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ORBITS ON VERTICES AND EDGES OF FINITE GRAPHS

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Given two integers $\nu > 0$ and $\varepsilon \geqslant 0$, we prove that there exists a finite graph (resp. a finite connected graph) whose automorphism group has exactly ν orbits on the set of vertices and ε orbits on the set of edges if and only if $\nu \leqslant 2\varepsilon + 1$ (resp. $\nu \leqslant \varepsilon + 1$).

1. Introduction

In this paper, we shall answer two questions raised by J. Doyen. All graphs are supposed to be undirected, without loops and multiple edges.

Let P (resp. P_c) be the set of all ordered pairs (ν, ε) of integers $\nu > 0$ and $\varepsilon \geqslant 0$ for which there exists a finite graph (resp. a finite connected graph) whose automorphism group has exactly ν orbits on the set of vertices and ε orbits on the set of edges. We shall prove the following theorems:

Theorem 1. *(ν, ε) is in P if and only if $\nu \leqslant 2\varepsilon + 1$.*

Theorem 2. *(ν, ε) is in P_c if and only if $\nu \leqslant \varepsilon + 1$.*

For every integer $k \geqslant 2$, C_k will denote the *k-claw*, that is the graph with k edges and $k+1$ vertices, one of which is adjacent to all others. Note that the automorphism group Aut C_k has two orbits on vertices and one orbit on edges.

2. Proof of Theorem 1

(i) If $(\nu, \varepsilon) \in P$, then $\nu \leqslant 2\varepsilon + 1$. Indeed, let G be any graph. Since very edge of G has two vertices, every edge orbit of Aut G gives rise to at most two vertex orbits. Moreover, if there is an isolated vertex (that is a vertex belonging to no edge), the set of all such vertices is another orbit of Aut G. Hence $\nu \leqslant 2\varepsilon + 1$.

(ii) We first show that $(1, \varepsilon) \in P$ for every $\varepsilon \geqslant 0$. Let $n = 3\varepsilon + 1$ and let $G(1, \varepsilon)$ be the graph whose vertices are the elements of the additive group $Z_n = \{0, \ldots, n-1\}$ of integers modulo n (so that $G(1, 0)$ has one vertex and no edge) and whose edges (for $\varepsilon \geqslant 1$) are all subsets $\{i, i+j\}$ of Z_n with $i \in Z_n$ and $1 \leqslant j \leqslant \varepsilon$. Since $\alpha: Z_n \to Z_n: x \to x+1$ is an automorphism of $G(1, \varepsilon)$, all vertices are in one orbit. Since the edge $\{i, i+j\}$ is in the same orbit as $\{0, j\}$, which is contained in exactly $2\varepsilon - j - 1$ triangles, we conclude that $\operatorname{Aut} G(1, \varepsilon)$ has ε orbits on edges.

It remains to prove that $(\nu, \varepsilon) \in P$ whenever $2 \leqslant \nu \leqslant 2\varepsilon + 1$. If $\nu = 2t+1$ is odd, let $G(\nu, \varepsilon)$ be the graph whose $t+1$ connected components are isomorphic respectively to $C_2, C_3, \ldots, C_{t+1}$ and $G(1, \varepsilon - t)$ (note that $\varepsilon - t \geqslant 0$ since $\nu \leqslant 2\varepsilon + 1$). Clearly, $\operatorname{Aut} G(\nu, \varepsilon)$ has $2t+1 = \nu$ vertex orbits and $t + (\varepsilon - t) = \varepsilon$ edge orbits. If $\nu = 2t$ is even, let $G(\nu, \varepsilon)$ be the graph obtained by adding one new vertex and no new edge to $G(\nu - 1, \varepsilon)$. It is easy to check that $G(\nu - 1, \varepsilon)$ has no isolated vertex. Therefore $\operatorname{Aut} G(\nu, \varepsilon)$ has $(\nu - 1) + 1 = \nu$ vertex orbits and ε edge orbits. □

3. Proof of Theorem 2

(i) If $(\nu, \varepsilon) \in P_c$, then $\nu \leqslant \varepsilon + 1$. Indeed, let G be a finite connected graph such that $\operatorname{Aut} G$ has ν vertex orbits $O_1, \ldots, O_\nu$ and ε edge orbits. Let G' be the graph whose vertices are $O_1, \ldots, O_\nu$, where O_i and O_j $(i \neq j)$ are adjacent in G' if and only if there are two vertices $v_i \in O_i$ and $v_j \in O_j$ such that v_i and v_j are adjacent in G. The connectedness of G implies the connectedness of G'. As a finite connected graph with ν vertices, G' has at least $\nu - 1$ edges. On the other hand, if we associate with each edge $\{O_i, O_j\}$ of G' an edge $\{v_i, v_j\}$ of G, where $v_i \in O_i$ and $v_j \in O_j$, the edges of G associated with two distinct edges of G' are necessarily in distinct edge orbits of $\operatorname{Aut} G$. Hence the number of edge orbits of $\operatorname{Aut} G$ is not less than the number of edges of G', and so $\varepsilon \geqslant \nu - 1$. This inequality has been proved in a different way by Siemons [1, Theorem 3.3].

(ii) For every integer $\nu \geqslant 1$, the automorphism group of a path of length $2\nu - 2$ has clearly ν vertex orbits and $\nu - 1$ edge orbits, so that $(\nu, \nu - 1) \in P_c$.

It remains to prove that $(\nu, \varepsilon) \in P_c$ whenever $1 \leqslant \nu \leqslant \varepsilon$. If $\nu = 1$, the connected graphs $G(1, \varepsilon)$ described in the proof of Theorem 1 show that $(1, \varepsilon) \in P_c$ for every $\varepsilon \geqslant 1$. If $\nu \geqslant 2$, let $G_c(\nu, \varepsilon)$ be the connected graph whose vertices are the elements of $Z_n \times Z_\nu$ with $n = 3(\varepsilon - \nu + 1) + 1$, and whose edges are

(1) the edges of a graph $G(1, \varepsilon - \nu + 1)$ constructed on the n vertices $(i, 0)$, where $i \in Z_n$, and

(2) all subsets $\{(i, r), (i, r+1)\}$ of $Z_n \times Z_\nu$ where $i \in Z_n$ and $0 \leqslant r \leqslant \nu - 2$. It is not difficult to show that $\operatorname{Aut} G_c(\nu, \varepsilon)$ has ν vertex orbits and ε edge orbits.

Therefore $(\nu, \varepsilon) \in P_c$ whenever $1 \leqslant \nu \leqslant \varepsilon + 1$. □

4. Remarks

It is an easy exercise to adapt the above proofs in order to get the corresponding results for finite directed graphs.

However, as far as we know, similar problems arising in a natural way for other classes of undirected graphs (for example infinite graphs, finite regular graphs, finite planar graphs, etc . . .) are still unsolved.

References

[1] J. Siemons, Automorphism groups of graphs, Arch. Math. 41 (1983) 379–384.