JOURNAL OF DIFFERENTIAL EQUATIONS **143**, 243–266 (1998)
ARTICLE NO. DE973371

Dynamics of a Nonlocal Kuramoto–Sivashinsky Equation*

Jinqiao Duan and Vincent J. Ervin

Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29643-1907
E-mail: duan@math.clemson.edu; ervin@math.clemson.edu

Received October 26, 1996; revised August 20, 1997

In this paper we study the effects of a "nonlocal" term on the global dynamics of the Kuramoto–Sivashinsky equation. We show that the equation possesses a "family of maximal attractors" parameterized by the mean value of the initial data.

dimensional dynamical system.

1. INTRODUCTION

In this article we investigate the impact of a "nonlocal" term on the global dynamics of the well-known Kuramoto–Sivashinsky (K–S) equation. The equation under consideration arises in the modeling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field [13]. The application of a uniform electric field at infinity, perpendicular to the inclined plane, is to destabilize the liquid films on the surface of the plane. In an industrial setting it is hoped that this destabilization will lead to an enhancement of heat transfer. The nonlocal operator arising is the Hilbert transform applied to the third spatial derivative of the unknown solution u. The modeling equation is

$$u_t + u_{xxxx} + u_{xx} + uu_x + \alpha H(u_{xxx}) = 0, \tag{1.1}$$

where α is a positive coefficient, and

$$H(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{x - \xi}\, d\xi, \tag{1.2}$$

(the integral is understood in the sense of the Cauchy Principle Value).

* This work was supported by the National Science Foundation Grant DMS-9704345.

0022-0396/98 $25.00

The constant $\alpha := W_e/\sqrt{W\,|\cot\beta - \frac{4}{5}R_e|}$, where W_e is the electrical Weber number, W the Weber number, R_e the Reynolds number and β the angle between the plane and the horizontal. The critical Reynolds number is $R_e^* = \frac{5}{4}\cot\beta$.

When the electrical field is absent, i.e. $W_e = 0$, the Hilbert transform term is gone and we have the usual K–S equation.

Throughout this paper we restrict our attention to the case of u periodic on the interval $I := (-l, l)$. Then, (1.2) is replaced by (see [1]) the "periodic" Hilbert transform

$$H(f) = -\frac{1}{2l}\int_I \cot\frac{\pi(x-\xi)}{2l}\, f(\xi)\, d\xi.$$

Observe that $u = C$, a constant, satisfies (1.1). Hence (1.1) cannot have a bounded attractor. Moreover, for

$$u(x,0) = u_0(x) \qquad \text{for} \quad x \in I, \qquad \text{and} \qquad \bar{u}_0 := \frac{1}{2l}\int_I u_0(x)\, dx, \tag{1.3}$$

integrating (1.1) over the interval I yields

$$\frac{d}{dt}\int_I u(x,t)\, dx = 0, \qquad \text{i.e.} \quad \frac{1}{2l}\int_I u(x,t)\, dx = \bar{u}_0.$$

Thus, we have that the mean of the solution is conserved with respect to time. This may be interpreted in the sense that the "dynamics" of u satisfying (1.1) are centered around the mean value of the initial data. Therefore, in order to study the motion, with regard to having a bounded attractor, it is appropriate to consider the dynamics of

$$v(x,t) := u(x,t) - \bar{u}_0. \tag{1.4}$$

Hence our investigations focus on:

$$v_t + v_{xxxx} + v_{xx} + v v_x + \bar{u}_0 v_x + \alpha H(v_{xxx}) = 0, \tag{1.5}$$

$$v(x,t) \text{ periodic on } I, \text{ with } v(x,0) = v_0(x), \text{ and } \int_I v_0(x)\, dx = 0. \tag{1.6}$$

The usual K–S equation (i.e., the Eq. (1.1) without the Hilbert transform term) has been a prototypical example of infinite dimensional systems with interesting behavior under appropriate boundary and initial conditions. For example, it has finite dimensional maximal attractor ([21, 5, 19, 14]) and inertial manifolds ([12, 20, 28, 24]). The K–S equation and its variants are regarded as simple models for fluid flow down a solid surface (e.g. [8, 18]).

A previous investigation on the impact of a dispersive term u_{xxx} on the dynamics of the K–S equation by Ercolani *et al.*, [11], found that this term has a stablizing effect on the system. Chang *et al.*, [7], and Alfaro *et al.*, [4], studied the influence of the term u_{xxx} on the bifurcation structure of the usual K–S equation. In [10], Duan *et al.* showed that the addition of the dispersive term u_{xxx} to the usual K–S equation does not alter the dimensions of the global attractor or inertial manifold, and does not change the number of the determining nodes.

In this paper we study the effects of a "nonlocal" term on the global dynamics of the Kuramoto–Sivashinsky equation. This work extends that in [22] where several nonlocal operators were considered for the primitive form of the K–S equation, restricted to the even periodic case (which corresponds to the odd periodic case for (1.1)). This paper is also inspired by the work of Duan *et al.* ([9]) on the effect of nonlocal interactions on the dynamics of the Ginzburg–Landau equation. In order to establish boundedness of the solution a coercive estimate is needed and this is obtained by the introduction of a suitably constructed gauge function, combining the approaches used in [26] and [5]. We show that the (1.1) possesses a "family of maximal attractors" parameterized by the mean value of the initial data. The dimension of the attractor is estimated as a function of the coefficient of the nonlocal term, and the width of the periodic domain.

In Section 2, we prove the existence and uniqueness of global classical solutions for both (1.1) and (1.5). In Section 3, we consider maximal attractors and estimate their Hausdorff and fractal dimensions. Finally, we comment on the impact of the nonlocal operator in Section 4.

2. GLOBAL CLASSICAL SOLUTION

In this section we show that the solution v of (1.5), (1.6) is a classical solution, i.e. satisfies the differential equation "pointwise", and exists for all $t > 0$. To do this we follow the usual steps:

(i) establish *local* existence and uniqueness (Section 2.2),

(ii) show that the solution remains bounded (a priori) on all finite intervals $[0, T]$ (Section 2.3).

From (i) and (ii) we then have:

THEOREM 2.1. *There exists a unique, global, classical solution* $v(x, t)$ *to* (1.5), (1.6), *for* $v(x, 0) \in \dot{H}^1_{per}(I)$.

2.1. *Preliminaries*

We denote by $L^2_{per}(I)$, $H^k_{per}(I)$, $k=1, 2, \ldots$, the usual Sobolev spaces of periodic functions on I. Let $\dot{L}^2_{per}(I)$, $\dot{H}^k_{per}(I)$ denote the spaces of functions g in $L^2_{per}(I)$, $H^k_{per}(I)$, respectively, with mean zero, i.e. $\bar{g} := (1/2l)\int_I g(x)\,dx = 0$. In the following, $\|\cdot\|$ denotes the usual $L^2_{per}(I)$ norm. Due to the Poincaré inequality, $\|D^k u\|$ is an equivalent norm in $\dot{H}^k_{per}(I)$. All integrals are with respect to $x \in I$, unless specified otherwise.

Following are several inequalities we utilize in our analysis of (1.5), (1.6).

Young's inequality ([23], p. 180).

$$ab \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q}, \text{ in particular } ab \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \tag{2.1}$$

for $a, b \in \mathbb{R}$, $p, q, \varepsilon > 0$, and $1/p + 1/q = 1$.

Uniform Gronwall inequality ([27], p. 89). Assume that positive locally integrable functions $y(t)$, $g(t)$, $h(t)$ satisfy

$$\frac{dy}{dt} \leqslant gy + h, \qquad t \geqslant 0,$$

and moreover,

$$\int_t^{t+r} g(s)\,ds \leqslant a_1, \qquad \int_t^{t+r} h(s)\,ds \leqslant a_2, \qquad \int_t^{t+r} y(s)\,ds \leqslant a_3,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \leqslant \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \qquad t \geqslant 0. \tag{2.2}$$

Poincaré inequality ([27], p. 49).

$$\int_I g^2\,dx \leqslant (2l)^2 \int_I g_x^2\,dx, \tag{2.3}$$

for $g \in \dot{H}^1_{per}(I)$.

Gagliardo–Nirenberg inequality ([2], p. 79).

$$\|D^j g\| \leqslant C\,\|D^m g\|^{j/m}\,\|g\|^{(m-j)/m}, \qquad 0 \leqslant j \leqslant m, \tag{2.4}$$

for $g \in \dot{H}^m_{per}(I)$. The positive constant C depends on m only. This inequality follows from Corollary 4.16 in Adams [2] and the fact that $\|D^m g\|$ is an equivalent norm for $\dot{H}^m_{per}(I)$.

Agmon inequality ([27], p. 50).

$$\|g\|_\infty^2 \leqslant 2\,\|g\|\,\|g_x\|, \tag{2.5}$$

for $g \in \dot{H}^1_{per}(I)$. This inequality also follows from $g^2(x) = 2\int_{x_0}^x gg_x\,dx$, where $g(x_0) = 0$ (as g is continuous and has zero mean).

The Hilbert transform (1.2) is a linear, invertible, bounded operator from L^2 to L^2, and from Sobolev space H^k to H^k. Several noteworthy properties of the transform are (see [1, 3]; also [25, 29]):

$$\begin{gathered} D_x H = H D_x, \\ H^{-1} = -H, \\ \int vH(u) = -\int uH(v), \\ \int H(u)\,H(v) = \int uv, \\ \int uH(u) = 0, \\ \|H(u)\| = \|u\|. \end{gathered} \tag{2.6}$$

These properties hold for the Hilbert transformation on both the real line and periodic intervals [1]. On the periodic interval $(-l, l)$, the Hilbert transformation has a simple representation

$$H(f)(x) = i\sum_{k\in Z} \operatorname{sgn}(k)\, f_k e^{ik\pi x/l},$$

for $f(x) = \sum_{k\in Z} f_k e^{ik\pi x/l}$, with f_k's the Fourier coefficients of f.

2.2 *Local Existence and Uniqueness*

LEMMA 2.1. *For $v_0(x) \in \dot{H}^1_{per}$, there exists a unique, classical solution $v(x, t)$ satisfying* (1.5) (1.6) *valid for* $0 < t < \tau$, *where* $\tau = \tau(u_0)$.

Proof. To establish local existence and uniqueness for v we rewrite (1.5) as

$$v_t + (A + B)\,v = f(v), \tag{2.7}$$

where $A = D_x^4 + D_x^2 + a$, $B = \alpha H(D_x^3)$ and $f(v) = -vv_x - \bar{u}_0 v_x + av$. In [17] Hsieh remarks that the elemental instability mechanism is the negative diffusion $-v_{xx}$ term. Observe that $Domain(A) = \dot{H}^4_{per}$ and $Domain(B) = \dot{H}^3_{per}$. Thus, the operator A is sectorial in $\dot{L}^2_{per}$, ([16], p. 19). For a chosen sufficiently large the eigenvalues of A,

$$\left(\frac{k\pi}{l}\right)^4 - \left(\frac{k\pi}{l}\right)^2 + a,$$

are all positive. As $A^{-(3/4)}: \dot{L}^2_{per} \to \dot{H}^3_{per}$ is a bounded linear operator ([16], Theorem 1.4.2), then so is $BA^{-(3/4)}: \dot{L}^2_{per} \to \dot{L}^2_{per}$. Thus, applying Corollary 1.4.5 in Henry [16], $A+B$ is sectorial in $\dot{L}^2_{per}$, ([16], Corollary 1.4.5).

A straight forward calculation shows that the nonlinear operator $f: \dot{H}^1_{per} \to \dot{L}^2_{per}$ is locally Lipschitzian. Hence, there exists a unique, local, classical solution to (1.5), (1.6), and the solution either exists for all positive time or becomes unbounded in the $\dot{H}^1_{per}$ norm at some finite time, ([16], Theorems 3.3.3, 3.3.4, 3.5.2). ▮

Next, in order to establish global existence, we need to show that the latter alternative can not occur.

2.3. *Uniform Boundness*

To establish that $\|v\|$ is (uniformly) bounded in $\dot{H}^1_{per}$ we firstly establish its boundness in $\dot{L}^2_{per}$ (Theorem 2.3). This bound is achieved in two steps. We firstly consider the case for v anti–symmetric. A *time independent* gauge function $\phi(x)$ is introduced and explicitly constructed to yield an important coercive estimate, (2.13). For the general case a time dependent function $b(t)$ is introduced in the argument of ϕ. This function, $b(t)$, is chosen such that a similiar coercive estimate, (2.28) is satisfied.

2.3.1. *Anti-symmetric Case*

In this section we consider the function v satisfying (1.5) subject to

$$v(x,t) \text{ periodic on } I, \quad \text{with} \quad v(x,0) = v_0(x), \quad \text{and} \quad v(x,t) = -v(-x,t) \quad \text{for all} \quad x \text{ in } I. \tag{2.8}$$

Proceeding as in [21, 22, 26] we introduce a "gauge" function by means of the change of variable

$$v(x,t) = z(x,t) + \phi(x), \tag{2.9}$$

where both z and ϕ are anti-symmetric, $z(x,t) \in H^4_{per}$, and $\phi \in C^4_{per}(I)$ is to be chosen appropriately.

With this change of variables (1.5) becomes

$$z_t + z_{xxxx} + z_{xx} + zz_x + \phi z_x + \phi' z + \bar{u}_0 z_x + \alpha H(z_{xxx}) = F, \tag{2.10}$$

where $F = -\phi^{iv} - \phi'' - \phi\phi' - \bar{u}_0 \phi' - \alpha H(\phi''')$.

Multiplying (2.10) through by z and integrating over I, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \|z_{xx}\|^2 - \|z_x\|^2 + \frac{1}{2}\int_I \phi' z^2\,dx + \alpha \int_I zH(z_{xxx})\,dx = \int_I Fz\,dx. \tag{2.11}$$

We now choose the function ϕ so that the term $\frac{1}{2}\int_I \phi' z^2\,dx$ neutralizes the effects of the terms $-\|z_x\|^2$ and $\alpha \int_I zH(z_{xxx})\,dx$.

LEMMA 2.2. *For* $w(x,t), z(x,t) \in H^2_{per}(I)$ *satisfying* $w(l,t) = z(l,t) = 0$, *consider the bi-linear form*

$$\langle w, z\rangle_{\beta,\phi} := \int_I w''z''\,dx - \beta \int_I w'z'\,dx + \int_I wz\phi'\,dx. \tag{2.12}$$

Then, given $\beta^* > 0$, *there exists an anti-symmetric function* $\phi \in C^4_{per}(I)$ *and* $\mu > 0$, *such that for all* β *satisfying* $0 \leqslant \beta \leqslant \beta^*$

$$\langle w, w\rangle_{\beta,\phi} \geqslant 2\mu^2 \|w\|^2. \tag{2.13}$$

This lemma implies that for such chosen β and ϕ, the bilinear form $\langle w, z\rangle_{\beta,\phi}$ defines an inner product, and subsequently, the Cauchy–Schwarz inequality holds for this inner product.

The construction of ϕ follows analogously to that presented in [26]. Instead of the usual C^∞ "cut-off functions", based upon the quotient of exponential functions, we use polynomial functions. The polynomial functions enable us to more accurately estimate the "radius of the bounding ball", and subsequently the dimension of the attractor.

Outline of Proof. We have that $z(l) = 0$. Following the construction described in Lemma 2.2 of [26] we obtain $\phi(x)$ which is $2l$ periodic with the following properties:

(i) $\phi \in C^4_0(I)$, (2.14)

(ii) $\phi(x) = 2\gamma x$ on $[0, (1-\delta)l]$, where δ satisfies $0 < \delta < 1$, (2.15)

(iii) $\phi(x) = -\phi(x)$, for $-l < x < 0$, (2.16)

(iv) $|\phi'(x)| \leqslant 4\gamma/\delta$ for all $x \in I$. (2.17)

For specificity, we use for ϕ:

$$\phi(x) = 2\gamma x + \left(-\frac{4\gamma}{\delta} - 2\gamma\right) \int_0^x \Lambda(\xi, (1-\delta)l, (1-(1-\sigma_1)\delta)l)\, d\xi$$

$$+ \frac{4\gamma}{\delta} \int_0^x \Lambda(\xi, (1-\sigma_2\delta)l, l)\, d\xi \qquad \text{for} \quad 0 \leqslant x \leqslant l, \tag{2.18}$$

where $\Lambda(x; a, b)$ is described in (A.1), and $0 < \sigma \leqslant \sigma_1, \sigma_2 \leqslant 1/2$ chosen such that $\phi(l) = 0$.

Such a choice for ϕ yields

$$\langle w, w \rangle_{\beta, \phi} \geqslant \left\{ 2\gamma - \frac{(\beta + 4\gamma l^2 \delta)^2}{4} \right\} \|w\|^2. \tag{2.19}$$

(See Fig. 2.1.) Finally, a routine calculation shows that for

$$\gamma = \frac{3}{2}(\beta^*), \qquad \text{and} \qquad \delta = \frac{1}{3l^2\beta^*}, \tag{2.20}$$

$$\langle w, w \rangle_{\beta, \phi} \geqslant \frac{3}{4}(\beta^*)^2 \|w\|^2, \tag{2.21}$$

for all β satisfying $0 \leqslant \beta \leqslant \beta^*$. ∎

THEOREM 2.2. *Let v be a solution of* (1.5), (2.8). *Then, there exists coonstants $C_1, C_2 > 0$, independent of v_0 and t, such that*

$$\|v(x, t)\| \leqslant \|v_0\| \exp(-C_1 t) + C_2. \tag{2.22}$$

Proof. Proceeding as above, we introduce the transformation $v = z + \phi$. We note that since $\|\phi\|$ is independent of z, the boundness of $\|v\|$ is equivalent to that of $\|z\|$.

Choose ϕ satisfying (2.13) with

$$\beta^* = 4(1 + \alpha^2). \tag{2.23}$$

Using (2.6) we have

$$\begin{aligned} \left| \alpha \int_I z H(z_{xxx})\, dx \right| &= \left| -\alpha \int_I z_x H(z_{xx})\, dx \right| \\ &\leqslant |\alpha|\, \|z_x\|\, \|H(z_{xx})\| \\ &\leqslant \tfrac{1}{4} \|z_{xx}\|^2 + \alpha^2 \|z_x\|^2. \end{aligned} \tag{2.24}$$

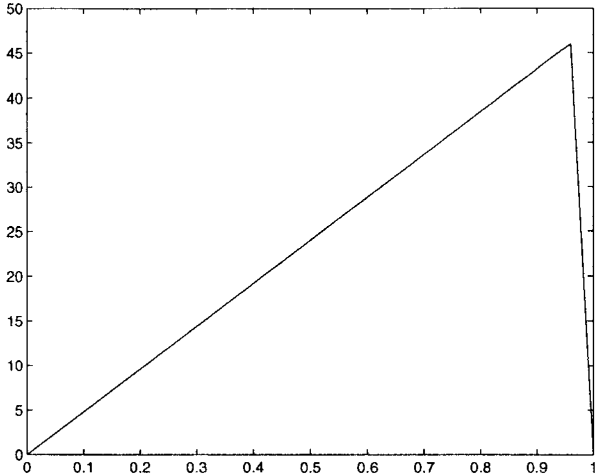

Fig. 2.1. Graph of $\phi(x)$ for $0 \leqslant x \leqslant l$.

Thus, from (2.11), using (2.24) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|^2 + \frac{3}{4}\|z_{xx}\|^2 - (1+\alpha^2)\|z_x\|^2 + \frac{1}{2}\int_I \phi' z^2\,dx \\
&\leqslant -\left[\int_I z_{xx}\phi''\,dx - \int_I z_x\phi'\,dx + \int_I z\phi\phi'\,dx\right] \\
&\quad - \bar{u}_0\int_I z\phi'\,dx - \alpha\int_I zH(\phi''')\,dx \\
&= -\langle z, \phi\rangle_{1,\phi} + \bar{u}_0\int_I z_x\phi\,dx + \alpha\int_I z_x H(\phi'')\,dx \\
&\leqslant \frac{1}{4}\langle z, z\rangle_{1,\phi} + \langle \phi, \phi\rangle_{1,\phi} + \frac{1}{8}\|z_x\|^2 + 2\bar{u}_0^2\|\phi\|^2 + \frac{1}{8}\|z_x\|^2 + 2\alpha^2\|\phi''\|^2,
\end{aligned}$$

where in the last step, we have used the Cauchy–Schwarz inequality (and then the Young's inequality) for the inner product $\langle z, \phi\rangle_{1,\phi}$; see the remark after the Lemma 2.2.

The above estimates imply that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|^2 + \frac{1}{2}\|z_{xx}\|^2 - (1+\alpha^2)\|z_x\|^2 + \frac{1}{4}\int_I \phi' z^2\,dx \\
\leqslant (1+2\alpha^2)\|\phi''\|^2 - \|\phi'\|^2 + 2\bar{u}_0^2\|\phi\|^2.
\end{aligned}$$

Using (2.13) with $\beta = 4(1+\alpha^2)$ and

$$R^2 := \frac{2}{\mu^2}\,[(1+2\alpha^2)\,\|\phi''\|^2 - \|\phi'\|^2 + 2\bar{u}_0^2\,\|\phi\|^2], \tag{2.25}$$

implies

$$\frac{1}{2}\frac{d}{dt}\,\|z\|^2 + \frac{1}{4}\,\|z_{xx}\|^2 + \frac{1}{2}\mu^2\,\|z\|^2 \leqslant \frac{\mu^2}{2}\,R^2.$$

By integration, it then follows that

$$\|z(x,t)\|^2 \leqslant \|z_0\|^2 \exp(-\mu^2 t) + \|R\|^2, \tag{2.26}$$

which implies (2.22). ∎

2.3.2. *General Case*

We now consider the general case for v satisfying (1.5), (1.6). The coercive estimate (2.13) established above does not hold in this case. The gauge function $\phi(x)$ constructed above is linear on I except near the end-points of the interval, where $v(x,t)$ (and $z(x,t)$) was assumed to be zero. For the zero mean case considered here we have, by the intermediate value theorem, that there exists $\eta(t) \in I$ such that $z(\eta, t) = 0$. The shift function $b(t)$ is introduced in the argument of ϕ to compensate for the fact that η is a function of t. The method we use here follows the approach used in [5].

We begin by establishing the following coercive property.

LEMMA 2.3. *Let ϕ, μ, β, β^* be as described in Lemma 2.2. and Theorem 2.2. For $w(x,t)$, $z(x,t) \in \dot{H}^4_{per}$, consider the bi-linear form*

$$\begin{aligned}\langle\!\langle w, z\rangle\!\rangle_{\beta,\phi} := &\int_I w''z''\,dx - \beta\int_I w'z'\,dx + \int_I wz\phi'\,dx\\ &+ \frac{1}{4\mu^2 l}\int_I w\phi'\,dx\int_I z\phi'\,dx.\end{aligned} \tag{2.27}$$

Then,

$$\langle\!\langle z, z\rangle\!\rangle_{\beta,\phi} \geqslant 2\mu^2\,\|z\|^2. \tag{2.28}$$

This lemma implies that the bilinear form $\langle\!\langle w, z\rangle\!\rangle_{\beta,\phi}$ defines an inner product, and subsequently, the Cauchy–Schwarz inequality holds for this inner product.

Proof. For $z \in \dot{H}^4_{per}$, let $z0(x,t) = z(x,t) - \chi(t)$, where $\chi(t) = z(l,t)$. Note that

$$\int_I \phi' z^2\,dx = \int_I \phi'(z0+\chi)^2\,dx = \int_I \phi' z0^2\,dx + 2\chi \int_I \phi' z0\,dx$$

$$= \int_I \phi' z0^2\,dx + 2\chi \int_I \phi' z\,dx.$$

Using (2.13) (for $z0$), and the fact that $z \in \dot{H}^4_{per}$,

$$\|z_{xx}\|^2 - \beta\,\|z_x\|^2 + \int_I \phi' z^2\,dx = \|z0_{xx}\|^2 - \beta\,\|z0_x\|^2 + \int_I \phi' z0^2\,dx + 2\chi \int_I \phi' z\,dx$$

$$\geqslant 2\mu^2\,\|z0\|^2 + 2\chi \int_I \phi' z\,dx$$

$$= 2\mu^2\,\|z\|^2 + 4\mu^2 l\chi^2 + 2\chi \int_I \phi' z\,dx$$

$$\geqslant 2\mu^2\,\|z\|^2 - \frac{1}{4\mu^2 l}\left(\int_I \phi' z\,dx\right)^2$$

from which (2.28) readily follows. ∎

Note that the inequality (2.28) holds with ϕ replaced by any translation, $\phi_b(x) := \phi(x+b)$, of ϕ.

We are now in a position to establish the following uniform in time L_2 estimate.

THEOREM 2.3. *Let v be a solution of* (1.5), (1.6). *Then, there exists constants $C_1, C_2 > 0$, independent of v_0 and t, such that*

$$\|v(x,t)\| \leqslant \|v_0\|\,\exp(-C_1 t) + C_2. \tag{2.29}$$

Proof. In a similar fashion to the anti-symmetric case, consider the transformation

$$v(x,t) = z(x,t) + \phi_b(x), \tag{2.30}$$

where $z(x,t) \in \dot{H}^4_{per}$, and $\phi_b(x) = \phi(x+b(t)) \in \dot{C}^4_{per}(I)$.

With this change of variables (1.5) becomes

$$z_t + z_{xxxx} + z_{xx} + z z_x + \phi_b z_x + \phi_b' z + b'\phi_b' + \bar{u}_0 z_x + \alpha H(z_{xxx}) = F,$$

where $F = -\phi_b^{iv} - \phi_b'' - \phi_b \phi_b' - \bar{u}_0 \phi_b' - \alpha H(\phi_b''')$.

Multiplying through by z and integrating over I, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+\|z_{xx}\|^2-\|z_x\|^2+\frac{1}{2}\int_I \phi_b' z^2\,dx+b'\int_I \phi_b' z\,dx$$
$$+\alpha\int_I zH(z_{xxx})\,dx=\int_I Fz\,dx. \tag{2.31}$$

In this case

$$\int_I Fz\,dx=-\left[\int_I z_{xx}\phi_b''\,dx-\int_I z_x\phi_b'\,dx+\int_I z\phi_b\phi_b'\,dx\right]$$
$$+\bar{u}_0\int_I z'\phi_b\,dx+\alpha\int_I z'H(\phi_b'')\,dx$$
$$\leqslant-\left[\int_I z_{xx}\phi_b''\,dx-\int_I z_x\phi_b'\,dx+\int_I z\phi_b\phi_b'\,dx\right.$$
$$\left.+\frac{1}{4\mu^2 l}\int_I \phi_b\phi_b'\,dx\int_I z\phi_b'\,dx\right]$$
$$+\frac{1}{8}\|z_x\|^2+2\bar{u}_0^2\|\phi_b\|^2+\frac{1}{8}\|z_x\|^2+2\alpha^2\|\phi_b''\|^2$$
$$\leqslant\frac{1}{4}\langle\!\langle z,z\rangle\!\rangle_{1,\phi_b}+\langle\!\langle \phi_b,\phi_b\rangle\!\rangle_{1,\phi_b}+\frac{1}{8}\|z_x\|^2+2\bar{u}_0^2\|\phi_b\|^2$$
$$+\frac{1}{8}\|z_x\|^2+2\alpha^2\|\phi_b''\|^2,$$
$$\leqslant\frac{1}{4}\langle\!\langle z,z\rangle\!\rangle_{1,\phi_b}+\frac{\mu^2}{2}R^2. \tag{2.32}$$

Now, define $b(t)$ via the equation

$$b'(t)=\frac{1}{2}\frac{1}{4\mu^2 l}\int_I v(x,t)\,\phi_b'(x)\,dx:=f(t,b), \tag{2.33}$$

$$\text{subject to } b(0)=0. \tag{2.34}$$

Note that

$$\int_I v(x,t)\,\phi_b'(x)\,dx=\int_I z(x,t)\,\phi_b'(x)\,dx.$$

The existence and uniqueness of solutions to (2.33), (2.34) follow from the continuity of f, and f being Lipschitz with respect to b.

Substituting (2.32), (2.33) into (2.31), we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+\frac{1}{2}\|z_{xx}\|^2-(1+\alpha^2)\|z_x\|^2+\frac{1}{4}\int_I \phi_b' z^2\,dx$$
$$+\frac{1}{4}\frac{1}{4\mu^2 l}\left(\int_I z\phi_b'\,dx\right)^2\leqslant\frac{\mu^2}{2}R^2.$$

Applying (2.28) we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+\frac{1}{4}\|z_{xx}\|^2+\frac{1}{2}\mu^2\|z\|^2\leqslant\frac{\mu^2}{2}R^2. \tag{2.35}$$

from which the stated result follows. ∎

We are now in a position to establish the global (in time) existence of the solution.

Proof of Theorem 2.1. What remains, to establish the uniform boundness of v in $\dot{H}^1_{per}$, is the uniform boundness of $\|v_x\|$. Due to the Poincaré inequality, $\|v_x\|\leqslant 2l\,\|v_{xx}\|$, we need only estimate $\|v_{xx}\|$.

Note that from (2.35) it follows that $\int_t^{t+\tau/2}\|v_{xx}\|^2\,ds$ is uniformly bounded. (Here τ is as in Lemma 2.1).

Taking the L^2 scalar product of the Eq. (1.5) with v_{xxxx}, we get

$$\frac{1}{2}\frac{d}{dt}\|v_{xx}\|^2+\|v_{xxxx}\|^2=-\int_I v_{xx}v_{xxxx}\,dx-\int_I vv_xv_{xxxx}\,dx$$
$$-\alpha\int_I v_{xxxx}H(v_{xxx})\,dx. \tag{2.36}$$

The terms on the right hand side of (2.36) may be bounded as follows.

$$-\int_I v_{xx}v_{xxxx}\,dx\leqslant\|v_{xx}\|\,\|v_{xxxx}\|\leqslant\varepsilon_1\|v_{xxxx}\|^2+\frac{\varepsilon_1^{-1}}{4}\|v_{xx}\|^2, \tag{2.37}$$

$$\begin{aligned}-\int_I vv_xv_{xxxx}\,dx&\leqslant\|v\|_\infty\,\|v_x\|\,\|v_{xxxx}\|\\&\leqslant\sqrt{2}\,\|v\|^{1/2}\,\|v_x\|^{3/2}\,\|v_{xxxx}\|\ (\text{using (2.5)})\\&\leqslant C\,\|v_x\|^{3/2}\,\|v_{xxxx}\|\\&\leqslant C\,\|v_{xx}\|^{3/4}\,\|v_{xxxx}\|\ (\text{using (2.4)})\\&\leqslant\varepsilon_2\|v_{xxxx}\|^2+C\varepsilon_2^{-1}\|v_{xx}\|^{3/2}\\&\leqslant\varepsilon_2\|v_{xxxx}\|^2+C\|v_{xx}\|^2+C,\end{aligned} \tag{2.38}$$

and

$$
\begin{aligned}
-\alpha \int_I v_{xxxx} H(v_{xxx})\,dx &\leqslant |\alpha|\ \|v_{xxx}\|\ \|v_{xxxx}\| \\
&\leqslant |\alpha| \left[\varepsilon_3\ \|v_{xxxx}\|^2 + \frac{\varepsilon_3^{-1}}{4}\ \|v_{xxx}\|^2 \right] \\
&\leqslant |\alpha| \left[\varepsilon_3\ \|v_{xxxx}\|^2 + \frac{\varepsilon_3^{-1}}{4}\ C\ \|v_{xxxx}\|\ \|v_{xx}\| \right] \\
&\leqslant |\alpha|\ [\varepsilon_3\ \|v_{xxxx}\|^2 + \varepsilon_4\ \|v_{xxxx}\|^2 + C\varepsilon_3^{-2}\varepsilon_4^{-1}\ \|v_{xx}\|^2)]
\end{aligned}
\tag{2.39}
$$

Inserting estimates (2.37)–(2.39) into (2.36), we obtain

$$
\frac{1}{2}\frac{d}{dt}\ \|v_{xx}\|^2 + [1 - \varepsilon_1 - \varepsilon_2 - |\alpha|(\varepsilon_3 + \varepsilon_4)]\ \|v_{xxxx}\|^2 \leqslant C_1\ \|v_{xx}\|^2 + C_2. \tag{2.40}
$$

For ε_i's sufficiently small we have $1 - \varepsilon_1 - \varepsilon_2 - |\alpha|\ (\varepsilon_3 + \varepsilon_4) > 0$. This then implies

$$
\frac{1}{2}\frac{d}{dt}\ \|v_{xx}\|^2 \leqslant C_1\ \|v_{xx}\|^2 + C_2. \tag{2.41}
$$

By the uniform Gronwall inequality (2.2) it follows that $\|v_{xx}\|$ is uniformly bounded (with respect to time). Hence $\|v_x\|$ is also uniformly bounded, as $\|v_x\| \leqslant 2l\ \|v_{xx}\|$. Thus, v is uniformly bounded in $\dot{H}^1_{per}$, i.e. there exists a ball of radius $\rho_1 > 0$ in $\dot{H}^1_{per}$ such that $v(x, t) \in B(0, \rho_1)$ for all $t > 0$, and therefore the local solution actually exists for all positive time. ∎

Now, having established the global existence of the solution, from Theorem 2.3 it follows that:

COROLLARY 2.1. *There exists a positive bound* $\mathscr{R}$, *independent of the initial conditions*, v_0, *such that* $\limsup_{t \to \infty} \|v(x, t)\| \leqslant \mathscr{R} := \mathscr{R}(\alpha, l, \bar{u}_0)$. *Furtermore, for every such solution* v *there exists a time* $T_v > 0$ *such that* v *enters a fixed ball* B *in* $\dot{L}^2_{per}(I)$ *of radius* $2\mathscr{R}$ *in time* T_v, *and stays in* B *thereafter.*

Finally, for the solution $u(x, t)$ to (1.1) we have:

COROLLARY 2.2. *There exists a unique, global, classical solution* $u(x, t)$ *to* (1.1) *for* $u(x, 0) \in H^1_{per}(I)$.

3. GLOBAL DYNAMICS

As mentioned in the introduction, the nonlocal Kuramoto–Sivashinsky equation (1.1) does not have 'a' bounded maximal attractor. However, we show that Eq. (1.5) for the "fluctuation", $v = u - \bar{u} = u - \bar{u}_0$, has a bounded, finite dimensional attractor.

In this section we present results relating to the dimension of the maximal attractor as a function of the length of the interval l, and the "nonlocal" parameter α. Our interest is in determining the behavior of the dimension of the attractor as l and α becomes large. Physically $\alpha \to \infty$ corresponds to the flow approaching the critical Reynolds number.

Below we establish the following behavior for the attractor:

THEOREM 3.1. *There exists a finite dimensional compact, connected, and maximal attractor in* $\dot{H}^1_{per}(I)$ *for the dynamical system* (1.5), (1.6) *for each* $\bar{u}_0$. *Moreover the upper bound for the dimension of the maximal attractor is* $O(\alpha^{3/2} l^{3/2} + l^2)$, *for* α *and* l *large.*

In order to establish Theorem 3.1 we need to establish the existence of the maximal attractor, and obtain explicit estimates for $\limsup_{t \to \infty} (1/t) \int_0^t \|v_{xx}\| \, ds$, and the *size* of the "absorbing ball", $\mathscr{R} = R + \|\phi\|$. This will be done in the following four lemmas.

LEMMA 3.1. *For the dynamical system* (1.5), (1.6), *there exists a compact, connected, and maximal attractor in* $\dot{H}^1_{per}(I)$.

Proof. Let $S(t)$: $\dot{H}^1_{per}(I) \to \dot{H}^2_{per}(I)$ denote the solution operator for (1.5) (1.6), defined via $v(x, t) = S(t)\, v_0(x)$. It follows from the $\dot{H}^1_{per}$ and $\dot{H}^2_{per}$ estimates in the last section, that $S(t)$ is a bounded nonlinear operator from $\dot{H}^1_{per}(I)$ into $\dot{H}^2_{per}(I)$, for every $t > 0$. As $\dot{H}^2_{per}$ is compactly imbedded in $\dot{H}^1_{per}$ ([23], Theorem 6.98), then $S(t)$ is a compact mapping from $\dot{H}^1_{per}$ into $\dot{H}^1_{per}$ for every $t > 0$. Thus the ω-limit set of the absorbing set $B(0, \rho_1)$ under $S(t)$, which we denote $\mathscr{A}$,

$$\mathscr{A} = \bigcap_{s > 0} \left(\overline{\bigcup_{t \geqslant s} S(t)\, B(0, \rho_1)} \right), \tag{3.1}$$

is the maximal attractor, where the closure is taken in the $\dot{H}^1_{per}$ topology (see [27] or [15] and references therein). The maximal attractor $\mathscr{A}$ is necessarily a non-empty compact connected subset of $\dot{H}^1_{per}(I)$. ▮

LEMMA 3.2. *For v satisfying* (1.5), (1.6), *we have*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|v_{xx}(s)\|^2\, ds \leqslant 2(1+\alpha^2)^2\, \mathscr{R}^2. \tag{3.2}$$

Proof. Multiplying (1.5) through by v and integrating over I we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v_{xx}\|^2 &= \|v_x\|^2 - \alpha\int_I vH(v_{xxx})\,dx\\
&\leqslant \|v_x\|^2 + \alpha\,\|v_x\|\,\|v_{xx}\|\\
&\leqslant (1+\alpha^2)\,\|v_x\|^2 + \frac{1}{4}\|v_{xx}\|^2\\
&\leqslant (1+\alpha^2)\,\|v\|\,\|v_{xx}\| + \frac{1}{4}\|v_{xx}\|^2\\
&\leqslant (1+\alpha^2)^2\,\|v\|^2 + \frac{1}{2}\|v_{xx}\|^2\end{aligned}$$

$$\text{i.e.}\quad \frac{d}{dt}\|v\|^2 + \|v_{xx}\|^2 \leqslant 2(1+\alpha^2)^2\,\|v\|^2. \tag{3.3}$$

Now, taking the limit supremum as $t\to\infty$ of the time average of (3.3), and in view of Theorem 2.3, we obtain (3.2). ∎

LEMMA 3.3. *For α large, the radius of the absorbing ball, $\mathscr{R}\sim R+\|\phi\|$, satisfies*

$$\mathscr{R}\sim O(\alpha^5 l^{5/2}). \tag{3.4}$$

Proof. From (2.25), (2.23), (2.21), (2.13), we have

$$\begin{aligned}&R^2 = [(1+2\alpha^2)\,\|\phi''\|^2 - \|\phi'\|^2 + 2\bar{u}_0^2\,\|\phi\|^2]/\mu^2,\\
&\mu^2 = 6(1+\alpha^2)^2, \qquad \gamma = 24(1+\alpha^2)^2, \qquad \delta = l^{-2}(1+\alpha^2)^{-1}/12.\end{aligned} \tag{3.5}$$

Using (2.14)–(2.17), (2.18), one readily obtains, for α large,

$$\int_I (\phi)^2\,dx \sim O(\alpha^8 l^3), \qquad \int_I (\phi')^2\,dx \sim O(\alpha^8 l^3). \tag{3.6}$$

Observe that ϕ'' is only non–negative on I in the subintervals $(-l, -(1-\sigma_2\delta)\,l)$, $(-(1-(1-\sigma_1)\delta)l,\ -(1-\delta)l)$, $((1-\delta)l,\ (1-(1-\sigma_1)\delta)l)$,

$((1-\sigma_2\delta)l, l)$. Using this observation together with (A.3) and (3.5) we obtain

$$\int_I (\phi'')^2 \, dx \sim O(\alpha^{14} l^5). \tag{3.7}$$

Thus, it follows from (3.6), (3.7) that for α, l large R and $\mathcal{R}$ have the same behavior

$$\mathcal{R} \sim R \sim O(\alpha^5 l^{5/2}). \quad \blacksquare$$

As in [6, 12, or 27], we may use the so-called Constantin–Foias–Temam trace formula (which works for the semiflow $S(t)$ here) to estimate the sum of the global Lyapunov exponents of $\mathcal{A}$. The sum of these Lyapunov exponents can then be used to estimate the upper bounds of $\mathcal{A}$'s Hausdorff and fractal dimensions, $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$. Now we use the trace formula to estimate the sum of the global Lyapunov exponents of $\mathcal{A}$. To this end, we linearize Eq. (1.5) about a solution $v(x, t)$ in the maximal attractor to obtain the following equation for $V(x, t) \in \dot{H}^1_{per}(I)$:

$$V_t = -V_{xxxx} - V_{xx} - vV_x - Vv_x - \bar{u}_0 V_x - \alpha H(V_{xxx}) := N'(v) V \tag{3.8}$$

with $V(x, 0) = \xi(x) \in \dot{H}^1_{per}(I)$. Denote by $\xi_1(x), ..., \xi_n(x)$, n linearly independent functions in $\dot{H}^1_{per}(I)$, and $V_i(x, t)$ the solution of (1.5) satisfying $V_i(x, 0) = \xi_i(x)$, $i = 1, ..., n$. Let $P_n(t)$ represent the orthogonal projection of $\dot{H}^1_{per}(I)$ onto the subspace spanned by $\{V_1(x, t), ..., V_n(x, t)\}$.

We need to estimate the following quantities

$$q_n = \limsup_{t\to\infty} \sup_{\substack{\xi_j \in \dot{H}^1_{per}(-l, l) \\ \|\xi_j\| \leqslant 1 \\ j = 1, 2, ..., n}} \frac{1}{t} \int_0^t \{\mathrm{Trace}(N'(v(s)) \circ P_n(s))\} \, ds, \tag{3.9}$$

for $n = 1, 2, ...$. The quantity q_n is an upper bound for the sum of the first n global Lyapunov exponents of the maximal attactor $\mathcal{A}$, which we use to estimate the fractal and Hausdorff dimensions of $\mathcal{A}$, $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$, (see [27, Theorem V.3.3]). Also note that $d_H(\mathcal{A}) \leqslant d_F(\mathcal{A})$.

LEMMA 3.4. *The quantity q_n satisfies*

$$q_n \sim -\kappa_1 n^5 + \kappa_2, \tag{3.10}$$

where κ_1, κ_2 are positive constants depending on α, l and $\bar{u}_0$.

Proof. At any given time t, let $\{\phi_j(t)\}_{j=1}^n$ be a subset of $\dot{H}^2_{per}(I)$, which is an orthonormal basis of the linear space $P_n(t)\, \dot{H}^1_{per}(I)$, with respect to the L^2-inner product, i.e. $(\phi_i(t), \phi_j(t)) = \delta_{ij}$. Thus

$$\operatorname{span}\{\phi_1(t), ..., \phi_n(t)\} = \operatorname{span}\{V_1(t), ..., V_n(t)\}.$$

Note that

$$\operatorname{Trace}(N'(v(t)) \circ P_n(t)) = \sum_{j=1}^{n} (N'(v(t))\, \phi_j(t), \phi_j(t)) \tag{3.11}$$

and

$$\begin{aligned}
(N'(v)\, \phi_j, \phi_j) &= -\left(\frac{\partial^4 \phi_j}{\partial x^4}, \phi_j\right) - \left(\frac{\partial^2 \phi_j}{\partial x^2}, \phi_j\right) - \alpha \left(H\left(\frac{\partial^3 \phi_j}{\partial x^3}\right), \phi_j\right) \\
&\quad - \bar{u}_0 \left(\frac{\partial \phi_j}{\partial x}, \phi_j\right) - \left(v\, \frac{\partial \phi_j}{\partial x}, \phi_j\right) - (\phi_j v_x, \phi_j) \\
&= -\left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2 + \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 + \alpha \left(H\left(\frac{\partial^2 \phi_j}{\partial x^2}\right), \frac{\partial \phi_j}{\partial x}\right) \\
&\quad - \bar{u}_0 \left(\frac{\partial \phi_j}{\partial x}, \phi_j\right) - \frac{1}{2}(\phi_j{}^2, v_x).
\end{aligned} \tag{3.12}$$

Let $f = \sum_{j=1}^n \phi_j^2$. Thus

$$\begin{aligned}
\sum_{j=1}^{n} (N'(v(t))\, \phi_j, \phi_j) &= -\sum_{j=1}^{n} \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2 + \sum_{j=1}^{n} \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 + \alpha \sum_{j=1}^{n} \left(H\left(\frac{\partial^2 \phi_j}{\partial x^2}\right), \frac{\partial \phi_j}{\partial x}\right) \\
&\quad - \bar{u}_0 \sum_{j=1}^{n} \left(\frac{\partial \phi_j}{\partial x}, \phi_j\right) - \frac{1}{2}(f, v_x).
\end{aligned} \tag{3.13}$$

We now estimate the last three terms on the right hand side of (3.13) separately.

$$\begin{aligned}
\alpha \sum_{j=1}^{n} \left(H\left(\frac{\partial^2 \phi_j}{\partial x^2}\right), \frac{\partial \phi_j}{\partial x}\right) &\leqslant \alpha \sum_{j=1}^{n} \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\| \left\|\frac{\partial \phi_j}{\partial x}\right\| \\
&\leqslant \frac{1}{4} \sum_{j=1}^{n} \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2 + \alpha^2 \sum_{j=1}^{n} \left\|\frac{\partial \phi_j}{\partial x}\right\|^2.
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
-\bar{u}_0 \sum_{j=1}^{n} \left(\frac{\partial \phi_j}{\partial x}, \phi_j\right) &\leqslant |\bar{u}_0| \sum_{j=1}^{n} \left\|\frac{\partial \phi_j}{\partial x}\right\| \|\phi_j\| \\
&\leqslant \frac{1}{4} \sum_{j=1}^{n} \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 + n\, |\bar{u}_0|^2.
\end{aligned} \tag{3.15}$$

Finally, by Sobolev–Lieb–Thirring inequality ([27], p. 461) there exist an absolute constant a_1 which is independent of n and of the functions $\{\phi_j\}_{j=1}^n$, such that

$$\|f\|_{L^5(I)}^5 \leqslant a_1 \sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2. \tag{3.16}$$

Thus using (3.16) and Hölder and Young's inequalities we get

$$\begin{aligned}\frac{1}{2}\,|(f, v_x)| &\leqslant \frac{1}{2}\,\|f\|_{L^5}\,\|v_x\|_{L^{5/4}} \leqslant \frac{1}{2}\left(a_1 \sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2\right)^{1/5} \|v_x\|_{L^{5/4}} \\ &\leqslant \frac{1}{4} \sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2 + a_2\,\|v_x\|_{L^{5/4}}^{5/4} \end{aligned}\tag{3.17}$$

where a_2 is a positive constant independent of n. Substitute (3.14), (3.15) and (3.17) into (3.13) we obtain

$$\begin{aligned}&\sum_{j=1}^n (N'(v(t))\,\phi_j, \phi_j) \\ &\quad\leqslant -\frac{1}{2} \sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2 + (\alpha^2 + n\,|\bar{u}_0|^2 + 1) \sum_{j=1}^n \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 + a_2\,\|v_x\|_{L^{5/4}}^{5/4}. \end{aligned}\tag{3.18}$$

The right hand side of this inequality can be further estimated as follows. By the Hölder inequality we have, noting the fact that $\|\phi_j\| = 1$,

$$\sum_{j=1}^n \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 \leqslant \sum_{j=1}^n \|\phi_j\|\,\left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\| \leqslant \sqrt{n}\left(\sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2\right)^{1/2}.$$

Therefore it follows from the Young's inequality (2.1) that

$$(\alpha^2 + n\,|\bar{u}_0|^2 + 1) \sum_{j=1}^n \left\|\frac{\partial \phi_j}{\partial x}\right\|^2 \leqslant n(\alpha^2 + n\,|\bar{u}_0|^2 + 1)^2 + \frac{1}{4} \sum_{j=1}^n \left\|\frac{\partial^2 \phi_j}{\partial x^2}\right\|^2. \tag{3.19}$$

Now we estimate the last term in (3.18). Due to the Hölder inequality and interpolation $\|v_x\| \leqslant \|v\|\,\|v_{xx}\|$, we have

$$\|v_x\|_{L^{5/4}}^{5/4} \leqslant (2l)^{3/8}\,\|v_x\|^{5/8} \leqslant (2l)^{3/8}\,\|v\|^{5/8}\,\|v_{xx}\|^{5/8}. \tag{3.20}$$

Moreover because $\|\phi_j\| = 1$, we have

$$n = \sum_{j=1}^n \|\phi_j\|^2 = \sum_{j=1}^n \int_{-l}^{l} \phi_j^2\,dx = \int_{-l}^{l} f(x)\,dx \leqslant (2l)^{4/5}\,\|f\|_{L^5}.$$

Thus it follows from (3.16) that

$$\frac{n^5}{16a_1 l^4} \leqslant \sum_{j=1}^{n} \left\| \frac{\partial^2 \phi_j}{\partial x^2} \right\|^2. \tag{3.21}$$

Combining (3.19), (3.20) and (3.21) in (3.18) we obtain

$$\begin{aligned}\operatorname{Trace}(N'(v(t)) \circ P_n(t)) \leqslant & -\frac{1}{64a_1 l^4} n^5 + n(\alpha^2 + n\,|\bar{u}_0|^2 + 1)^2 \\ & + a_2 (2l)^{3/8} \|v\|^{5/8} \|v_{xx}\|^{5/8}. \end{aligned} \tag{3.22}$$

Using Lemmas 3.2 and 3.3 we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|u(s)\|^{5/8} \|u_{xx}(s)\|^{5/8}\,ds &\leqslant \left(\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|u_{xx}(s)\|^2\,ds \right)^{5/16} \mathscr{R}^{5/8} \\ &\leqslant [2(1+\alpha^2)^2\,\mathscr{R}^2]^{5/16}\,\mathscr{R}^{5/8} \\ &= 2^{5/16}(1+\alpha^2)^{5/8}\,\mathscr{R}^{5/4}. \end{aligned} \tag{3.23}$$

Recall the quantity q_n,

$$q_n = \limsup_{t\to\infty} \sup_{\substack{\xi_j \in \dot{H}^1_{per}(-l,\,l) \\ \|\xi_j\| \leqslant 1 \\ j=1,2,\ldots,n}} \frac{1}{t} \int_0^t \{\operatorname{Trace}(N'(v(s)) \circ P_n(s))\}\,ds. \tag{3.24}$$

It follows that, using (3.22), (3.23) and applying Young's inequality to $n = n \cdot 1$, $n^2 = n^2 \cdot 1$, $n^3 = n^3 \cdot 1$,

$$\begin{aligned} q_n \leqslant & -\frac{1}{64a_1 l^4} n^5 + n(\alpha^2 + n\,|\bar{u}_0|^2 + 1)^2 + a_2(2l)^{3/8}\,2^{5/16}(1+\alpha^2)^{5/8}\,\mathscr{R}^{5/4} \\ = & -\frac{1}{64a_1 l^4} n^5 + a_2 l^{3/8} 2^{11/16}(1+\alpha^2)^{5/8}\,\mathscr{R}^{5/4} \\ & + n(1+\alpha^2)^2 + 2n^2\,|\bar{u}_0|^2(1+\alpha^2) + n^3\,|\bar{u}_0|^4 \\ \leqslant & -\frac{1}{256a_1 l^4} n^5 + a_2 l^{3/8} 2^{11/16}(1+\alpha^2)^{5/8}\,\mathscr{R}^{5/4} \\ & + \frac{4}{5}\left(\frac{256}{5}\right)^{1/4} a_1^{(1/4)}\, l(1+\alpha^2)^{5/2} \\ & + \frac{3}{5}\left(\frac{512}{5}\right)^{2/3} a_1^{2/3} l^{8/3} 2^{5/3}\,|\bar{u}_0|^{10/3}\,(1+\alpha^2)^{5/3} + \frac{2}{5}\left(\frac{768}{5}\right)^{3/2} a_1^{3/2} l^6\,|\bar{u}_0|^{10} \\ := & -\kappa_1 n^5 + \kappa_2, \end{aligned} \tag{3.25}$$

where

$$\kappa_1 = \frac{1}{256 a_1 l^4}$$

and

$$\kappa_2 = a_2 l^{3/8} 2^{11/16} (1+\alpha^2)^{5/8} \mathscr{R}^{5/4} + \frac{4}{5}\left(\frac{256}{5}\right)^{1/4} a_1^{(1/4)} l (1+\alpha^2)^{5/2}$$
$$+ \frac{3}{5}\left(\frac{512}{5}\right)^{2/3} a_1^{2/3} l^{8/3} 2^{5/3} |\bar{u}_0|^{10/3} (1+\alpha^2)^{5/3} + \frac{2}{5}\left(\frac{768}{5}\right)^{3/2} a_1^{3/2} l^6 |\bar{u}_0|^{10}. \quad \blacksquare$$

Now we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. From [27], Theorem V.3.3 and Lemma VI.2.2, the maximal attractor $\mathscr{A}$ has Hausdorff dimension, $d_H(\mathscr{A})$, less than or equal to n, and fractal dimension, $d_F(\mathscr{A})$, less than or equal to $2n$, where n is defined as

$$n - 1 \leqslant \left(\frac{2\kappa_2}{\kappa_1}\right)^{1/5} \leqslant n, \tag{3.26}$$

or

$$n^5 \sim l^{35/8} (1+\alpha^2)^{5/8} \mathscr{R}^{5/4} + l^5 (1+\alpha^2)^{5/2} + l^{20/3} (1+\alpha^2)^{5/3} |\bar{u}_0|^{10/3} + l^{10} |\bar{u}_0|^{10}. \tag{3.27}$$

Thus, from Lemma 3.3, for α and l large, we have

$$n \sim O(\alpha^{3/2} l^{3/2} + l^2). \quad \blacksquare \tag{3.28}$$

4. REMARKS

In this paper, we have proved the existence of a family of maximal attractors, parameterized by the mean of initial data, for a nonlocal Kuramoto–Sivashinsky equation on periodic intervals. We have also estimated the dimensions of these attractors in terms of the coefficient of the nonlocal term, the width of the periodic domain, and the mean of the initial data.

To demonstrate the influence of the nonlocal term, consider (1.5) with the Hilbert operator replaced by the Identity operator:

$$v_t + v_{xxxx} + v_{xx} + v v_x + \bar{u}_0 v_x + \alpha v_{xxx} = 0. \tag{4.1}$$

As remarked in Duan *et al.*, [10], the dispersive term αv_{xxx} has no impact on the size of absorbing set or dimension of the maximal attractor of the usual K–S equation, as $\int v_{xxx} D^k v \, dx = 0$ for $k = 0$ and 4, which are the terms that arise in the analysis. For the case considered herein, $\int H(v_{xxx}) \, v \, dx \neq 0$, so the non-local term $H(v_{xxx})$ influences the evolution of the L^2-energy and the global dynamics of the solution. However, in view of properties of Hilbert transforms, it is clear that $\int H(D_x^{2k} v) \, v \, dx = 0$, $k \in \mathbb{N}$, so terms of the form $H(D_x^{2k} v)$ do not influence the L^2-energy, or the dimension of the maximal attractor.

We also remark that the analysis presented herein extends readily to other linear integral or differential operators, provided they satisfy some boundedness conditions.

APPENDIX A: C^3 POLYNOMIAL CUTOFF FUNCTION

Define

$$\Lambda(x; a, b) = \begin{cases} 0, & x \leqslant a, \\ (x-a)^4 \, (x-b)^4 - \dfrac{140}{(b-a)^7} \displaystyle\int_a^x (t-a)^3 \, (t-b)^3 \, dt, & a \leqslant x \leqslant b, \\ 1 & x \geqslant b. \end{cases} \tag{A.1}$$

(See Fig. A.2.) Observe that (i) $\Lambda(a; a, b) = 0$, $\Lambda(b; a, b) = 1$,

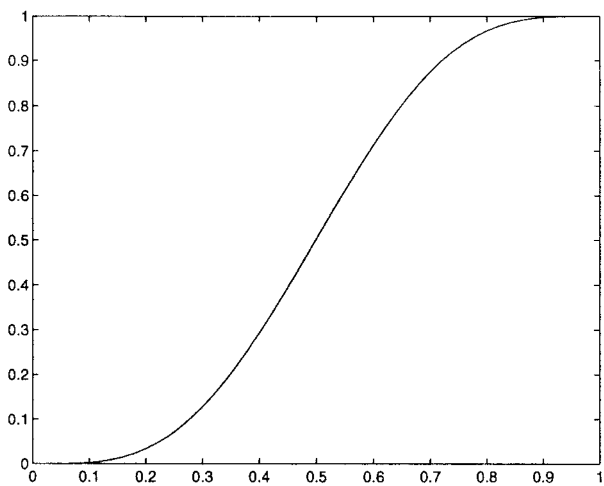

Fig. A.2. Graph of $\Lambda(x; 0, 1)$ for $0 \leqslant x \leqslant l$.

(ii) $\Lambda'(x; a, b) \geqslant 0$, provided $(b-a) \leqslant (35)^{1/8}$,

(iii) $\Lambda(x; a, b) \in C^3(\mathbb{R})$.

In addition,

$$\int_0^b (\Lambda(x; 0, b))^2 \, dx = \frac{1}{218790} b^{17} + \frac{1}{630} b^9 + \frac{521}{1287} b, \tag{A.2}$$

$$\int_0^b (\Lambda'(x; 0, b))^2 \, dx = \frac{4}{45045} b^{15} + \frac{700}{429} b^{-1}. \tag{A.3}$$

REFERENCES

1. A. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, Nonlocal models for nonlinear dispersive waves, *Phys. D* **40** (1989), 360–392.
2. R. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
3. E. A. Alarcon, Existence and finite dimensionality of the global attractor for a class of nonlinear dissipative equations, *Proc. R. Soc. Edinburgh A* **123** (1993), 893–916.
4. C. M. Alfaro and M. C. Depassier, A five-mode bifurcation analysis of a Kuramoto–Sivashinsky equation with dispersion, *Phys. Lett. A* **184** (1994), 184–189.
5. P. Collet, J.-P. Eckmann, H. Epstein, and J. Stubbe, A global attracting set for the Kuramoto–Sivashinsky equation, *Comm. Math. Phys.* **152** (1993), 203–214.
6. P. Constantin, C. Foias, and R. Temam, Attractor representing turbulent flows, *Memoirs of AMS* **53**, No. 314 (1985).
7. H. C. Chang, E. A. Demekhin, and D. I. Kopelevich, Laminarizing effects of dispersion in an active-dissipative nonlinear medium, *Phys. D* **63** (1993), 299–320.
8. H. C. Chang, Wave evolution on a falling film, *Annu. Rev. Fluid Mech.* **26** (1994), 103–136.
9. J. Duan, H. V. Ly, and E. S. Titi, The effect of nonlocal interactions on the dynamics of Ginzburg–Landau equation, *Z. Angew. Math. Phys.* **47** (1996), 433–455.
10. J. Duan, H. V. Ly, and E. S. Titi, The effect of dispersion on the dynamics of the Kuramoto–Sivashinsky equation, preprint, 1995.
11. N. M. Ercolani, D. W. McLaughlin, and H. Poitner, Attractors and transients for a perturbed periodic KdV equation: a nonlinear spectral analysis, *J. Nonlinear Sci.* **3** (1993), 477–539.
12. C. Foias, B. Nicolaenko, G. R. Sell, and R. Temam, Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension, *J. Math. Pures Appl.* **67** (1988), 197–226.
13. A. Gonzalez and A. Castellanos, Nonlinear electrohydrodynamic waves on films falling down an inclined plane, *Phys. Rev. E* **53** (1996), 3573–3578.
14. J. Goodman, Stability of the Kuramoto–Sivashinsky and related systems, *Comm. Pure Appl. Math.* **47** (1994), 293–306.
15. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," American Math. Soc., 1988.
16. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Springer-Verlag, Berlin, 1981.
17. D. Y. Hsieh, Elemental mechanisms of hydrodynamic instabilities, *Acta Mech. Sinica* **10** (1994), 193–202.
18. D. Y. Hsieh, Mechanism for instability of fluid flow down an inclined plane, *Phys. Fluids A* **2** (1990), 1145–1148.

19. J. S. Il'yashenko, Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation, *J. Dynamics Diff. Eqn.* **4** (1992), 585–615.
20. M. S. Jolly, I. G. Kevrekidis, and E. S. Titi, Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: analysis and computations, *Physica D* **44** (1990), 38–60.
21. B. Nicolaenko, B. Scheurer, and R. Temam, Some global dynamical properties of the Kuramoto–Sivashinsky equation: nonlinear stability and attractors, *Physica D* **16** (1985), 155–183.
22. B. Nicolaenko, B. Scheurer, and R. Temam, Some global dynamical properties of a class of pattern formation equations, *Comm. PDEs* **14** (1989), 245–297.
23. M. Renardy and R. C. Rogers, "An Introduction to Partial Differential Equations," Springer-Verlag, New York, 1993.
24. J. C. Robinson, Inertial manifolds for the Kuramoto–Sivashinsky equation, *Phys. Lett. A* **184** (1994), 190–193.
25. E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
26. M. Taboada, Finite-dimensional asymptotic behavior for the Swift–Hohenberg model of convection, *Nonlin. Anal.* **14** (1990), 43–54.
27. R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.
28. R. Temam and X. Wang, Estimates on the lowest dimension of inertial manfilds for the Kuramoto–Sivashinsky equation in the general case, *Diff. Integral Eqns* **7** (1994), 1095–1108.
29. F. G. Tricomi, "Integral Equations," Interscience, New York, 1957.