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Connectivity of Kronecker products by K_2

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ABSTRACT

Let $\kappa(G)$ be the connectivity of G. The Kronecker product $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. In this paper, we prove that $\kappa(G \times K_2) = \min\{2\kappa(G), \min\{|X| + 2|Y|\}\}$, where the second minimum is taken over all disjoint sets $X, Y \subseteq V(G)$ satisfying (1) $G - (X \cup Y)$ has a bipartite component C, and (2) $G[V(C) \cup \{x\}]$ is also bipartite for any $x \in X$.

1. Introduction

Throughout this paper, a graph G always means a finite undirected graph without loops or multiple edges. It is well known that a network is often modeled as a graph and the classical measure of the reliability is the connectivity and the edge connectivity.

The connectivity of a simple graph $G = (V(G), E(G))$, denoted by $\kappa(G)$, is the smallest number of vertices whose removal from G results in a disconnected or trivial graph. A set $S \subseteq V(G)$ is a separating set of G, if either $G - S$ is disconnected or has only one vertex. Let G_1 and G_2 be two graphs, the Kronecker product $G_1 \times G_2$ is the graph defined on the Cartesian product of vertex sets of the factors, with two vertices (u_1, v_1) and (u_2, v_2) adjacent if and only if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. This product is one of the four standard graph products [1] and is known under many different names, for instance as the direct product, the cross product and conjunction.

Recently, Brešar and Špacapan [2] obtained an upper bound and a lower bound on the edge connectivity of Kronecker products with some exceptions; they also obtained several upper bounds on the vertex connectivity of the Kronecker product of graphs. Mamut and Vumar [3] determined the connectivity of Kronecker product of two complete graphs. Guji and Vumar [4] obtained the connectivity of Kronecker product of a bipartite graph and a complete graph. These two results are generalized in [5], where the author proved a formula for the connectivity of Kronecker product of an arbitrary graph and a complete graph of order ≥ 3, which was conjectured in [4]. We mention that a different proof of the same result can be found in [6]. For the left case $G \times K_2$, Yang [7] determined an explicit formula for its edge connectivity. Bottreau and Métivier [8] derived a criterion for the existence of a cut vertex of $G \times K_2$; see also [9]. In this paper, based on a similar argument of [6], we determine a formula for $\kappa(G \times K_2)$. For more study on the connectivity of Kronecker product graphs, we refer to [10,11].

Let $X, Y \subseteq V(G)$ be two disjoint sets with $V(G) - (X \cup Y) \neq \emptyset$. We shall call (X, Y) a *b-pair* of G if it satisfies:

(1) $G - (X \cup Y)$ has a bipartite component C, and
(2) $G[V(C) \cup \{x\}]$ is also bipartite for any $x \in X$.

Fig. 1 shows two examples, where (X, Y) and (X', Y') are b-pairs of the two graphs G_1 and G_2, respectively.

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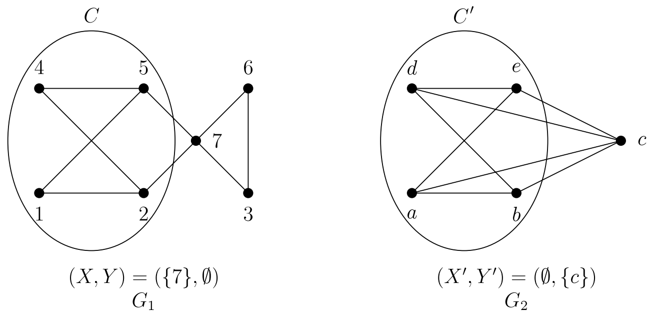

Fig. 1. Two graphs and their b-pairs.

Denote $b(G) = \min\{|X| + 2|Y| : (X, Y) \text{ is a } b\text{-pair of } G\}$. We note that $b(G_1) = 1$ and $b(G_2) = 2$ in Fig. 1. Our main result is the following.

Theorem 1.1. $\kappa(G \times K_2) = \min\{2\kappa(G), b(G)\}$.

We end this section by giving some useful properties of $b(G)$. Let $v \in V(G)$, we use $N(v)$, $d(v)$ and $\delta(G)$ to denote the neighbor set of v, the degree of v, and the minimum degree of G, respectively.

Lemma 1.1. *Let $m = |G| \geq 2$ and u be any vertex of G. Then*

(1) $b(G) = 0$, *if G is bipartite.*
(2) $b(G) \leq \delta(G)$.
(3) $b(G) \leq b(G - u) + 2$.

Proof. Part (1) is clear since $(\emptyset, \emptyset)$ is a b-pair of any bipartite graph by the definition of b-pairs. For each $v \in V(G)$, $(N(v), \emptyset)$ is a b-pair of G. (Take the isolated vertex v in $G - N(v)$ as the bipartite component C.) Therefore $b(G) \leq d(v)$ and part (2) is verified. Similarly, let (X', Y') be any b-pair of $G - u$. It is straightforward to show that $(X', Y' \cup \{u\})$ is a b-pair of G. Therefore, $b(G) \leq |X'| + 2|Y'| + 2$ and part (3) is verified. □

2. Proof of the main result

We first recall some basic results on the connectivity of Kronecker product of graphs [12]; see also [1].

Lemma 2.1. *The Kronecker product of two nontrivial graphs is connected if and only if both factors are connected and at least one factor is nonbipartite. In particular, $G \times K_2$ is connected if and only if G is a connected nonbipartite graph.*

Lemma 2.2. *Let G be a connected bipartite graph with bipartition (P, Q) and $V(K_2) = \{a, b\}$. Then $G \times K_2$ has exactly two connected components isomorphic to G, with bipartitions $(P \times \{a\}, Q \times \{b\})$ and $(P \times \{b\}, Q \times \{a\})$, respectively.*

From Lemmas 1.1(1) and 2.1, we only need to prove Theorem 1.1 for connected and nonbipartite graphs. For each $u \in V(G)$, set $S_u = \{u\} \times V(K_2) = \{(u, a), (u, b)\}$. Let $S \subseteq V(G \times K_2)$ satisfy the following two assumptions.

Assumption 1. $|S| < \min\{2\kappa(G), b(G)\}$, and

Assumption 2. $S'_u := S_u - S \neq \emptyset$ for each $u \in V(G)$.

Let G^* be the graph whose vertices are the classes S'_u for all $u \in V(G)$ and in which two different vertices S'_u and S'_v are adjacent if $G \times K_2 - S$ contains an $(S'_u - S'_v)$ edge, that is, an edge with one end in S'_u and the other one in S'_v.

Under the two assumptions on $S \subseteq V(G \times K_2)$, the connectedness of $G \times K_2 - S$ is verified by the following two lemmas.

Lemma 2.3. *If G is a connected nonbipartite graph, then G^* is connected.*

Proof. Suppose to the contrary that G^* is disconnected. Then the vertices of G^* can be partitioned into two nonempty parts, U^* and V^*, such that there are no $(U^* - V^*)$ edges. Let $U = \{u \in V(G) : S'_u \in U^*\}$, $V = \{v \in V(G) : S'_v \in V^*\}$ and Z be the collection of ends of all $(U - V)$ edges. Let $Z^* = \{S'_u : u \in V(G), |S'_u| = 1\}$. For any $u \in Z$, there exists an edge $uv \in E(U, V)$. It follows that both S'_u and S'_v contain exactly one element, since otherwise $G \times K_2 - S$ contains an $(S'_u - S'_v)$ edge, i.e., $S'_u S'_v \in E(G^*)$, which is contrary to the fact that there are no $(U^* - V^*)$ edges. Therefore, $S'_u \in Z^*$ and we have $|Z| \leq |Z^*|$ by the arbitrariness of u in Z.

Case 1: Either $U \subseteq Z$ or $V \subseteq Z$. We may assume $U \subseteq Z$. Let u be any vertex in U, then $d(u) \leq |Z| - 1$, and hence $\delta(G) \leq |Z| - 1$. Therefore, by Lemma 1.1(2), we have $|S| = |Z^*| \geq |Z| > \delta(G) \geq b(G)$, a contradiction.

Case 2: $U \nsubseteq Z$ and $V \nsubseteq Z$. Either of $U \cap Z$ and $V \cap Z$ is a separating set of G. Therefore, $\kappa(G) \leq \min\{|U \cap Z|, |V \cap Z|\} \leq |Z|/2$. Similarly, we have $|S| = |Z^*| \geq |Z| \geq 2\kappa(G)$, again a contradiction. □

Lemma 2.4. *Any vertex S'_w of G^*, as a subset of $V(G \times K_2 - S)$ is contained in the vertex set of some component of $G \times K_2 - S$.*

Proof. If $|S'_w| = 1$, then the assertion holds trivially. Now assume $|S'_w| = 2$. Let $U = \{u \in V(G) : |S'_u| = 2\}$, $V = \{v \in V(G) : |S'_v| = 1\}$ be the partitions of $V(G)$ and C the component of $G - V$ containing $w \in U$.

Since $|V| = |S| < b(G)$ by Assumption 1, it follows that $(V, \emptyset)$ is not a b-pair of G. Note $S'_w \subseteq V(C \times K_2)$. We may assume that the component C containing w is bipartite, since otherwise $C \times K_2$ is connected by Lemma 2.1 and hence the result follows. Therefore, by the definition of b-pairs, there exists a vertex $v \in V$ such that $G[V(C) \cup \{v\}]$ is nonbipartite.

Let (P, Q) be the bipartition of C and $V(K_2) = \{a, b\}$. Then, by Lemma 2.2, $C \times K_2$ has exactly two components C_1 and C_2 isomorphic to C, with bipartitions $(P \times \{a\}, Q \times \{b\})$ and $(P \times \{b\}, Q \times \{a\})$, respectively. The nonbipartiteness of $G[V(C) \cup \{v\}]$ implies that v is a common neighbor of P and Q. By symmetry, we may assume $S'_v = \{(v, a)\}$. It is easy to see that the subgraph induced by $V(C \times K_2) \cup S'_v$ is connected since (v, a) is a common neighbor of C_1 and C_2. □

Proof of Theorem 1.1. We apply induction on $m = |V(G)|$. It trivially holds when $m = 1$. We therefore assume $m \geq 2$ and the result holds for all graphs of order $m - 1$.

Let S_0 be a minimum separating set of G and $S = S_0 \times V(K_2) = \{(u, a), (u, b) : u \in S_0\}$. Then $G \times K_2 - S \cong (G - S_0) \times K_2$ is disconnected by Lemma 2.1. Therefore, $\kappa(G \times K_2) \leq 2\kappa(G)$.

Let (X, Y) be a b-pair of G with $|X| + 2|Y| = b(G)$. Let C be a bipartite component of $G - (X \cup Y)$ with bipartition (P, Q) such that $G[V(C) \cup \{x\}]$ is also bipartite for any $x \in X$. Let C_1 and C_2 be the two components of $C \times K_2$ with bipartitions $(P \times \{a\}, Q \times \{b\})$ and $(P \times \{b\}, Q \times \{a\})$, respectively. Define an injection $\varphi : X \to V(G \times K_2)$ as follows:

$$\varphi(x) = \begin{cases} (x, b) & \text{if } x \text{ has a neighbor in } P, \\ (x, a) & \text{otherwise.} \end{cases}$$

Let $S' = \varphi(X)$ and $S'' = \{(u, a), (u, b) : u \in Y\}$. Then $S' \cup S''$ is a separating set since C_1 is a component of $G \times K_2 - (S' \cup S'')$, which implies $\kappa(G \times K_2) \leq |S' \cup S''| = |X| + 2|Y| = b(G)$.

To show the reversed inequality, we may assume G is a connected nonbipartite graph. Let $S \subseteq V(G \times K_2)$ satisfy Assumption 1, i.e., $|S| < \min\{2\kappa(G), b(G)\}$.

Case 1: S satisfies Assumption 2. It follows by Lemmas 2.3 and 2.4 that $G \times K_2 - S$ is connected.

Case 2: S does not satisfy Assumption 2, i.e., there exists a vertex $u \in V(G)$ such that $S_u = \{(u, a), (u, b)\} \subseteq S$. Therefore,

$$\begin{aligned} |S - S_u| &= |S| - 2 \\ &< \min\{2\kappa(G), b(G)\} - 2 \\ &= \min\{2(\kappa(G) - 1), b(G) - 2\} \\ &\leq \min\{2\kappa(G - u), b(G - u)\}, \end{aligned}$$

where the last inequality above follows from Lemma 1.1(3).

By the induction assumption,

$$\kappa((G - u) \times K_2) = \min\{2\kappa(G - u), b(G - u)\}.$$

Hence, $(G - u) \times K_2 - (S - S_u)$ is connected. It follows by isomorphism that $G \times K_2 - S$ is connected.

Either of the two cases implies that $(G \times K_2 - S)$ is connected. Thus, $\kappa(G \times K_2) \geq \min\{2\kappa(G), b(G)\}$.

The proof of the theorem is complete by induction. □

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