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Ideals of quasi-symmetric functions and super-covariant polynomials for $\mathcal{S}_n$

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Abstract

The aim of this work is to study the quotient ring $\mathbf{R}_n$ of the ring $\mathbb{Q}[x_1, \ldots, x_n]$ over the ideal $\mathcal{J}_n$ generated by non-constant homogeneous quasi-symmetric functions. This article is a sequel of Aval and Bergeron (Proc. Amer. Math. Soc., to appear), in which we investigated the case of infinitely many variables. We prove here that the dimension of $\mathbf{R}_n$ is given by C_n, the nth Catalan number. This is also the dimension of the space $\mathbf{SH}_n$ of super-covariant polynomials, defined as the orthogonal complement of $\mathcal{J}_n$ with respect to a given scalar product. We construct a basis for $\mathbf{R}_n$ whose elements are naturally indexed by Dyck paths. This allows us to understand the Hilbert series of $\mathbf{SH}_n$ in terms of number of Dyck paths with a given number of factors.

1. Introduction

We study, in this paper, a natural analog of the space $\mathbf{H}_n$ of covariant polynomials of $\mathcal{S}_n$. Let X denote the n variables $x_1, \ldots, x_n$ and $\mathbb{Q}[X]$ denote the ring of polynomials in the variables X. Let $\mathcal{I}_n$ denote the ideal of $\mathbb{Q}[X]$ generated by all

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symmetric polynomials with no constant term. That is

$$\mathcal{I}_n = \langle h_k(X),\ k>0 \rangle,$$

where $h_k(X)$ is the kth homogeneous symmetric polynomials in the variables X (cf. [12]). We consider the following scalar product on $\mathbb{Q}[X]$:

$$\langle P, Q \rangle = P(\partial X)Q(X)|_{X=0}, \tag{1.1}$$

where ∂X stands for $\partial x_1, \dots, \partial x_n$ and in the same spirit $X = 0$ stands for $x_1 = \cdots = x_n = 0$. The space $\mathbf{H}_n$ is defined as the orthogonal complement, denoted by $\mathcal{I}_n^\perp$, of the ideal $\mathcal{I}_n$ in $\mathbb{Q}[X]$.

Equivalently (cf. [6, Proposition I.2.3]), covariant polynomials (also known as $\mathcal{S}_n$-harmonic polynomials) can be defined as polynomials P such that $Q(\partial X)P = 0$, for any symmetric polynomial Q with no constant term. Since elements of $\mathbf{H}_n$ satisfy the Laplace equation

$$(\partial x_1^2 + \cdots + \partial x_n^2)P = \Delta P = 0,$$

every covariant polynomial is also harmonic.

Classical results [1,16] state that the space $\mathbf{H}_n$ affords a graded $\mathcal{S}_n$-module structure and is isomorphic (as a representation of $\mathcal{S}_n$) to the left regular representation. Furthermore, as a graded $\mathcal{S}_n$-module, $\mathbf{H}_n$ is isomorphic to the quotient

$$Q_n = \mathbb{Q}[X]/\mathcal{I}_n.$$

The space Q_n appears naturally in other contexts; for instance, as the cohomology ring of the variety of complete flags [5]. The discussion above implies that

$$dim\, \mathbf{H}_n = n!. \tag{1.2}$$

Part of the interesting results surrounding the study of $\mathbf{H}_n$ involve the fact that it can also be described as the linear span of all partial derivatives of the Vandermonde determinant. This is a special case of a general result for finite groups generated by reflections [16].

By analogy, we consider here the space $\mathbf{SH}_n = \mathcal{J}_n^\perp$ of *super-covariant* polynomials, where $\mathcal{J}_n$ is the idea generated by *quasi-symmetric* polynomials with no constant term. Since the ring of symmetric polynomials is a subring of the ring of quasi-symmetric polynomials, we have $\mathcal{I}_n \subseteq \mathcal{J}_n$ hence $\mathcal{J}_n^\perp \subseteq \mathcal{I}_n^\perp$, thus

$$\mathbf{SH}_n \subseteq \mathbf{H}_n,$$

which justifies the terminology. Quasi-symmetric polynomials were introduced by Gessel in 1984 [8] and have since appeared as a crucial tool in many interesting algebraico-combinatorial contexts (cf. [4,7,13–15]).

As in the corresponding symmetric setup, we have a graded isomorphism

$$\mathbf{SH}_n \simeq \mathbf{R}_n = \mathbb{Q}[X]/\mathcal{J}_n \tag{1.3}$$

and the approach used in the following work concentrates on this alternate description. We construct a basis of $\mathbf{R}_n$ by giving an explicit set of monomial representatives. This set is naturally indexed by *Dyck paths* of length n, hence we obtain the following main theorem.

Theorem 1.1. *The dimension of* $\mathbf{SH}_n$ *is given y the well-known Catalan numbers*:

$$dim\,\mathbf{SH}_n = dim\,\mathbf{R}_n = C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.4}$$

In fact, taking into account the grading (*with respect to degree*), *we have the Hilbert series*

$$\sum_{k=0}^{n-1} dim\,\mathbf{SH}_n^{(k)} t^k = \sum_{k=0}^{n-1} \frac{n-k}{n+k}\binom{n+k}{k} t^k. \tag{1.5}$$

The article contains five section. In Section 2 we recall useful definitions and basic properties. In Section 3 we construct a family $\mathcal{G}$ of generators for the ideal $\mathcal{J}_n$ and state useful properties of this set. Section 4 is devoted to the proof of the first part of Theorem 1.1. We construct an explicit basis for $\mathbf{R}_n$ which allows us in Section 5 to obtain the Hilbert series of $\mathbf{SH}_n$.

Before we begin, let us remark that Hivert [9] has developed an action of the Hecke algebra on $\mathbb{Q}[X]$ for which a polynomial is invariant if and only if it is quasi-symmetric. One way to reformulate his result is to consider the generators $e_i = \frac{q-T_i}{(1+q)}$ of the Hecke algebra, where T_i are the standard generators and q is an arbitrary parameter. Then

$$e_i e_{i\pm1} e_i - \frac{q}{(1+q)^2} e_i \tag{1.6}$$

acts, via Hivert's action, as zero on the polynomial ring and generates the kernel of this action. Hence, the Temperley–Lieb algebra $TL_n(q)$ (cf. [10]) classically defined as the quotient of the Hecke algebra by relation (1.6), faithfully acts on polynomials. The algebra $TL_n(q)$ is known to have dimension equal to C_n and at $q = 1$ this is a quotient of the symmetric group algebra. The quasi-symmetric polynomials are thus identified as the polynomial invariants $\mathbb{Q}[X]^{TL_n}$ of the algebra $TL_n = TL_n(1)$.

The action of Hivert is not compatible with multiplication and does not preserve the ideal $\mathcal{J}_n$, yet there are some striking facts related to TL_n-invariants. The quasi-symmetric functions are closed under multiplication [14], in particular they form a subring of $\mathbb{Q}[X]$. Moreover, if we let n go to infinity, there is a graded Hopf algebra structure on quasi-symmetric functions [8] that is free and cofree with cogenerators in every degree [13]. That is, the graded dual is isomorphic to a free non-commutative Hopf algebra $\mathbb{Q}\langle h_1, h_2, \dots \rangle$ where $deg(h_k) = k$. Moreover, in this paper, we show that the space R_n of TL_n-covariants has dimension equal to $C_n = dim(TL_n)$.

These facts are very similar to the classical theory of group invariants [16]. Unfortunately the analogy is incomplete as Hivert's action does not induce an action on R_n. This raises new open questions for future investigation: how can we explain that $dim(R_n) = dim(TL_n)$?

2. Basic definitions

A *composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of a positive integer d is an ordered list of positive integers (>0) whose sum is d. We denote this by $\alpha \vDash d$ and also say that α is a composition of *size* d and denote this by $|\alpha|$. The integers α_i are the *parts* of α, and the length $\ell(\alpha)$ is set to be the number of parts of α. We denote by 0 the unique empty composition of size $d = 0$.

There is a natural one-to-one correspondence between compositions of d and subsets of $\{1, 2, \dots, d-1\}$. Let $S = \{a_1, a_2, \dots, a_k\}$ be such a subset, with $a_1 < \dots < a_k$, then the composition associated to S is $\alpha_d(S) = (a_1 - a_0, a_2 - a_1, \dots, a_{k+1} - a_k)$, where we set $a_0 := 0$ and $a_{k+1} := d$. We denote by $D(\alpha)$ the set associated to α through this correspondence. For compositions α and β, we say that β is a *refinement* of α, if $D(\alpha) \subset D(\beta)$, and denote this by $\beta \succcurlyeq \alpha$.

We use vector notation for monomials. More precisely, for $v = (v_1, \dots, v_n) \in \mathbb{N}^n$, we denote X^v the monomial

$$x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}. \tag{2.1}$$

For a polynomial $P \in \mathbb{Q}[X]$, we further denote $[X^v]P(X)$ as the coefficient of the monomial X^v in $P(X)$.

For a *vector* $v \in \mathbb{N}^n$, let $c(v)$ represent the composition obtained by erasing zero (if any) in v. A polynomial $P \in \mathbb{Q}[X]$ is said to be *quasi-symmetric* if and only if, for any v and μ in $\mathbb{N}^n$, we have

$$[X^v]P(X) = [X^\mu]P(X)$$

whenever $c(v) = c(\mu)$. The space of quasi-symmetric polynomials in n variables is denoted by $Qsym_n$. The space $Qsym_n^{(d)}$ of homogeneous quasi-symmetric polynomials of degree d admits as linear basis the set of *monomial* quasi-symmetric polynomials indexed by compositions of d. More precisely, for each composition α

of d with at most n parts, we set

$$M_\alpha = \sum_{c(v)=\alpha} X^v. \tag{2.2}$$

For the 0 composition, we set $M_0 = 1$. Another important linear basis is that of the *fundamental* quasi-symmetric polynomials (cf. [8]):

$$F_\alpha = \sum_{\beta \succcurlyeq \alpha} M_\beta \tag{2.3}$$

with $\alpha \vDash n$ and $\ell(\alpha) \leqslant n$. For example, with $n = 4$,

$$\begin{aligned} F_{21}(x_1, x_2, x_3, x_4) &= M_{21}(x_1, x_2, x_3, x_4) + M_{111}(x_1, x_2, x_3, x_4) \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4 \\ &\quad + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4. \end{aligned}$$

Part of the interest of fundamental quasi-symmetric functions comes from the following properties. The first is trivial, but very useful and the second comes from the theory of P-partitions [14,15].

Proposition 2.1. *For* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vDash d$,

$$F_\alpha(X) = \begin{cases} x_1 F_{(\alpha_1 - 1, \alpha_2, \dots, \alpha_k)}(X) + F_\alpha(x_2, \dots, x_n) & \text{if } \alpha_1 > 1, \\ x_1 F_{(\alpha_2, \alpha_3, \dots, \alpha_k)}(x_2, \dots, x_n) + F_\alpha(x_2, \dots, x_n) & \text{if } \alpha_1 = 1. \end{cases} \tag{2.4}$$

Let $u = u_1 \cdots u_l \in \mathcal{S}_\ell$ and $v = v_1 \cdots v_m \in \mathcal{S}_{[\ell+1, \ell+m]}$. Let $u \sqcup\!\sqcup v$ denote the set of *shuffles* of the words u and v, i.e. $u \sqcup\!\sqcup v$ is the set of all permutations w of $\ell + m$ such that u and v are subwords of w. In particular $u \sqcup\!\sqcup v$ contains $\binom{\ell+m}{m}$ permutations. Let $\mathcal{D}(u) = \{i,\ u_i > u_{i+1}\}$ denote the *descent set* of u. If β and γ are the two compositions such that $D(\beta) = \mathcal{D}(u)$ and $D(\gamma) = D(v)$, then

Proposition 2.2 (Stanley [15, Exercise 7.93]).

$$F_\beta\, F_\gamma = \sum_{w \in u \sqcup\!\sqcup v} F_{\alpha_{\ell+m}(\mathcal{D}(w))}. \tag{2.5}$$

In (2.1), the monomials are in correspondence with vectors $v \in \mathbb{N}^n$. Just as for compositions, the size $v_1 + \cdots + v_n$ of v is denoted by $|v|$. It is also convenient to denote by $\ell(v)$ the position of its last non-zero component. As usual, $v + \mu$ is the componentwise addition of vectors.

For ease of reading, we reserve the use of α, β and γ to represent compositions, and the other Greek letters to represent vectors. We use the same symbol α for both the composition $(\alpha_1, \dots, \alpha_\ell)$ and the word $\alpha_1 \cdots \alpha_\ell$, likewise for vectors. In general, the length of vectors (or number of variables) is fixed and equal to n. If w is a word of integers (that is an element of $\mathbb{N}^k$ for $0 \leqslant k \leqslant n$) we denote by $w0^* = w0^{n-k}$ the vector whose first k parts are the *letters* of w, to which are added $n-k$ zeros at the end. If $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_m$ are words of integers, the word

$$uv := u_1 \cdots u_k v_1 \cdots v_m$$

is the *concatenation* of u and v.

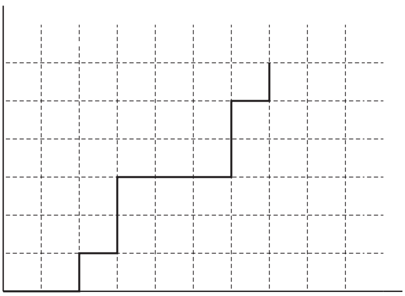
We next associate to any vector v a path $\pi(v)$ in the $\mathbb{N} \times \mathbb{N}$ plane with steps going north or east as follows. If $v = (v_1, \dots, v_n)$, the path $\pi(v)$ is

$$(0,0) \to (v_1, 0) \to (v_1, 1) \to (v_1 + v_2, 1) \to (v_1 + v_2, 2) \to \cdots$$

$$\to (v_1 + \cdots + v_n, n-1) \to (v_1 + \cdots + v_n, n).$$

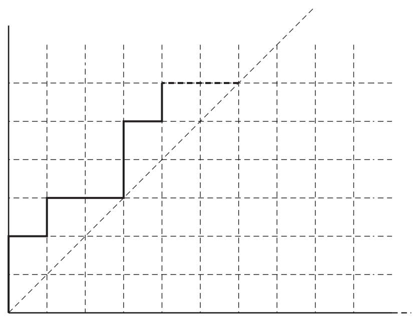
For example the path associated to $v = (2, 1, 0, 3, 0, 1)$ is

$\pi(v) =$

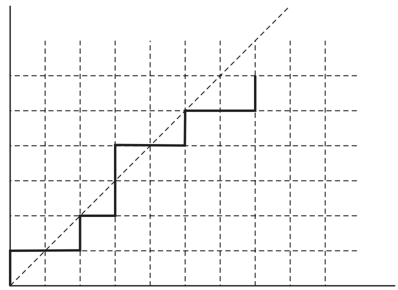


Observe that the height of the path is always n, whereas its width is $|v|$.

We distinguish two kinds of paths, thus two kinds of vectors, with respect to their "behavior" with respect to the diagonal $y = x$. If the path remains above the diagonal, we call it a *Dyck path*, and say that the corresponding vector is *Dyck*. If not, we say that the path (or equivalently the associated vector) is *transdiagonal*. For example $\eta = (0, 0, 1, 2, 0, 1)$ is Dyck and $\varepsilon = (0, 2, 1, 0, 2, 2)$ is transdiagonal.



η



ε

Observe that $v = v_1 \cdots v_n$ is transdiagonal if and only if there exists $1 \leqslant m \leqslant n$ such that

$$m < v_1 + \cdots + v_m. \tag{2.6}$$

Recall that the classical lexicographic order, on monomials of same degree, is

$$X^v \geqslant_{\text{lex}} X^\mu \quad \text{iff} \quad v \geqslant_{\text{lex}} \mu, \tag{2.7}$$

where we say that v is lexicographically larger than μ, $v >_{\text{lex}} \mu$, if the first non-zero part of the vector $v - \mu$ is positive. For example

$$x_1^3 >_{\text{lex}} x_1^2 x_2 >_{\text{lex}} x_1 x_2^2 >_{\text{lex}} x_2^3 \quad \text{since} \quad (3,0) >_{\text{lex}} (2,1) >_{\text{lex}} (1,2) >_{\text{lex}} (0,3).$$

3. The $\mathcal{G}$ basis

Following [2], we exploit relations (2.4) to construct a family

$$\mathcal{G} = \{G_\varepsilon\} \subset \mathcal{J}_n$$

indexed by vectors that are transdiagonal. For α any composition of $k \leqslant n$, the polynomial G_ε, with $\varepsilon := \alpha 0^*$, is defined to be

$$G_\varepsilon := F_\alpha. \tag{3.1}$$

When $\alpha \neq 0$, the vector $\varepsilon = \alpha 0^*$ is clearly transdiagonal. For a general vector ε (not of the form $\alpha 0^*$), the polynomial G_ε is defined recursively in the following way. Let $\varepsilon = w0a\beta 0^*$ be the unique factorization of ε such that w is a word of $k-1$ non-negative integers, $a > 0$ is a positive integer, and β is a composition (parts > 0). Then we set

$$G_\varepsilon = G_{wa\beta 0^*} - x_k G_{w(a-1)\beta 0^*}. \tag{3.2}$$

By induction on the length of the indexing vectors, both terms on the right of (3.2) are well defined, and we have

- $\ell(wa\beta 0^*) = \ell(w(a-1)\beta 0^*) = \ell(\varepsilon) - 1$;
- $wa\beta 0^*$ and $w(a-1)\beta 0^*$ are transdiagonal as soon as ε is transdiagonal.

In fact, let m be the first ordinate where $\pi(\varepsilon)$ crosses the diagonal, this is to say the smallest integer such that $m < \varepsilon_1 + \cdots + \varepsilon_m$. Then the second assertion follows from

$$\varphi_1 + \cdots + \varphi_m > \psi_1 + \cdots + \psi_m = \varepsilon_1 + \cdots + \varepsilon_m - 1 > m - 1,$$

where $\varphi = wa\beta 0^*$ and $\psi = w(a-1)\beta 0^*$.

For example,

$$\begin{aligned}
G_{1020} &= G_{1200} - x_2 G_{1100} \\
&= F_{12}(x_1, x_2, x_3, x_4) - x_2 F_{11}(x_1, x_2, x_3, x_4) \\
&= x_1x_2^2 + x_1x_3^2 + x_1x_4^2 + x_2x_3^2 + x_2x_4^2 + x_3x_4^2 + x_1x_2x_3 + x_1x_2x_4 \\
&\quad + x_1x_3x_4 + x_2x_3x_4 - x_2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\
&= x_1x_3^2 + x_1x_3x_4 + x_1x_4^2 - x_2^2x_3 - x_2^2x_4 + x_2x_3^2 + x_2x_4^2 + x_3x_4^2.
\end{aligned}$$

We observe in this example that the leading monomial (in lex order) of G_{1020} is $X^{1020} = x_1^1 x_2^0 x_3^2 x_4^0$. This holds in general for the $\mathcal{G}$ family as stated in the following proposition, for which all technical details can be found in [2].

Proposition 3.1 (Aval and Bergeron [2, Corollary 3.4]). *The leading monomial $LM(G_\varepsilon)$ of G_ε is X^ε.*

4. Proof of the main theorem

We now give an explicit basis for the space $\mathbf{R}_n$ naturally indexed by Dyck paths. This proves the first part of Theorem 1.1.

Theorem 4.1. *The set of monomials*

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\} \tag{4.1}$$

is a basis of the space $\mathbf{R}_n$.

The proof is achieved in a few steps. We start with the following lemma.

Lemma 4.2. *Any* $\mathcal{P}(X) \in \mathbb{Q}[X]$ *is in the linear span of* $\mathcal{B}_n$ *modulo* $\mathcal{J}_n$. *That is*

$$P(X) \equiv \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta \pmod{\mathcal{J}_n}. \tag{4.2}$$

Proof. It clearly suffices to show that (4.2) holds for any monomial X^ν, with ν transdiagonal. We assume that there exists X^ν not reducible of the form (4.2) and we choose X^ε to be the smallest amongst them with respect to the lexicographic order. Let us write

$$\begin{aligned} X^\varepsilon &= LM(G_\varepsilon) \\ &= (X^\varepsilon - G_\varepsilon) + G_\varepsilon \\ &\equiv X^\varepsilon - G_\varepsilon \pmod{\mathcal{J}_n}. \end{aligned}$$

All monomials in $(X^\varepsilon - G_\varepsilon)$ are lexicographically smaller than X^ε, thus they are reducible. This contradicts our assumption on X^ε and completes our proof. $\square$

Thus $\mathcal{B}_n$ spans the space $\mathbf{R}_n$. We now prove its linear independence. This is equivalent to showing that the set $\mathcal{G}$ is a Gröbner basis of the ideal $\mathcal{J}_n$. A crucial lemma is the following one, which is the quasi-symmetric analogue of a classical result is the case of symmetric polynomials ([6, Theorem II.2.2]).

Lemma 4.3. *If we denote by* $\mathcal{L}[S]$ *the linear span of a set S, then*

$$\mathbb{Q}[X] = \mathcal{L}[X^\eta F_\alpha \mid X^\eta \in \mathcal{B}_n,\ \alpha \models r \geqslant 0]. \tag{4.3}$$

Proof. We have already obtained the following reduction for any monomial X^ε in $\mathbb{Q}[X]$.

$$X^\varepsilon \equiv \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta \pmod{\mathcal{J}_n},$$

which is equivalent to

$$X^\varepsilon = \sum_{X^\eta \in \mathcal{B}_n} c_\eta X^\eta + \sum_{\alpha \models r \geqslant 1} Q_\alpha F_\alpha. \tag{4.4}$$

We then apply reduction (4.4) to each monomial of the Q_α's and use Proposition 2.2 to reduce products of fundamental quasi-symmetric functions. We obtain (4.3) in a finite number of operations since degrees strictly decrease at each operation, because $\alpha \models r \geqslant 1$ implies $deg\, Q_\alpha < |\varepsilon|$. $\square$

The next lemma is the final step in our proof of Theorem 4.1.

Lemma 4.4. *The set $\mathcal{G}$ is a linear basis of the ideal $\mathcal{J}_n$, i.e.*

$$\mathcal{J}_n = \mathcal{L}[G_\varepsilon \mid \varepsilon \textit{ transdiagonal}]. \tag{4.5}$$

Proof. Let us denote by $\mathcal{A}_n$ the set

$$\mathcal{A}_n = \{X^\xi \mid x_1^{\xi_n} x_2^{\xi_{n-1}} \cdots x_n^{\xi_1} \in \mathcal{B}_n\}. \tag{4.6}$$

Now the algebra endomorphism of $\mathbb{Q}[X]$ that *reverses* the variables,

$$x_i \mapsto x_{n-i+1},$$

clearly fixes the subalgebra *Qsym*. In fact it maps F_α to $F_{\alpha'}$, where α' is the reverse composition.

It follows from Lemma 4.3 and the endomorphism above that

$$\mathbb{Q}[X] = \mathcal{L}[X^\xi F_\alpha \mid X^\xi \in \mathcal{A}_n,\ \alpha \models r \geqslant 0]. \tag{4.7}$$

Now to prove Lemma 4.4, we reduce the problem as follows. We first use (4.7) and Proposition 2.2 to write

$$\begin{aligned}\mathcal{J}_n &= \langle F_\alpha,\ \alpha \models s \geqslant 0 \rangle_{\mathbb{Q}[X]} = \mathcal{L}[X^\xi F_\alpha F_\beta \mid X^\xi \in \mathcal{A}_n, \alpha \models s \geqslant 0, \beta \models t \geqslant 1] \\ &= \mathcal{L}[X^\xi F_\gamma \mid X^\xi \in \mathcal{A}_n, \gamma \models r \geqslant 1].\end{aligned}$$

It is now sufficient to prove that for all $X^\xi \in \mathcal{A}_n$ and all $\gamma \models r \geqslant 1$

$$X^\xi F_\alpha \in \mathcal{L}[G_\varepsilon \mid \varepsilon \text{ transdiagonal}]. \tag{4.8}$$

But Lemma 4.2 implies that any monomial of degree greater than n is in $\mathcal{J}_n$. Hence to prove (4.8), we need only show it for ξ and γ such that $|\xi| + |\gamma| \leqslant n$. To do that, we reduce the product

$$x_n^{\xi_n}(x_{n-1}^{\xi_{n-1}}(\cdots(x_2^{\xi_2}(x_1^{\xi_1}F_\alpha)))) \tag{4.9}$$

recursively, using

$$x_k G_{wb\beta 0^*} = G_{w(b+1)\beta 0^*} - G_{w0(b+1)\beta 0^*} \tag{4.10}$$

or

$$x_k G_{w0^*00^*} = G_{w0^*10^*} - G_{w0^*010^*}. \tag{4.11}$$

Relations (4.10) and (4.11) are immediate consequences of the definition of the $\mathcal{G}$ basis (relation (3.2)).

We have to show that the vectors ε generated in this process are all transdiagonal and that the length $\ell(\varepsilon)$ always remains at most equal to n. Let us first check that the transdiagonal part. This is obvious in the case of relation (4.11). In the other case (relation (4.10)), for $\varphi = wb\beta 0^*$, it is sufficient to observe that if m is such that

$$\varphi_1 + \cdots + \varphi_m > m$$

with $m > \ell(w)$ (if not, it is evident), then

$$\varphi'_1 + \cdots + \varphi'_m > m+1 > m \quad \text{and} \quad \varphi''_1 + \cdots + \varphi''_{m+1} > m+1,$$

where $\varphi' = w(b+1)\beta 0^*$, and $\varphi'' = w0(b+1)\beta 0^*$. We shall now prove that the length of the ε's always remains at most equal to n. For this we need to keep track of the term $\varepsilon_{\ell(\varepsilon)}$. Two cases have to be considered.

- First case: $\varepsilon_{\ell(\varepsilon)}$ comes from $\alpha_{\ell(\alpha)}$ that has shifted to the right by relation (4.10). It could move at most $|\xi|$ steps to the right, whence

$$\ell(\varepsilon) \leqslant \ell(\alpha) + |\xi| \leqslant |\alpha| + |\xi| \leqslant n.$$

- Second case: $\varepsilon_{\ell(\varepsilon)}$ is a "1" generated by relation (4.11) that has shifted to the right. If it is generated by a multiplication by x_k, then we consider the vector

$$\eta = \xi_n \xi_{n-1} \cdots \xi_k 0^*.$$

Since $X^\xi \in \mathcal{A}_n$ implies $\pi(\eta)$ is a Dyck path, we have

$$|\eta| < \ell(\eta) = n - k + 1$$

hence the generated "1" can shift at most to position

$$k + |\eta| \leqslant k + n - k = n. \qquad \square$$

The recursive process used to reduce a product of form (4.9) is illustrated in the following example, where $n = 5$.

$$\begin{aligned} x_1 x_3 F_{21} &= x_3(x_1 F_{21}) \\ &= x_3(G_{31000} - G_{03100}) \end{aligned}$$

$$= x_3 G_{31000} - x_3 G_{03100}$$

$$= G_{31100} - G_{31010} - G_{03200} + G_{03020}.$$

End of proof of Theorem 4.1: By Lemma 4.2, the set $\mathcal{B}_n$ spans the quotient $\mathbf{R}_n$. Assume we have a linear dependence relation modulo $\mathcal{J}_n$, i.e. there exists P

$$P = \sum_{X^\xi \in \mathcal{B}_n} a_\xi X^\xi \in \mathcal{I}_n.$$

By Lemma 4.4, $\mathcal{J}_n$ is linearly spanned by the G_ε's, thus

$$P = \sum_{\varepsilon \text{ transdiagonal}} b_\varepsilon G_\varepsilon.$$

This implies $LM(P) = X^\varepsilon$, with ε transdiagonal, which is absurd. $\square$

A consequence of Lemma 4.4 and Theorem 4.1 is that the set $\mathcal{G}$ is a Gröbner basis of $\mathcal{J}_n$ with respect to the lex order. From this we see below that a minimal Gröbner basis of $\mathcal{J}_n$ is obtained from $\mathcal{G}$ if we select the $G_\varepsilon \in \mathcal{G}$ such that $\pi(\varepsilon)$ has exactly one step under the line $y = x$ and no other horizontal steps after that.

Corollary 4.5. *A minimal Gröbner basis for $\mathcal{J}_n$ is given by*

$$\{G_\varepsilon \in \mathcal{G} \mid \varepsilon = w0^*,\ \ell(w) = |w| + 1,\ w_1 + \cdots + w_s \leqslant s,\ \textit{for}\ s < \ell(w)\}. \tag{4.12}$$

Proof. Theorem 4.1 implies that the monomial ideal $LT(\mathcal{J}_n)$ of leading terms of $\mathcal{J}_n$ is generated by all monomials X^η where $\pi(\eta)$ is transdiagonal. For any such η let m be the smallest integer such that $m < \eta_1 + \cdots + \eta_m$ and let $\varepsilon = \eta_1 \cdots \eta_{m-1} a 0^*$ where $a = m - 1 - \eta_1 - \cdots - \eta_{m-1}$. The monomial X^ε divides X^η which shows that $LT(\mathcal{J}_n)$ is generated by the leading monomial of the G_ε in (4.12). This gives that (4.12) is a Gröbner basis. To show minimality, consider X^ξ a monomial that strictly divides the leading monomial of a G_ε in (4.12). Since $\pi(\varepsilon)$ has exactly one step under the line $y = x$, we have that $\pi(\xi)$ is not transdiagonal and $X^\xi \notin LT(\mathcal{J}_n)$. Hence the leading monomials of the G_ε in (4.12) is a minimal set of generators for $LT(\mathcal{J}_n)$. $\square$

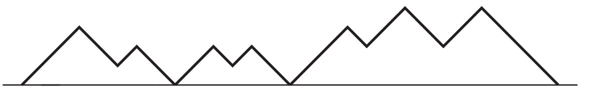
5. Hilbert series

Since Theorem 4.1 gives us an explicit basis for the quotient $\mathbf{R}_n$, which is isomorphic to $\mathbf{SH}_n$ as a graded vector space, we are able to refine relation (1.4) by giving the Hilbert series of the space of super-covariant polynomials. For $k \in \mathbb{N}$, let

$\mathbf{SH}_n^{(k)}$ and $\mathbf{R}_n^{(k)}$ denote the projections

$$\mathbf{SH}_n^{(k)} = \mathbf{SH}_n \cap \mathbb{Q}^{(k)}[X] \simeq \mathbf{R}_n \cap \mathbb{Q}^{(k)}[X] = \mathbf{R}_n^{(k)}, \tag{5.1}$$

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree k together with zero. Here, we represent Dyck paths horizontally, with n rising steps $(1,1)$ and n falling steps $(1,-1)$. Let us denote by $D_n^{(k)}$ the number of Dyck paths of length $2n$ ending by exactly k falling steps and by $C_n^{(k)}$ the number of Dyck paths of length $2n$ which have exactly k factors, i.e. $k+1$ points on the axis. The next figure gives an example of a Dyck path of length 28, ending with four falling steps and made of three factors.



It is well known that

$$D_n^{(k)} = C_n^{(k)} = \frac{k(2n-k-1)!}{n!\,(n-k)!}, \tag{5.2}$$

where the first equality is classical (cf. [17] for example for a bijective proof), and the second corresponds to [11, formula (7)].

Let us denote by $F_n(t)$ the Hilbert series of $\mathbf{SH}_n$, i.e.

$$F_n(t) = \sum_{k \geqslant 0} dim\, \mathbf{SH}_n^{(k)} t^k. \tag{5.3}$$

Theorem 5.1. *For* $0 \leqslant k \leqslant n-1$, *the dimension of* $\mathbf{SH}_n^{(k)}$ *is given by*

$$dim\, \mathbf{SH}_n^{(k)} = dim\, \mathbf{R}_n^{(k)} = D_n^{(n-k)} = C_n^{(n-k)} = \frac{n-k}{n+k}\binom{n+k}{k}. \tag{5.4}$$

For $k \geqslant n$ *the dimension of* $\mathbf{SH}_n^{(k)}$ *is* 0.

Proof. By Theorem 4.1, we know that the set

$$\mathcal{B}_n = \{X^\eta \mid \pi(\eta) \text{ is a Dyck path}\}$$

is a basis for $\mathbf{R}_n$. It is then sufficient to observe that the path $\pi(\eta)$ associated to η ends by exactly $n - |\eta|$ falling steps. $\square$

For example, we have:

n	$F_n(t)$
1	1
2	$1+t$
3	$1+2t+2t^2$
4	$1+3t+5t^2+5t^3$
5	$1+4t+9t^2+14t^3+14t^4$
6	$1+5t+14t^2+28t^3+42t^4+42t^5$
7	$1+6t+20t^2+48t^3+90t^4+132t^5+132t^6$

This gives

$$F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k \tag{5.5}$$

from which one easily deduces that the generating series for the $F_n(t)$'s is

$$\sum_n F_n(t) x^n = \frac{1-\sqrt{1-4tx}-2x}{2(t+x-1)}. \tag{5.6}$$

Remark 5.2. The study of various filtrations of the space $\mathbb{Q}[X]$, with respect to family of ideals of quasi-symmetric polynomials, will be the object of a forthcoming paper [3].

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