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The inverse eigenvalue problem for symmetric anti-bidiagonal matrices

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Abstract

The inverse eigenvalue problem for real symmetric matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & * & * \\ 0 & 0 & 0 & \cdots & * & * & 0 \\ \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

is solved. The solution is shown to be unique. The problem is also shown to be equivalent to the inverse eigenvalue problem for a certain subclass of Jacobi matrices.

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1. Introduction

The goal of this paper is to characterize completely the spectra of real symmetric *anti-bidiagonal* matrices, i.e., matrices of the form

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & . & \vdots & \vdots \\ 0 & a_{n-2} & \cdots & 0 & 0 \\ a_n & a_{n-1} & \cdots & 0 & 0 \end{bmatrix}, \quad a_1, \dots, a_n \in \mathbb{R}. \tag{1}$$

This work is motivated by the author's ongoing work on the nonnegative inverse eigenvalue problem and is inspired by well-known results on Jacobi matrices due to Hochstadt [5,6], Hald [4], Gray and Wilson [3], as well as by the classical connection between the Jacobi matrices and orthogonal polynomials (see, e.g., [1, p. 267]).

The blanket assumption for the rest of the paper is that all a_j are positive. This restriction is clearly unimportant, since the sign of any a_j, $j > 1$, can be changed using a unitary similarity of the form

$$\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n), \qquad \varepsilon_j = \pm 1,$$

and the problem for $a_1 < 0$ can be solved by switching from A to $-A$. The assumption $a_j > 0$, $j = 1, \dots, n$, is however just right to guarantee uniqueness of a matrix that realizes a given n-tuple as its spectrum.

2. Definitions and notation

Notation used in the paper is rather standard. The spectrum of a matrix A is denoted by $\sigma(A)$. A submatrix of A with rows indexed by an increasing sequence α and columns indexed by another sequence β is denoted by $A(\alpha, \beta)$. For simplicity, a principal submatrix of A with rows and columns indexed by α is denoted by $A(\alpha)$. (A typical choice for such an α will be $i{:}j$, the sequence of consecutive integers i through j.) The size of a sequence α is denoted by $\#\alpha$. If $\#\alpha = \#\beta$, then $\det A(\alpha, \beta)$ is denoted by $A[\alpha, \beta]$; $\det A(\alpha)$ is denoted by $A[\alpha]$. The elementary symmetric functions of an n-tuple $\Lambda =: (\lambda_1, \dots, \lambda_n)$ are denoted as $\sigma_j(\Lambda)$. Thus

$$\sigma_1(\Lambda) := \sum_{j=1}^{n} \lambda_j, \qquad \sigma_2(\Lambda) := \sum_{i<j} \lambda_i \lambda_j, \text{ etc.}$$

The term anti-bidiagonal matrix was already introduced. Other requisite definitions are listed next.

A *Jacobi matrix* is a tridiagonal matrix with positive codiagonal entries.

A *sign-regular matrix of class $d \leqslant n$ with signature sequence* $\varepsilon_1, \dots, \varepsilon_d$, where $\varepsilon_j = \pm 1$ for all j, is a matrix satisfying

$$\varepsilon_j A[\alpha] \geqslant 0 \quad \text{whenever } \#\alpha = j, \quad j = 1, \dots, d.$$

If in addition all minors of order at most d are nonzero, the matrix is called *strictly sign-regular*. Finally, if a certain power of a sign-regular matrix of class d is strictly sign-regular, then the matrix is called sign-regular of class d^+. A particular case of strict sign regularity is *total positivity* when all minors of a matrix are positive.

A sequence $\mu_1 < \cdots < \mu_k$ is said to *interlace* a sequence $\lambda_1 < \cdots < \lambda_{k+1}$ if

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_k < \lambda_{k+1}.$$

3. Results

The following theorem is the main result of this paper.

Theorem 1. *A real n-tuple Λ can be realized as the spectrum of a symmetric anti-bidiagonal matrix (1) with all a_j positive if and only if $\Lambda = (\lambda_1, \ldots, \lambda_n)$ where*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0. \tag{2}$$

The realizing matrix is necessarily unique.

Proof. *Necessity*. Let J denote the anti-diagonal unit matrix

$$J := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & . & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Note that J is sign-regular of class n with the signature sequence

$$1, -1, -1, 1, 1, \ldots, (-1)^{\lceil n-1/2 \rceil}. \tag{3}$$

Next, note that $B := JA$ is a nonnegative bidiagonal matrix, hence all its minors are nonnegative. Now, by the Cauchy–Binet formula

$$A[\alpha] = (JB)[\alpha] = \sum_{\#\beta=\#\alpha} J[\alpha, \beta]B[\beta, \alpha].$$

We conclude that the matrix A is sign-regular of class n with the same signature sequence (3). Since A^2 is a positive definite Jacobi matrix, a high enough power of A^2 is totally positive, hence A is sign-regular of type n^+.

By a theorem of Gantmacher and Krein [2, p. 301], the eigenvalues of A therefore can be arranged to form a sequence with alternating signs and strictly decreasing absolute values whose first element is positive, i.e., the spectrum $\sigma(A)$ satisfies (2).

Sufficiency. First reduce the inverse problem for anti-bidiagonal matrices to the inverse problem for certain Jacobi matrices. Consider a matrix of the form (1). To stress its dependence on n parameters a_1 through a_n, let us denote it by A_n. The argument will involve the collection of all matrices $A_n, n \in \mathbb{Z}$, determined by a single sequence $a_1, a_2, \ldots$ Denote the characteristic polynomial of A_n by p_n:

$$p_n(\lambda) := \det(\lambda I - A_n).$$

Expanding it by its first row yields

$$p_n(\lambda) = \lambda p_{n-1}(\lambda) - a_n^2 p_{n-2}(\lambda), \quad n \geqslant 2, \tag{4}$$

$$p_0(\lambda) = 1, \quad p_1(\lambda) = \lambda - a_1, \tag{5}$$

since the matrix A_{n-1} is similar to its reflection about the anti-diagonal.

This three-term recurrence relation (4) with initial conditions (5) is also satisfied (see, e.g., [1, p. 267] or check directly) by the characteristic polynomials of the Jacobi matrices

$$B_n := \begin{bmatrix} a_1 & a_2 & 0 & \cdots & 0 & 0 \\ a_2 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & 0 & \cdots & a_n & 0 \end{bmatrix} \tag{6}$$

if each of them is expanded by its last row. Thus the inverse eigenvalue problem for anti-bidiagonal matrices A_n is equivalent to the inverse eigenvalue problem for Jacobi matrices B_n.

Now comes the crucial step in the proof. Consider expanding the characteristic polynomials of matrices B_n in the opposite order, i.e., starting from the first row. Precisely, let us denote by q_{n-j+1} the characteristic polynomial of the principal submatrix $B_n(j{:}n)$, with $q_n = p_n$. The corresponding recurrence relation is

$$q_n(\lambda) = (\lambda - a_1)q_{n-1}(\lambda) - a_2^2 q_{n-2}(\lambda), \tag{7}$$

$$q_{n-j}(\lambda) = \lambda q_{n-j-1}(\lambda) - a_{j+2}^2 q_{n-j-2}(\lambda), \quad j = 1, \ldots, n-2, \tag{8}$$

$$q_0(\lambda) = 1, \quad q_1(\lambda) = \lambda. \tag{9}$$

Let Λ be an n-tuple satisfying (2). Define the polynomial q_n as

$$q_n(\lambda) := \prod_{j=1}^{n} (\lambda - \lambda_j)$$

and show that one can define polynomials q_{n-j} for all $j = 1, \ldots, n$ so as to meet the requirements (7)–(9). To this end, first define

$$a_1 := \sigma_1(\Lambda), \qquad q_{n-1}(\lambda) := \frac{(-1)^n q_n(-\lambda) - q_n(\lambda)}{2a_1}. \tag{10}$$

Note that $a_1 > 0$ due to the properties of Λ and that the (monic) polynomial q_{n-1} is even or odd depending on whether $n-1$ is even or odd. Also note that the coefficient of λ^{n-3} in q_{n-1} is equal to

$$\frac{\sigma_3(\Lambda)}{a_1} = \frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} < 0.$$

On the other hand, the coefficient of λ^{n-2} in $q_n(\lambda)$ is $\sigma_2(\Lambda) < 0$. Therefore, it remains to show that the quantity $\frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} - \sigma_2(\Lambda)$ is positive, so a_2 can be defined as its (positive) square root:

$$a_2 := \sqrt{\frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} - \sigma_2(\Lambda)}.$$

Indeed, let us prove that

$$\sigma_3(\Lambda) > \sigma_1(\Lambda)\sigma_2(\Lambda) \tag{11}$$

by induction. The base case is $n = 3$, where $\lambda_1 > -\lambda_2 > \lambda_3 > 0$. Then (11) reduces to the inequality

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)(\lambda_1 + \lambda_2 + \lambda_3) > 1. \tag{12}$$

Differentiating the left-hand side of (12), one can check that it is an increasing function of λ_1 for $\lambda_1 \geqslant -\lambda_2$. Since the left-hand side is exactly 1 when $\lambda_1 = -\lambda_2$, this proves (12) and therefore proves (11). If $n > 3$, also notice that inequality (11) turns into equality for $\lambda_1 = -\lambda_2$, so it remains to argue that the difference $\sigma_3(\Lambda) - \sigma_1(\Lambda)\sigma_2(\Lambda)$ is an increasing function of λ_1 for $\lambda_1 \geqslant -\lambda_2$. But this is indeed the case, as can be seen by considering symmetric functions of the set $\Lambda' := (-\lambda_2, \ldots, -\lambda_n)$. Since

$$\begin{aligned}
\sigma_1(\Lambda) &= \lambda_1 - \sigma_1(\Lambda'),\\
\sigma_2(\lambda) &= -\lambda_1\sigma_1(\Lambda') + \sigma_2(\Lambda'),\\
\sigma_3(\Lambda) &= \lambda_1\sigma_2(\Lambda') - \sigma_3(\Lambda'),
\end{aligned}$$

the inequality (11) amounts to

$$\lambda_1^2\sigma_1(\Lambda') - \lambda_1\sigma_1^2(\Lambda') + \sigma_1(\Lambda')\sigma_2(\Lambda') - \sigma_3(\Lambda') > 0,$$

and the derivative of the last left-hand side is positive, since $\lambda_1 \geqslant \sigma_1(\Lambda')$. This completes the proof of (11). Thus, a_2 is well-defined.

With these definitions in place, define q_{n-2} from (7), i.e., let

$$q_{n-2}(\lambda) := -\frac{q_n(\lambda) - (\lambda - a_1)q_{n-1}(\lambda)}{a_2^2}.$$

Note that q_{n-2} is a monic polynomial and is odd or even (precisely, it has the same parity as its leading term).

Now show that the roots of q_{n-1} interlace those of q_n and the roots of q_{n-2} interlace those of q_{n-1}. Note that the polynomials $q_n(\lambda)$ and $(-1)^{n-1}q_n(-\lambda)$ have the same sign on the intervals

$$(-|\lambda_1|, -|\lambda_2|),\ (-|\lambda_3|, -|\lambda_4|), \ldots,\ (|\lambda_2|, |\lambda_1|).$$

Moreover, the sequence of these signs is alternating. The polynomial q_{n-1} defined by (10) therefore has exactly $n-1$ real zeros, each of them between two consecutive zeros of q_n. The implication for the root interlacing of q_{n-2} and q_{n-1} is immediate and is a standard argument on orthogonal polynomials (cf. [1, Section 5.4]): Due to the root interlacing of q_{n-1} and q_n and due to (7), the values of q_{n-2} at the zeros of q_{n-1} form an alternating sequence. Therefore, the roots of q_{n-2} interlace those of q_{n-1}.

The rest of the argument is quite straightforward. With q_{n-j} and q_{n-j-1} defined, one defines q_{n-j-2} from (8) making sure that a_{j+2}^2 is indeed positive, for each $j = 1, \ldots, n-2$. The resulting monic polynomials will have alternating parities and interlacing roots. The quantity a_{j+2}^2 is to be set equal to the difference between the second elementary symmetric function σ_2 of the roots of q_{n-j-1} and the second elementary symmetric function of the roots of q_{n-j}. With a slight abuse of notation, this may be denoted by

$$a_{j+2}^2 = \sigma_2(q_{n-j-1}) - \sigma_2(q_{n-j}).$$

The roots of either polynomial are symmetric about 0, therefore, the corresponding second elementary symmetric function is simply

$$(-1) \cdot \text{ the sum of squares of all positive roots.}$$

By the interlacing property, the sum of squares for q_{n-j} exceeds that for q_{n-j-1}, hence $\sigma_2(q_{n-j-1}) - \sigma_2(q_{n-j}) > 0$ and hence a_{j+2} is well-defined.

The argument also shows the uniqueness of the realizing matrix (6), therefore the uniqueness of the realizing matrix (1), provided, of course, that a_j are chosen to be positive. □

The following corollary was established in the course of the above proof.

Corollary 2. *A real n-tuple Λ can be realized as the spectrum of a Jacobi matrix* (6) *if and only if $\Lambda = (\lambda_1, \ldots, \lambda_n)$ where*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0.$$

The realizing matrix is necessarily unique.

Finally, another simple consequence of Theorem 1 is the following result.

Corollary 3. *Let $\mathscr{M}$ be a real positive n-tuple. Then there exists a Jacobi matrix that realizes $\mathscr{M}$ as its spectrum and has a symmetric anti-bidiagonal square root of the form* (1) *with all a_j positive.*

Proof. Let the elements of $\mathscr{M}$ be ordered $\mu_1 > \mu_2 > \cdots > \mu_n (> 0)$. Define

$$\lambda_j := (-1)^{j-1}\sqrt{\mu_j}, \quad j = 1, \ldots, n, \qquad \Lambda := (\lambda_j : j = 1, \ldots, n).$$

Then

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0.$$

By Theorem 1, there exists a symmetric anti-bidiagonal matrix A with spectrum $\sigma(A) = \Lambda$. But then $B := A^2$ is a Jacobi matrix with spectrum $\mathscr{M}$. $\square$

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