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Discrete Mathematics 275 (2004) 67–85
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Non-central generalized q-factorial coefficients and q-Stirling numbers

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Received 2 April 2002; received in revised form 20 February 2003; accepted 10 March 2003

Abstract

The q^s-differences of the non-central generalized q-factorials of t of order n, scale parameter s and non-centrality parameter r, at $t=0$, are thoroughly examined. These numbers for $s \to 0$ and $s \to \infty$ converge to the non-central q-Stirling numbers of the first and the second kind, respectively. Explicit expressions, recurrence relations, generating functions and other properties of these q-numbers are derived. Further, a sequence of Bernoulli trials is considered in which the conditional probability of success at the nth trial, given that k successes occur before that trial, varies geometrically with n and k. Then, the probability functions of the number of successes in n trials and the number of trials until the occurrence of the kth success are deduced in terms of the q^s-differences of the non-central generalized q-factorials of t of order n, scale parameter s and non-centrality parameter r.

MSC: Primary 05A30; Secondary 60J10

Keywords: q-Stirling numbers; Generalized q-factorial coefficients; q-distributions

1. Introduction

Carlitz [1,2] introduced the q-Stirling numbers of the second kind in connection with an enumeration problem in abelian groups and explored some of their properties. In the second paper he found it convenient, for some purposes, to generalize these numbers; he introduced what in this paper we call the non-central q-Stirling numbers of the second kind. Gould [14] studied the q-Stirling numbers of the first and second kind, which were defined as sums of all k-factor products that are formed from the first n q-natural numbers, without and with repeated factors, respectively. Milne [17,18] expressed the

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0012-365X/$ - see front matter
doi:10.1016/S0012-365X(03)00099-2

number of certain maps of $\{1, 2, \ldots, n\}$ into the set of all lines in a vector space over a finite field of q elements, q being a power of a prime, in terms of the q-Stirling numbers of the second kind. He also showed that these numbers could be viewed as the generating function of an inversion statistic on partitions. Garsia and Remmel [11] expressed the q-Stirling numbers of the second kind as q-rook numbers for a triangular Ferrers board, providing another interesting combinatorial interpretation. The signless (absolute) q-Lah numbers (or q-Laguerre numbers in the terminology of Hahn [15] and Garsia and Remmel [10]) are the q-rook numbers for a rectangular Ferrers board. Charalambides [3] studied the kth q-difference of the generalized q-factorial of order n and increment a, which for $a = 1$ reduces to the q-Lah number. Wachs and White [19] introduced the p, q-Stirling numbers of the second kind as a generating function of the joint distribution of inversion and non-inversion numbers. Leroux [16] and De Medicis and Leroux [7] further studied these numbers, along with the p, q-Stirling numbers of the first kind. Further, a generalization of the p, q-Stirling numbers of the first and second kind, inspired from their interpretation in terms of the $0-1$ tableaux, is thoroughly examined by De Medicis and Leroux [8].

Recently, Crippa et al. [6] considered a sequence of Bernoulli trials with $\lambda_{n,k}$ the conditional probability of success at the nth trial, given that k successes occur before that trial. Then, the probability function of the number of successes up to the nth trial was expressed in terms of the q-Stirling numbers of the first kind if $\lambda_{n,k} = q^n$ and in terms of the q-Stirling numbers of the second kind if $\lambda_{n,k} = q^k$. A graph theoretical interpretation of these q-distributions was provided. Also, the probability function of the value of the counting register in the approximate counting algorithm, derived by Flajolet [9], is merely the second of these q-distributions. Crippa and Simon [5] examined the distribution of the number of successes up to the nth trial for $\lambda_{n,k} = q^{an+bk+c}$, with a, b and c such that $0 \leqslant \lambda_{n,k} \leqslant 1$ for $k = 0, 1, \ldots, n, n = 0, 1, \ldots$. The probability function of this distribution was expressed as an alternate sum of products of q-binomial coefficients. Further, it was shown that the probability of k successes in n trials with $\lambda_{n,k} = 1 - q^{(a+b)n-bk+c}$ equals the probability of $n - k + 1$ successes in n trials with $\lambda_{n,k} = q^{an+bk+c}$. This alternate sum constitutes a generalization of both q-Stirling numbers of the first and second kind to which it reduces when $a = 1$, $b = c = 0$ and $b = 1$, $a = c = 0$, respectively. A thorough study of its properties is thus justified and useful.

In the present paper the coefficient $C_q(n, k; s, r)$ of kth order q^s-factorial of t in the expansion of the non-central generalized q-factorial of t of order n, scale parameter s and non-centrality parameter r is thoroughly investigated. The q-distributions studied by Crippa and Simon [5] are expressed in terms of the coefficients $|C_{q^{-a}}(n, k; -s, -r)| = [-1]^n_{q^a} C_{q^{-a}}(n, k; -s, -r)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots,$ with $s = b/a$ and $r = c/a$. In addition, the distribution of the number of trials until the occurrence of the kth success is also expressed in terms of these coefficients.

2. Preliminaries, definitions and notation

Let $0 < q < 1$, x a real number and k a positive integer. The number $[x]_q = (1 - q^x)/(1 - q)$ is called *q-real number* and in particular the number $[k]_q$ is called *q-positive integer*. The kth order factorial of the q-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k},$$

is called *q-factorial of x of order k*. In particular, $[k]_q! = [1]_q[2]_q \cdots [k]_q$ is called *q-factorial*. The *q-binomial coefficient* is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q!} = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Note that

$$\lim_{q\to 1} \begin{bmatrix} x \\ k \end{bmatrix}_q = \binom{x}{k}.$$

The general *q-binomial* and the *negative q-binomial formulae* may be expressed as

$$\prod_{i=1}^{\infty} \frac{1+tq^{i-1}}{1+tq^{x+i-1}} = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \begin{bmatrix} x \\ k \end{bmatrix}_q t^k, \quad |t|<1,\ 0<q<1 \tag{2.1}$$

and

$$\prod_{i=1}^{\infty} \frac{1-tq^{x+i-1}}{1-tq^{i-1}} = \sum_{k=0}^{\infty} \begin{bmatrix} x+k-1 \\ k \end{bmatrix}_q t^k, \quad |t|<1,\ 0<q<1, \tag{2.2}$$

respectively. In particular, for $x = n$ a positive integer, these formulae reduce to

$$\prod_{i=1}^{n} (u+tq^{i-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k u^{n-k}, \quad 0<q<1 \tag{2.3}$$

and

$$\prod_{i=1}^{n} (1-tq^{i-1})^{-1} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k, \quad |t|<1,\ 0<q<1, \tag{2.4}$$

respectively. In general, the transition from a formula to its q-analogue is not unique. Thus, another useful *q-binomial formula* is the following

$$q^{nt} = \sum_{k=0}^{n} (-1)^k (1-q)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [t]_{k,q}. \tag{2.5}$$

Also, useful are the following two *q-exponential functions*:

$$e_q(t) = \prod_{i=1}^{\infty} (1-(1-q)q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!}, \quad |t| < 1/(1-q), \tag{2.6}$$

$$E_q(t) = \prod_{i=1}^{\infty} (1+(1-q)q^{i-1}t) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{t^k}{[k]_q!}, \quad -\infty < t < \infty, \tag{2.7}$$

with $e_q(t)E_q(-t)=1$. The *q-Vandermonde's formula* may be expressed as

$$[u+t]_{n,q}=\sum_{k=0}^{n} q^{-k(n-k-u)} \begin{bmatrix} n \\ k \end{bmatrix}_q [u]_{n-k,q}[t]_{k,q}. \tag{2.8}$$

The *q-Newton expansion* of a polynomial $f_n(t)$ in q^t, of degree less than or equal to n, into a polynomial of q-factorials of t is given by

$$f_n(t)=\sum_{k=0}^{n}\frac{1}{[k]_q!}[\Delta_q^k f_n(t)]_{t=0}[t]_{k,q}, \tag{2.9}$$

where Δ_q is the q-difference operator defined, in terms of the usual shift operator E, by

$$\Delta_q^k=\prod_{i=1}^{k}(E-q^{i-1}), \quad k=1,2,\ldots\ . \tag{2.10}$$

The non-central q-factorial of t of order n and non-centrality parameter r, $[t-r]_{n,q}$, upon using the relation $[t-r-j]_q=q^{-r-j}([t]_q-[r+j]_q)$, $j=0,1,\ldots$, is expressed as

$$[t-r]_{n,q}=q^{-\binom{n}{2}-rn}([t]_q-[r]_q)([t]_q-[r+1]_q)\cdots([t]_q-[r+n-1]_q).$$

This is a polynomial of the q-number $[t]_q$ of degree n. Executing the multiplications and arranging the terms in ascending order of powers of $[t]_q$ we get

$$[t-r]_{n,q}=q^{-\binom{n}{2}-rn}\sum_{k=0}^{n}s_q(n,k;r)[t]_q^k, \quad n=0,1,\ldots\ . \tag{2.11}$$

Inversely, the nth power of the q-number $[t]_q$ may be expressed in the form of a polynomial of non-central q-factorials of t. Specifically

$$[t]_q^n=\sum_{k=0}^{n}q^{\binom{k}{2}+rk}S_q(n,k;r)[t-r]_{k,q}, \quad n=0,1,\ldots,$$

or equivalently

$$[t+r]_q^n=\sum_{k=0}^{n}q^{\binom{k}{2}+rk}S_q(n,k;r)[t]_{k,q}, \quad n=0,1,\ldots\ . \tag{2.12}$$

Note that the expansion of the non-central ascending q-factorial of t of order n and non-centrality parameter r,

$$\begin{aligned}[t+r+n-1]_{n,q}&=[t+r]_q[t+r+1]_q\cdots[t+r+n-1]_q\\&=[-1]_q^n[-t-r]_{n,q^{-1}},\end{aligned}$$

into a polynomial of the q-number $[t]_q$ is deduced from (2.11) as

$$[t+r+n-1]_{n,q}=q^{\binom{n}{2}+rn}\sum_{k=0}^{n}|s_{q^{-1}}(n,k;r)|[t]_q^k, \tag{2.13}$$

where $|s_{q^{-1}}(n,k;r)| = [-1]_q^n s_{q^{-1}}(n,k;r)$. More generally, the non-central generalized q-factorial of t of order n, scale parameter s and non-centrality parameter r,

$$[st+r]_{n,q} = [st+r]_q[st+r-1]_q \cdots [st+r-n+1]_q,$$

may be expressed as a polynomial of q^s-factorials of t as

$$[st+r]_{n,q} = q^{-\binom{n}{2}+rn}\sum_{k=0}^{n} q^{s\binom{k}{2}} C_q(n,k;s,r)[t]_{k,q^s}. \tag{2.14}$$

Further, the expansion of the non-central ascending generalized q-factorial of t of order n, scale parameter s and non-centrality parameter r,

$$\begin{aligned}[st+r+n-1]_{n,q} &= [st+r]_q[st+r+1]_q \cdots [st+r+n-1]_q \\ &= [-1]_q^n[-st-r]_{n,q^{-1}},\end{aligned}$$

into a polynomial of q^s-factorials of t may be deduced from (2.14) as

$$[st+r+n-1]_{n,q} = q^{\binom{n}{2}+rn}\sum_{k=0}^{n} q^{s\binom{k}{2}} |C_{q^{-1}}(n,k;-s,-r)[t]_{k,q^s}, \tag{2.15}$$

where $|C_{q^{-1}}(n,k;-s,-r)| = [-1]_q^n C_{q^{-1}}(n,k;-s,-r)$. Also, in the particular case of $s=-1$, upon replacing q by q^{-1} and introducing the coefficient $L_q(n,k;r) = C_{q^{-1}}(n,k;-1,r)$, we get the expression

$$[-(t-r)]_{n,q^{-1}} = q^{\binom{n}{2}-rn}\sum_{k=0}^{n} q^{\binom{k}{2}} L_q(n,k;r)[t]_{k,q}. \tag{2.16}$$

Further, since $[-(t-r)]_{n,q^{-1}} = [t-r+n-1]_{n,q}/[-1]_q^n$ and setting $|L_q(n,k;r)| = [-1]_q^n L_q(n,k;r)$, we find

$$[t-r+n-1]_{n,q} = q^{\binom{n}{2}-rn}\sum_{k=0}^{n} q^{\binom{k}{2}} |L_q(n,k;r)|[t]_{k,q}. \tag{2.17}$$

Note that, on using (2.9) with $f_n(t) = [t+r]_{n,q}$, it follows from (2.13) that

$$S_q(n,k;r) = q^{-\binom{k}{2}-rk}\left[\frac{1}{[k]_q!}\Delta_q^k[t+r]_q^n\right]_{t=0}. \tag{2.18}$$

Similarly,

$$C_q(n,k;s,r) = q^{\binom{n}{2}-s\binom{k}{2}-rn}\left[\frac{1}{[k]_{q^s}!}\Delta_{q^s}^k[st+r]_{n,q}\right]_{t=0}. \tag{2.19}$$

The coefficients $s_q(n,k;r)$ and $S_q(n,k;r)$ of expansions (2.11) and (2.12) are called *non-central q-Stirling numbers of the first and second kind*, respectively. The coefficients $C_q(n,k;s,r)$ of expansion (2.14) may be called *non-central generalized* *q-factorial coefficients*. In particular, the coefficients $L_q(n,k;r)$ and $|L_q(n,k;r)|$ of expansions (2.16) and (2.17) are called *non-central q-Lah* and *signless non-central q-Lah numbers*, respectively.

Clearly, from (2.11), (2.12) and (2.14) it follows that

$$s_q(n,k;r) = S_q(n,k;r) = C_q(n,k;s,r) = 0, \quad k > n,$$

$$s_q(0,0;r) = S_q(0,0;r) = C_q(0,0;s,r) = 1.$$

Further, from (2.13),

$$\sum_{k=0}^{n} |s_{q^{-1}}(n,k;r)|[t]_q^k$$
$$=([t]_q + q^{-1}[r]_{q^{-1}})([t]_q + q^{-1}[r+1]_{q^{-1}})\cdots([t]_q + q^{-1}[r+n-1]_{q^{-1}}),$$

it follows that

$$|s_{q^{-1}}(n,k;r)| = q^{-(n-k)} \sum [r+i_1]_{q^{-1}}[r+i_2]_{q^{-1}}\cdots[r+i_{n-k}]_{q^{-1}}$$

and so

$$|s_q(n,k;r)| = q^{n-k} \sum [r+i_1]_q[r+i_2]_q\cdots[r+i_{n-k}]_q,$$

where the summation is extended over all $(n-k)$-combinations $\{i_1, i_2, \ldots, i_{n-k}\}$ of the n indices $\{0, 1, \ldots, n-1\}$. Hence, for r a non-negative integer, the numbers $|s_q(n,k;r)|$, $k=0,1,\ldots,n$, $n=0,1,\ldots$, which may be called *signless non-central q-Stirling numbers of the first kind*, are non-negative q-integers. Also, expanding the ascending q-factorial $[t-r+n-1]_{n,q}$ into q-factorials of t, by the aid of the q-Vandermonde's formula, it follows that

$$[t-r+n-1]_{n,q} = \sum_{k=0}^{n} q^{k(k-1)-kr} \begin{bmatrix} n \\ k \end{bmatrix}_q [n-r-1]_{n-k,q}[t]_{k,q}$$

and so by (2.17),

$$|L_q(n,k;r)| = q^{-\binom{n}{2}+\binom{k}{2}+r(n-k)} \frac{[n]_q!}{[k]_q!} \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q. \tag{2.20}$$

Note that for $r=0$ the non-central q-Stirling numbers of the first and second kind reduce to the usual (central) q-Stirling numbers of the first and second kind, respectively, $s_q(n,k;0) = s_q(n,k)$, $S_q(n,k;0) = S_q(n,k)$. Similarly, for $r=0$ the non-central generalized q-factorial coefficients and in particular the non-central q-Lah numbers reduce to the usual (central) generalized q-factorial coefficients and the usual (central) q-Lah numbers, respectively, $C_q(n,k;s,0) = C_q(n,k;s)$, $L_q(n,k;0) = L_q(n,k)$. The generalized q-factorial coefficients $C_q(n,k;s)$ were studied in Charalambides [3], while the q-Lah numbers $L_q(n,k)$ appeared in Hahn [15] and discussed in Garsia and Remmel [10] as q-Laguerre numbers. For $r \neq 0$ the non-central q-Stirling numbers of the first kind may be expressed in terms of the usual q-Stirling numbers of the first kind. Specifically, expanding $[t-r]_{n,q}$ into powers of $[t-r]_q = q^{-r}([t]_q - [r]_q)$, by the aid of the usual

q-Stirling numbers of the first kind, and then expanding the powers of $[t]_q - [r]_q$ into powers of $[t]_q$, by the aid of Newton's binomial formula, we deduce the expression

$$s_q(n,k;r) = \sum_{j=k}^{n} (-1)^{j-k} q^{r(n-j)} \binom{j}{k} [r]_q^{j-k} s_q(n,j).$$

Also, expanding $[t-r]_{n,q}$ into q-factorials of t, by the aid of Vandermonde's formula, and then expanding the factorials of t into powers of t, by the aid of the usual q-Stirling numbers of the first kind, we conclude the expression

$$s_q(n,k;r) = \sum_{j=k}^{n} q^{\binom{n-j}{2}+r(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q [-r]_{n-j,q} s_q(j,k).$$

Similarly

$$S_q(n,k;r) = \sum_{j=k}^{n} q^{\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q [r]_{j-k,q} S_q(n,j)$$

and

$$S_q(n,k;r) = \sum_{j=k}^{n} q^{r(j-k)} \binom{n}{j} [r]_q^{n-j} S_q(j,k).$$

Also

$$C_q(n,k;s,r) = q^{-r(n-k)} \sum_{j=k}^{n} q^{s\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_{q^s} [r/s]_{j-k,q^s} C_q(n,j;s)$$

and

$$C_q(n,k;s,r) = \sum_{j=k}^{n} q^{\binom{n-j}{2}-r(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q [r]_{n-j,q} C_q(j,k;s).$$

Moreover, for $q \to 1$ the non-central q-Stirling numbers of the first and second kind converge to the non-central Stirling numbers of the first and second kind, respectively,

$$\lim_{q\to 1} s_q(n,k;r) = s(n,k;r), \qquad \lim_{q\to 1} S_q(n,k;r) = S(n,k;r).$$

Similarly, for $q \to 1$ the non-central generalized q-factorial coefficients and, in particular, the non-central q-Lah numbers converge to the non-central generalized factorial coefficients and the non-central Lah numbers, respectively,

$$\lim_{q\to 1} C_q(n,k;s,r) = C(n,k;s,r), \qquad \lim_{q\to 1} L_q(n,k;r) = L(n,k;r).$$

A review of the basic properties and combinatorial applications of the non-central Stirling numbers and the non-central generalized factorial coefficients is included in

Charalambides [4]. Further, expression (2.14) may be written as

$$[t+r]_{n,q} = q^{-\binom{n}{2}+rn} \sum_{k=0}^{n} q^{s\binom{k}{2}} \{[s]_q^{-k} C_q(n,k;s,r)\}\{[s]_q^k [t/s]_{k,q^s}\}$$

and since $\lim_{s\to 0}[s]_q^k [t/s]_{k,q^s} = [t]_q^k$ it follows, by virtue of (2.11), that

$$\lim_{s\to 0} [s]_q^{-k} C_q(n,k;s,r) = s_q(n,k;-r). \tag{2.21}$$

Similarly, writing (2.14) in the form

$$[s]_{q^{1/s}}^{-n}[s(t+r)]_{n,q^{1/s}} = q^{-\frac{1}{s}\binom{n}{2}} \sum_{k=0}^{n} q^{\binom{k}{2}+rk} \{q^{r(n-k)}[s]_{q^{1/s}}^{-n} C_{q^{1/s}}(n,k;s,rs)\}[t]_{k,q}$$

and since $\lim_{s\to\infty} [s]_{q^{1/s}}^{-n}[s(t+r)]_{n,q^{1/s}} = [t+r]_q^n$ it follows, by virtue of (2.12), that

$$\lim_{s\to\infty} q^{r(n-k)}[s]_{q^{1/s}}^{-n} C_{q^{1/s}}(n,k;s,rs) = S_q(n,k;r). \tag{2.22}$$

3. Explicit expressions and recurrence relations

Explicit expressions and a recurrence relation for the non-central generalized q-factorial coefficients are derived in the following theorems.

Theorem 3.1. *The non-central generalized q-factorial coefficients are given by*

$$C_q(n,k;s,r) = \frac{q^{\binom{n}{2}-s\binom{k}{2}-rn}}{[k]_{q^s}!} \sum_{j=0}^{k} (-1)^{k-j} q^{s\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^s} [sj+r]_{n,q}. \tag{3.1}$$

Also

$$C_q(n,k;s,r) = \frac{[s]_q^k}{(1-q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{\binom{n-j}{2}-r(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_{q^s}. \tag{3.2}$$

Proof. The q^s-difference operator of order k, $\Delta_{q^s}^k = \prod_{i=1}^{k}(E - q^{s(i-1)})$, on using the q-binomial theorem, is expressed as

$$\Delta_{q^s}^k = \sum_{j=0}^{k} (-1)^{k-j} q^{s\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^s} E^j.$$

Performing this operator on $[st+r]_{n,q}$, (2.19) yields (3.1).

The non-central generalized q-factorial of t of order n, scale parameter s and non-centrality parameter r, $[st+r]_{n,q}$, on using successively the q-binomial formulae (2.3) and (2.5), is expressed as a polynomial of q^s-factorials of t as

$$q^{\binom{n}{2}-rn}[st+r]_{n,q}$$

$$= \frac{1}{(1-q)^n} \prod_{i=1}^{n} (q^{-r+i-1} - q^{st})$$

$$=\frac{1}{(1-q)^n}\sum_{j=0}^{n}(-1)^j q^{\binom{n-j}{2}-r(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}_q (q^{st})^j$$

$$=\sum_{j=0}^{n}(-1)^j q^{\binom{n-j}{2}-r(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{k=0}^{j}(-1)^k \frac{[s]_q^k q^{s\binom{k}{2}}}{(1-q)^{n-k}}\begin{bmatrix} j \\ k \end{bmatrix}_{q^s}[t]_{k,q^s}$$

$$=\sum_{k=0}^{n}\frac{[s]_q^k q^{s\binom{k}{2}}}{(1-q)^{n-k}}\left\{\sum_{j=k}^{n}(-1)^{j-k} q^{\binom{n-j}{2}-r(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_{q^s}\right\}[t]_{k,q^s},$$

yielding (3.2). □

Theorem 3.2. *The non-central generalized q-factorial coefficients satisfy the triangular recurrence relation*

$$C_q(n,k;s,r)=[s]_q C_q(n-1,k-1;s,r) + ([sk]_q-[n-r-1]_q)C_q(n-1,k;s,r), \quad (3.3)$$

for $k=1,2,\ldots,n, n=1,2,\ldots,$ *with initial conditions*

$$C_q(0,0;s,r)=1, \quad C_q(0,k;s,r)=0, \ k>0,$$

$$C_q(n,0;s,r)=q^{\binom{n}{2}-rn}[r]_{n,q}, \quad n>0.$$

Proof. Expanding both members of the recurrence relation $[st+r]_{n,q}=[st+r-n+1]_q[st+r]_{n-1,q}$ into q^s-factorials of t, by the aid of (2.14) and since $[st+r-n+1]_q=q^{-(n-1)+r}([s]_q[t]_{q^s}-[n-r-1]_q)$, we get the relation

$$\sum_{k=0}^{n} q^{s\binom{k}{2}} C_q(n,k;s,r)[t]_{k,q^s}=\sum_{k=0}^{n-1} q^{s\binom{k}{2}} C_q(n-1,k;s,r)[s]_q[t]_{q^s}[t]_{k,q^s} + \sum_{k=0}^{n-1} q^{s\binom{k}{2}}[n-r-1]_q C_q(n-1,k;s,r)[t]_{k,q^s},$$

which, on using the expressions $[t]_{q^s}[t]_{k,q^s}=q^{sk}[t]_{k+1,q^s}+[k]_{q^s}[t]_{k,q^s}$, $[s]_q[k]_{q^s}=[sk]_q$, yields

$$\sum_{k=0}^{n} q^{s\binom{k}{2}} C_q(n,k;s,r)[t]_{k,q^s}=\sum_{k=0}^{n-1} q^{s\binom{k+1}{2}}[s]_q C_q(n-1,k;s,r)[t]_{k+1,q^s} + \sum_{k=0}^{n-1} q^{s\binom{k}{2}}([sk]_q-[n-r-1]_q)C_q(n-1,k;s,r)[t]_{k,q^s}.$$

Equating the coefficients of $[t]_{k,q^s}$ in both sides of the last relation we deduce (3.3). □

Explicit expressions and recurrence relations for the non-central q-Stirling numbers, on using (2.21) and (2.22), are deduced in the following corollaries of Theorems 3.1 and 3.2, respectively.

Corollary 3.1. (*a*) *The non-central q-Stirling numbers of the first kind are given by*

$$s_q(n,k;r)=\frac{1}{(1-q)^{n-k}}\sum_{j=k}^{n}(-1)^{j-k}q^{\binom{n-j}{2}+r(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}_q\binom{j}{k}. \tag{3.4}$$

(*b*) *The non-central q-Stirling numbers of the second kind are given by*

$$S_q(n,k;r)=\frac{1}{[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{j+1}{2}-(r+j)k}\begin{bmatrix} k \\ j \end{bmatrix}_q[r+j]_q^n. \tag{3.5}$$

Also

$$S_q(n,k;r)=\frac{1}{(1-q)^{n-k}}\sum_{j=k}^{n}(-1)^{j-k}q^{r(j-k)}\binom{n}{j}\begin{bmatrix} j \\ k \end{bmatrix}_q. \tag{3.6}$$

Corollary 3.2. (*a*) *The non-central q-Stirling numbers of the first kind satisfy the triangular recurrence relation*

$$s_q(n,k;r)=s_q(n-1,k-1;r)-[n+r-1]_q s_q(n-1,k;r), \tag{3.7}$$

for $k=1,2,\dots,n,\ n=1,2,\dots,$ *with initial conditions*

$$s_q(0,0;r)=1,\ \ s_q(0,k;r)=0,\ \ k>0,\ \ s_q(n,0;r)=q^{\binom{n}{2}+rn}[-r]_{n,q},\ \ n>0.$$

(*b*) *The non-central q-Stirling numbers of the second kind satisfy the triangular recurrence relation*

$$S_q(n,k;r)=S_q(n-1,k-1;r)+[r+k]_q S_q(n-1,k;r), \tag{3.8}$$

for $k=1,2,\dots,n,\ n=1,2,\dots,$ *with initial conditions*

$$S_q(0,0;r)=1,\quad S_q(0,k;r)=0,\ \ k>0,\quad S_q(n,0;r)=[r]_q^n,\ \ n>0.$$

Theorem 3.3. *The non-central generalized q-factorial coefficients are connected with the non-central q-Stirling numbers of the first and second kind by*

$$C_q(n,k;s,s\rho-r)=q^{-s\rho(n-k)}\sum_{j=k}^{n}s_q(n,j;r)S_{q^s}(j,k;\rho)[s]_q^j. \tag{3.9}$$

Proof. Expanding $[s(t+\rho)-r]_{n,q}$ into powers of $[s(t+\rho)]_q=[s]_q[t+\rho]_{q^s}$, by the aid (2.11), and then expanding the powers of $[t+\rho]_{q^s}$ into q^s-factorials of t, by the aid of

(2.12), we get the expression

$$
\begin{aligned}
&[s(t+\rho)-r]_{n,q}\\
&\quad = q^{-\binom{n}{2}-rn}\sum_{j=0}^{n} s_q(n,j;r)[s]_q^j[t+\rho]_{q^s}^j\\
&\quad = q^{-\binom{n}{2}-rn}\sum_{j=0}^{n} s_q(n,j;r)[s]_q^j\sum_{k=0}^{j} q^{s\binom{k}{2}+s\rho k}S_{q^s}(j,k;\rho)[t]_{k,q^s}\\
&\quad = q^{-\binom{n}{2}+(s\rho-r)n}\sum_{k=0}^{n} q^{s\binom{k}{2}}\left\{q^{-s\rho(n-k)}\sum_{j=k}^{n} s_q(n,j;r)S_{q^s}(j,k;\rho)[s]_q^j\right\}[t]_{k,q^s}
\end{aligned}
$$

and since, by (2.14),

$$[s(t+\rho)-r]_{n,q} = q^{-\binom{n}{2}+(s\rho-r)n}\sum_{k=0}^{n} q^{s\binom{k}{2}}C_q(n,k;s,s\rho-r)[t]_{k,q^s},$$

we deduce (3.9). □

Setting in (3.9) $s=1$ and $\rho=r$ and since $C_q(n,k;1,0)=\delta_{n,k}$ we deduce the following corollary.

Corollary 3.3. *The non-central q-Stirling numbers of the first and second kind satisfy the orthogonality relations*

$$\sum_{j=k}^{n} s_q(n,j;r)S_q(j,k;r)=\delta_{n,k}, \qquad \sum_{j=k}^{n} S_q(n,j;r)s_q(j,k;r)=\delta_{n,k}. \tag{3.10}$$

Theorem 3.4. *The non-central generalized q-factorial coefficients satisfy the following relation*

$$C_1(n,k;s_1s_2,r_1+s_1r_2)=\sum_{j=k}^{n} q^{-s_1r_2(n-j)}C_q(n,j;s_1,r_1)C_{q^{s_1}}(j,k;s_2,r_2). \tag{3.11}$$

In particular,

$$\sum_{j=k}^{n} q^{r(n-j)}C_q(n,j;s,r)C_{q^s}(j,k;1/s,-r/s)=\delta_{n,k}. \tag{3.12}$$

Proof. Expanding the generalized q-factorial $[s_1(s_2t+r_2)+r_1]_{n,q}$ into generalized q-factorials $[s_2t+r_2]_{j,q^{s_1}}$, $j=0,1,\ldots,n$ and then expanding these factorials into

q-factorials $[t]_{k,q^{s_1s_2}}$, $k=0,1,\ldots,j$, by the aid of (2.14), we get

$$\begin{aligned}[s_1(s_2t+r_2)+r_1]_{n,q} &= q^{-\binom{n}{2}+r_1n}\sum_{j=0}^{n} q^{s_1\binom{j}{2}} C_q(n,j;s_1,r_1)[s_2t+r_2]_{j,q^{s_1}} \\ &= q^{-\binom{n}{2}+r_1n}\sum_{j=0}^{n} C_q(n,j;s_1,r_1) q^{s_1r_2j}\sum_{k=0}^{j} q^{s_1s_2\binom{k}{2}} C_{q^{s_1}}(j,k;s_2,r_2)[t]_{k,q^{s_1s_2}}\end{aligned}$$

and so

$$[s_1(s_2t+r_2)+r_1]_{n,q} = q^{-\binom{n}{2}+r_1n}\sum_{k=0}^{n} q^{s_1s_2\binom{k}{2}}\left\{\sum_{j=k}^{n} q^{s_1r_2j} C_q(n,j;s_1,r_1) C_{q^{s_1}}(j,k;s_2,r_2)\right\}[t]_{k,q^{s_1s_2}}.$$

Also from (2.14) we find

$$[s_1s_2t+s_1r_2+r_1]_{n,q} = q^{-\binom{n}{2}+(r_1+s_1r_2)n}\sum_{k=0}^{n} q^{s_1s_2\binom{k}{2}} C_q(n,k;s_1s_2,r_1+s_1r_2)[t]_{k,q^{s_1s_2}}.$$

The last two expressions imply (3.11). □

4. Generating functions and other properties

Theorem 4.1. (*a*) *The generating function of the non-central q-Stirling numbers of the second kind* $S_q(n,k;r)$, $n=k,k+1,\ldots,$ *for fixed* k, *is given by*

$$\phi_{k;q}(u) = \sum_{n=k}^{\infty} S_q(n,k;r)u^n = u^k\prod_{i=0}^{k}(1-[r+i]_q u)^{-1}, \quad u<1/[r+k]_q. \tag{4.1}$$

(*b*) *The non-central q-Stirling numbers of the second kind are given by the sum*

$$S_q(n,k;r) = \sum [r+i_1]_q[r+i_2]_q\cdots[r+i_{n-k}]_q, \tag{4.2}$$

where the summation is extended over all $(n-k)$*-combinations with repetition of the* $k+1$ *non-negative integers* $\{0,1,\ldots,k\}$.

Proof. (a) Note first that the series $\sum_{n=k}^{\infty} a_n$ for $a_n = S_q(n,k;r)u^n$, $u<1/[r+k]_q$, is convergent, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{S_q(n,k;r)}{[k]_q![r+k]_q^n}([r+k]_q u)^n = \frac{1}{[k]_q!}\lim_{n\to\infty}([r+k]_q u)^n = 0.$$

Further, multiplying the triangular recurrence relation of the non-central q-Stirling numbers of the second kind by u^n and summing the resulting expression for $n=k,k+1,\ldots,$ we obtain for the generating function $\phi_{k;q}(u)$ the recurrence relation

$$\phi_{k;q}(u) = u\phi_{k-1;q}(u) + [r+k]_q u\phi_{k;q}(u), \quad k=1,2,\ldots\ .$$

Hence

$$\phi_{k;q}(u) = u(1-[r+k]_q u)^{-1}\phi_{k-1;q}(u), \quad k = 1,2,\ldots .$$

Applying successively this recurrence and since

$$\phi_{0;q}(u) = \sum_{n=k}^{\infty} S_q(n,0;r)u^n = \sum_{n=k}^{\infty} [r]_q^n u^n = (1-[r]_q u)^{-1},$$

we find (4.1).

(b) Expanding each factor in (4.1), by the aid of the geometric series, we get

$$\phi_{k;q}(u) = \sum_{n=k}^{\infty} S_q(n,k;r)u^n = u^k \prod_{i=0}^{k}\left(\sum_{j_i=0}^{\infty} [r+i]_q^{j_i} u^{j_i}\right)$$

$$= \sum_{n=k}^{\infty}\left(\sum [r]_q^{j_0}[r+1]_q^{j_1}\cdots[r+k]_q^{j_k}\right)u^n$$

and so

$$S_q(n,k;r) = \sum [r]_q^{j_0}[r+1]_q^{j_1}\cdots[r+k]_q^{j_k},$$

where the summation is extended over all integers $j_i \geqslant 0$, $i = 0,1,\ldots,k$, such that $j_0 + j_1 + \cdots + j_k = n-k$. Clearly, this expression is equivalent to (4.2). □

Corollary 4.1. *The reciprocal non-central q-factorial* $1/[t-r]_{k+1,q}$ *is expanded into reciprocal q-powers* $1/[t]_q^{n+1}$, $n = k, k+1, \ldots,$ *as*

$$\frac{1}{[t-r]_{k+1,q}} = q^{\binom{k+1}{2}+r(k+1)} \sum_{n=k}^{\infty} S_q(n,k;r)\frac{1}{[t]_q^{n+1}}, \quad t > k+r. \tag{4.3}$$

Inversely, the reciprocal q-power $1/[t]_q^{k+1}$, *is expanded into reciprocal non-central q-factorials* $1/[t-r]_{n+1,q}$, $n = k, k+1, \ldots,$ *as*

$$\frac{1}{[t]_q^{k+1}} = \sum_{n=k}^{\infty} q^{-\binom{n+1}{2}-r(n+1)} s_q(n,k;r)\frac{1}{[t-r]_{n+1,q}}. \tag{4.4}$$

Proof. Setting in (4.1) $u = 1/[t]_q$ and since

$$([t]_q - [r]_q)([t]_q - [r+1]_q)\cdots([t]_q - [r+k]_q) = q^{\binom{k+1}{2}+r(k+1)}[t-r]_{k+1,q}$$

we conclude (4.3). Inverting (4.3), by the aid of the second of (3.10), we get (4.4).

□

A more general expansion in terms of the non-central generalized q-factorial coefficients is derived in the following theorem.

Theorem 4.2. *The reciprocal q-factorial* $1/[t]_{k+1,q^s}$ *is expanded into reciprocal non-central generalized q-factorials* $1/[st+r]_{n+1,q}$, $n = k, k+1, \ldots,$ *as*

$$\frac{1}{[t]_{k+1,q^s}} = q^{s\binom{k+1}{2}} \sum_{n=k}^{\infty} q^{-\binom{n+1}{2}+r(n+1)}[s]_q C_q(n,k;s,r)\frac{1}{[st+r]_{n+1,q}}. \tag{4.5}$$

Proof. Consider the series

$$C_{k;q}(t)=\sum_{n=k}^{\infty}q^{-\binom{n+1}{2}+r(n+1)}C_q(n,k;s,r)\,\frac{1}{[st+r]_{n+1,q}}.$$

Multiplying both sides of the recurrence relation

$$C_q(n,k;s,r)=[s]_qC_q(n-1,k-1;s,r)+([sk]_q-[n-r-1]_q)C_q(n-1,k;s,r)$$

by

$$\frac{q^{-\binom{n}{2}+rn}}{[st+r]_{n,q}}=\frac{q^{-\binom{n+1}{2}+r(n+1)}([st]_q-[n-r]_q)}{[st+r]_{n+1,q}}$$

we find

$$\begin{aligned}
&q^{-\binom{n+1}{2}+r(n+1)}[st]_qC_q(n,k;s,r)\,\frac{1}{[st+r]_{n+1,q}}\\
&\quad-q^{-\binom{n+1}{2}+r(n+1)}[n-r]_qC_q(n,k;s,r)\,\frac{1}{[st+r]_{n+1,q}}\\
&\quad=q^{-\binom{n}{2}+rn}[sk]_qC_q(n-1,k;s,r)\,\frac{1}{[st+r]_{n,q}}\\
&\qquad-q^{-\binom{n}{2}+rn}[n-r-1]_qC_q(n-1,k;s,r)\,\frac{1}{[st+r]_{n,q}}\\
&\qquad+q^{-\binom{n}{2}+rn}[s]_qC_q(n-1,k-1;s,r)\,\frac{q}{[st+r]_{n,q}}.
\end{aligned}$$

Summing for $n=k,k+1,\dots$, we get the recurrence relation

$$C_{k;q}(t)=\frac{q^{-sk}}{[t-k]_{q^s}}C_{k-1;q}(t),\ \ k=1,2,\dots\ .$$

Hence

$$C_{k;q}(t)=C_{0;q}(t)\,\frac{q^{-s\binom{k+1}{2}}}{[t-1]_{k,q^s}}.$$

Since $C_q(n,0;s,r)=q^{\binom{n}{2}-rn}[r]_{n,q}$, $n>0$, we have

$$C_{0;q}(t)=\sum_{n=0}^{\infty}q^{-n+r}\,\frac{[r]_{n,q}}{[st+r]_{n+1,q}}=\frac{1}{[s]_q[t]_{q^s}},$$

whence

$$C_{k;q}(t)=\frac{q^{-\binom{k+1}{2}}}{[s]_q[t]_{k+1,q^s}}$$

and (4.5) is established. □

Bivariate generating functions for the non-central q-Stirling numbers of the first and second kind, analogous to the generating functions for the corresponding usual (central) q-Stirling numbers, derived by Gessel [13] and Garsia and Remmel [12], are deduced in the following theorem.

Theorem 4.3. *Let $s_q(n,k;r)$ and $S_q(n,k;r)$, $k=0,1,\ldots,n$, $n=0,1,\ldots,$ be the non-central q-Stirling numbers of the first and second kind, respectively. Then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} s_q(n,k;r)t^k \frac{u^n}{[n]_q!} = \prod_{i=1}^{\infty} \frac{1+uq^{r+i-1}}{1+(1-(1-q)t)uq^{i-1}} \tag{4.6}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{\binom{k}{2}+rk} S_q(n,k;r)t^k \frac{u^n}{n!} = E_q(-t)\sum_{j=0}^{\infty} e^{[r+j]_q u}\frac{t^j}{[j]_q!}, \tag{4.7}$$

where $E_q(t)=\sum_{k=0}^{\infty} q^{\binom{k}{2}} t^k/[k]_q!$ is a q-exponential function.

Proof. Multiplying both members of (2.11) by $u^n/[n]_q!$ and summing for $n=0,1,\ldots,$ we get the relation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} s_q(n,k;r)[t]_q^k \frac{u^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \begin{bmatrix} t-r \\ n \end{bmatrix}_q (uq^r)^n,$$

which, on using the q-binomial formula (2.1), yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} s_q(n,k;r)[t]_q^k \frac{u^n}{[n]_q!} = \prod_{i=1}^{\infty}\frac{1+uq^{r+i-1}}{1+uq^{t+i-1}}$$

$$= \prod_{i=1}^{\infty}\frac{1+uq^{r+i-1}}{1+(1-(1-q)[t]_q)uq^{i-1}}.$$

Replacing in the last expression $[t]_q$ by t we deduce (4.6). From the explicit expression (3.5) of the q-Stirling numbers of the second kind we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{\binom{k}{2}+rk} S_q(n,k;r)t^k \frac{u^n}{n!}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \frac{e^{[r+j]_q}t^k}{[k-j]_q![j]_q!}$$

$$= \sum_{j=0}^{\infty}\left\{\sum_{k=j}^{\infty} (-1)^{k-j} q^{\binom{k-j}{2}} \frac{t^{k-j}}{[k-j]_q!}\right\} e^{[r+j]_q u}\frac{t^j}{[j]_q!}.$$

Introducing the q-exponential function (2.7), we get (4.7). □

5. Probability distributions

Consider a sequence of Bernoulli trials and assume that the conditional probability of success at the nth trial, given that k successes occur before that trial, varies geometrically with n and k. Specifically, suppose that the probability of success at the $(n+1)$th trial, given that k successes occur up to the nth trial, is given by

$$\lambda_{n,k} = q^{an+bk+c}, \quad k=0,1,\dots,n, \ n=0,1,\dots,$$

with a, b and c such that $0 \leqslant \lambda_{n,k} \leqslant 1$ for $k=0,1,\dots,n$ and $n=0,1,\dots$. The particular case $b=0$, $k=0,1,\dots,n$, corresponds to the assumption that the probability of success at any trial depends only on the number of previous trials, while other particular case $a=0$, $n=0,1,\dots$, corresponds to the assumption that the probability of success at any trial depends only on the number of previous successes.

Theorem 5.1. *Consider a sequence of Bernoulli trials and assume that the probability of success at the* $(n+1)$*th trial, given that k successes occur up to the nth trial, is given by*

$$\lambda_{n,k} = q^{an+bk+c}, \quad k=0,1,\dots,n, \ n=0,1,\dots,$$

with a, b *and* c *such that* $0 \leqslant \lambda_{n,k} \leqslant 1$ *for* $k=0,1,\dots,n$ *and* $n=0,1,\dots$. *Then the probability function* $p_k(n)=P(X_n=k)$, $k=0,1,\dots,n$, *of the number* X_n *of successes up to the nth trial is given by*

$$p_k(n) = q^{a\binom{n}{2}+b\binom{k}{2}+cn}\frac{(1-q^a)^n}{(1-q^b)^k}|C_{q^{-a}}(n,k;-s,-r)|, \quad k=0,1,\dots,n, \tag{5.1}$$

where $|C_{q^{-a}}(n,k;-s,-r)| = [-1]^n_{q^a} C_{q^{-a}}(n,k;-s,-r)$, *with* $s=b/a$ *and* $r=c/a$.

Proof. The probability function $p_k(n)=P(X_n=k)$, $k=0,1,\dots,n$, $n=0,1,\dots$, satisfies the recurrence relation

$$p_k(n) = (1-q^{a(n-1)+bk+c})p_k(n-1) + q^{a(n-1)+b(k-1)+c}p_{k-1}(n-1),$$

for $k=1,2,\dots,n$, $n=1,2,\dots$, with initial conditions

$$p_0(0)=1, \ \ p_0(n) = q^{a\binom{n}{2}+cn}\prod_{i=0}^{n-1}(1-q^{-ai-c}), \ \ n>0, \ \ p_k(0)=0, \ \ k>0.$$

The recurrence relation and its initial conditions suggest considering the following expression of the probability function

$$p_k(n) = q^{a\binom{n}{2}+b\binom{k}{2}+cn}\frac{(1-q^a)^n}{(1-q^b)^k}c_{n,k}, \quad k=0,1,\dots,n, \ n=0,1,\dots,$$

where the sequence $c_{n,k}$, $k=0,1,\dots,n$, $n=0,1,\dots$, is to be determined. Clearly, this sequence satisfies the recurrence relation

$$c_{n,k} = ([-sk]_{q^{-a}} - [n+r-1]_{q^{-a}})c_{n-1,k} + [-s]_{q^{-a}}c_{n-1,k-1},$$

for $k = 1, 2, \ldots, n,\ n = 1, 2, \ldots,$ with initial conditions

$$c_{0,0} = 1,\ c_{n,0} = q^{-a\binom{n}{2}-cn}[-r]_{n,q^{-a}},\ n > 0,\ c_{0,k} = 0,\ k > 0$$

and $s = b/a,\ r = c/a$. Comparing the last recurrence relation with the recurrence relation (3.3) of the non-central generalized q-factorial coefficients, we get $c_{n,k} = C_{q^{-a}}(n, k; -s, -r)$ and since $(1 - q^{-a}) = [-1]_{q^a}(1 - q^a)$, (5.2) is established. □

Remark 5.1. The probability generating function (5.1), where, according to (3.2),

$$|C_{q^{-a}}(n, k; -s, -r)| = q^{-a\binom{n}{2}-cn} \frac{(1-q^b)^k}{(1-q^a)^n} \sum_{j=k}^{n} (-1)^{j-k} q^{a\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b},$$

was deduced by Crippa and Simon [5] from the corresponding probability generating function.

Theorem 5.2. *Consider a sequence of Bernoulli trials and assume that the probability of success at the* $(n+1)$*th trial, given that* k *successes occur up to the* n*th trial, is given by*

$$\lambda_{n,k} = q^{an+bk+c}, \quad k = 0, 1, \ldots, n, n = 0, 1, \ldots,$$

with a, b *and* c *such that* $0 \leqslant \lambda_{n,k} \leqslant 1$ *for* $k = 0, 1, \ldots, n$ *and* $n = 0, 1, \ldots$. *Then the probability function* $q_n(k) = P(W_k = n),\ n = k, k+1, \ldots,$ *of the number* W_k *of trials until the occurrence of the* k*th success is given by*

$$q_n(k) = q^{a\binom{n}{2}+b\binom{k}{2}+cn} \frac{(1-q^a)^{n-1}}{(1-q^b)^{k-1}} |C_{q^{-a}}(n-1, k-1; -s, -r)|, \tag{5.2}$$

for $n = k, k+1, \ldots$, *where* $|C_{q^{-a}}(n, k; -s, -r)| = [-1]_q^n C_{q^{-a}}(n, k; -s, -r)$, *with* $s = b/a$ *and* $r = c/a$.

Proof. Clearly, the probability function $q_n(k) = P(W_k = n),\ n = k, k+1, \ldots,$ is connected with the probability function $p_k(n) = P(X_n = k),\ k = 0, 1, \ldots, n$ by

$$q_n(k) = p_{k-1}(n-1)\lambda_{n-1,k-1},$$

which, by virtue of (5.1), implies (5.2). □

Corollary 5.1. *Consider a sequence of Bernoulli trials with varying success probability. Let* X_n *be the number of successes up to the* n*th trial and* $p_n(n) = P(X_n = k)$, $k = 0, 1, \ldots, n$ *its probability function.*

(a) *If the probability of success at the* $(n+1)$*th trial is given by* $\lambda_{n,k} = q^{n+r}$, $k = 0, 1, \ldots, n,\ n = 0, 1, \ldots,$ *then*

$$p_k(n) = q^{\binom{n}{2}+rn}(1-q)^{n-k}|s_{q^{-1}}(n, k; r)|, \quad k = 0, 1, \ldots, n, \tag{5.3}$$

where $|s_{q^{-1}}(n, k; r)|$ *is the signless non-central* q*-Stirling number of the first kind.*

(b) *If the probability of success at the* $(n+1)$*th trial, given that k successes occur up to the nth trial, is given by* $\lambda_{n,k} = q^{k+r}$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots,$ *then*

$$p_k(n) = q^{\binom{k}{2}+rk}(1-q)^{n-k} S_q(n,k;r), \quad k = 0, 1, \ldots, n, \tag{5.4}$$

where $S_q(n,k;r)$ *is the non-central q-Stirling number of the second kind.*

Corollary 5.2. *Consider a sequence of Bernoulli trials with varying success probability. Let* W_k *be the number of trials until the occurrence of the kth success and* $q_n(k) = P(W_k = n)$, $n = k, k+1, \ldots$ *its probability function.*

(a) *If the probability of success at the* $(n+1)$*th trial is given by* $\lambda_{n,k} = q^{n+r}$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots,$ *then*

$$q_n(k) = q^{\binom{n}{2}+rn}(1-q)^{n-k} |s_{q^{-1}}(n-1, k-1; r)|, \quad n = k, k+1, \ldots, \tag{5.5}$$

where $|s_{q^{-1}}(n,k;r)|$ *is the signless non-central q-Stirling number of the first kind.*

(b) *If the probability of success at the* $(n+1)$*th trial, given that k successes occur up to the nth trial, is given by* $\lambda_{n,k} = q^{k+r}$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots,$ *then*

$$q_n(k) = q^{\binom{k}{2}+rk}(1-q)^{n-k} S_q(n-1, k-1; r), \quad n = k, k+1, \ldots, \tag{5.6}$$

where $S_q(n,k;r)$ *is the non-central q-Stirling number of the second kind.*

Acknowledgements

The author expresses his sincere thanks to both referees for their comments and suggestions towards revising this paper. This research was partially supported by the University of Athens Research Special Account under Grant 70/4/3406.

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