilde{x}_2)\). Hence, the edge \((\tilde{x}_{m-i+1}, \tilde{x}_{m-j+1}) \in P_i^{-1}\), where \(P_i^{-1} = P_i\). Analogously, if \((x_i, x_j)\) is equivalent to \((x_1, x_m)\), we prove that \((\varphi(x_i), \varphi(x_j))\) is equivalent to \((\tilde{x}_m, \tilde{x}_1)\). It follows that \((x_i, x_j) \in P_i\) iff \((\tilde{x}_{m-i+1}, \tilde{x}_{m-j+1}) \in P_i\). This implies that \(\psi\) is an isomorphism between \(g\) and \(g'\).

As said before, an arbitrary 2-structure satisfies the \((n-2)\)-property if its reversible version has this property. Since all reversible 2-structures with \((n-2)\)-property have at most 2 features, we determine by Theorem 8.1, through their reversible version set, the nonreversible 2-structures with the \((n-2)\)-property.

**Theorem 8.2.** For each \(n\) even, where \(n > 4\), there are 4 nonreversible 2-structures on \(n\) elements satisfying the \((n-2)\)-property, up to isomorphism. For each \(n\) odd, where \(n \geq 3\), there are 3 nonreversible 2-structures on \(n\) elements with this property (up to isomorphism). In addition, if \(n = 4\), then there are 2 nonreversible 2-structures with the \((n-2)\)-property.
**Proof.** By Corollary 6.1 the reversible 2-structures satisfying the \((n-2)\)-property have at most 2 features. By Theorem 8.1, and Proposition 6.3, we can directly determine the number of different nonreversible 2-structures, up to isomorphism, with the \((n-2)\)-property. In fact, for \(n\) even, there are two (precisely one for \(n=4\)) reversible 2-structures having 1 symmetric and 1 antisymmetric feature, and one 2s with 2 antisymmetric features with the \((n-2)\)-property. If \(n > 3\) is odd, then by Proposition 6.3, there are one reversible 2s with 1 symmetric and 1 antisymmetric feature and one 2s with 2 antisymmetric features on \(n\) elements that satisfy the \((n-2)\)-property. Let \(h\) be the reversible 2s on \(n\) elements with 1 symmetric and 1 antisymmetric feature satisfying the \((n-2)\)-property, that is \(h=(D, \{R_{11}, R_{12}, R_{2}\})\). Then, by case (iii) of Theorem 8.1, \(\text{rvers}(h)=\{g, g'\}\), where \(g=(D_{a}, \{R_{11}, R_{2} \cup R_{12}\})\) and \(g'=(D_{a}', \{R_{12}, R_{2} \cup R_{11}\})\), with \(R_{12}=R_{12}^{-1}\), \(R_{2} \cup R_{11}=R_{2} \cup R_{12}^{-1}\). Hence, by Lemma 8.2, \(g\) and \(g'\) are isomorphic.

Now let \(h\) be the reversible 2s on \(n\) elements with 2 antisymmetric features. Thus by case (ii) of Theorem 8.1, \(\text{rvers}(h)=\{g_{1}, g', g_{2}, g_{2}'\}\), where \(g_{i}=(D, \{P_{i}\}), g_{i}'=(D, \{P_{i}^{-1}: P_{i} \in \mathcal{P}_{i}\})\), for \(i \in \{1, 2\}\). Hence, by Lemma 8.2, \(g_{1}, g_{1}'\) are isomorphic. Since \(g_{1}=(D, \{P_{1}, P_{2}, \{P_{1} \cup P_{2}\}^{-1}\}), g_{2}=(D, \{P_{1}, P_{2}^{-1}, \{P_{1} \cup P_{2}^{-1}\}^{-1}\})\), where \(P_{1} \neq P_{2}^{-1}\), then if there exists an isomorphism \(\varphi\) between \(g_{1}\) and \(g_{2}\), then there are edges in \(g_{1}\), corresponding under \(\varphi\) to edges \(\varphi(g_{2})\), that have opposite directions of the edges in \(\varphi(g_{2})\), thus yielding a contradiction (see in Fig. 5 and 7 the 2-structures \(g_{1}, g_{1}'\)). Similarly, it follows that \(g_{1}', g_{2}'\) are not isomorphic. Thus by Theorem 8.1, the theorem holds.

We observe, that by Theorem 6.2 and the previous Theorem 8.2, for each \(n > 4\) there is the same number of arbitrary 2-structures satisfying the \((n-2)\)-property: there are eight of such 2-structures on \(n\) elements.

**9. Discussion**

In this paper we determine the class of primitive 2-structures satisfying the \((n-2)\)-property. This result is based on the notion of a chain of clans by which we show that these 2-structures have a "regular" behavior with respect to how primitivity is violated by removing single elements from their domain.

We have proved that reversible 2-structures with the \((n-2)\)-property have at most 2 features. This fact is of great interest since it implies that these 2-structures represent graphs. Hence the characterization of 2-structures with the \((n-2)\)-property given in the paper has important connections with graph theory: from it we can investigate the class of primitive graphs with the \((n-2)\)-property.

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