Nontrivial solutions for perturbations of a Hardy–Sobolev operator on unbounded domains

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Abstract
In this paper we study the existence of nontrivial solution of the problem

\[-\Delta_p u - (\mu / [d(x)]^\rho) \times |u|^{p-2} u = f(u) \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) is the usual \(p\)-Laplacian, \(0 < \mu < (p/(p-1))^p\) and \(d(x) = \text{dist}(x, \partial \Omega)\). In this paper we study the existence of weak solution of the boundary value problem

\[L_\mu u = f(u) \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.\]  

Keywords: \(p\)-Laplacian; Hardy–Sobolev operator; Unbounded domain

1. Introduction

Let \(\Omega\) be an unbounded domain in \(\mathbb{R}^N\) such that \(\Omega = \Omega' \times \mathbb{R}^{N-k}\), \(\Omega'\) is a bounded domain with smooth boundary in \(\mathbb{R}^k\), \(1 < k < N\). We define a Hardy–Sobolev operator \(L_\mu\) on \(W_0^{1,p}(\Omega)\) as

\[L_\mu u := -\Delta_p u - \frac{\mu}{[d(x)]^\rho} |u|^{p-2} u,\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) is the usual \(p\)-Laplacian, \(0 < \mu < (p/(p-1))^p\) and \(d(x) = \text{dist}(x, \partial \Omega)\). In this paper we study the existence of weak solution of the boundary value problem

\[L_\mu u = f(u) \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.\]  

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Here \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function which is \( O(|s|^{p-1}) \) around 0 and has subcritical growth, i.e.,
\[
|f(s)| \leq a_0 |s|^{p-1} + b_0 |s|^{q-1}, \quad \forall s \in \mathbb{R},
\]
for some constants \( a_0 \) and \( b_0 > 0 \), where \( p < q < p^* = Np/(N - p) \) for \( p < N \) and equal to \( \infty \) for \( p \geq N \). The above problem in case of \( \mu = 0 \) has been studied in [4]. In our case \( (\mu \neq 0) \), the difficulty arises due to the noncompactness of the embedding \( W^{1,p}_0(\Omega) \) into \( L^p(\Omega, |d(x)|^{-p}) \). We overcome this difficulty by using a result of Boccardo and Murat [2] which is stated in Theorem 2.5, and some techniques from [7]. The ideas of [4] are used to tackle the noncompactness comes from the unboundedness of the domain. We assume the following on the nonlinearity \( f \):
\[
\limsup_{s \to 0} \frac{pF(s)}{|s|^p} \leq \alpha \leq \beta \leq \liminf_{|s| \to \infty} \frac{pF(s)}{|s|^p},
\]
for some constants \( \alpha > 0 \), \( p < q < p^* = Np/(N - p) \) for \( p < N \) and equal to \( \infty \) for \( p \geq N \). There exist \( a, q, \nu \), such that \( a > 0 \), \( p < q < p^* \) and \( 0 < \nu < p^* \),
\[
\limsup_{|s| \to \infty} \frac{F(s)}{|s|^q} \leq b,
\]
\[
sf(s) - pF(s) \geq a|s|^{\nu} > 0, \quad \forall s \in \mathbb{R}\setminus\{0\},
\]
\[
\limsup_{|s| \to 0} \frac{sf(s)}{|s|^p} \leq M,
\]
where \( F(s) = \int_0^s f(t) \, dt \) and \( \lambda_1 \) is given by
\[
\lambda_1 = \inf_{0 \neq u \in W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx - \mu \int_\Omega |u|^{p'} \, dx}{\int_\Omega |u|^p \, dx}.
\]
In Section 2 we show that \( \lambda_1 \) is equal to the first eigenvalue of \( L_\mu \) on \( W^{1,p}_0(\Omega') \) where \( \Omega = \Omega' \times \mathbb{R}^{N-k} \). The solutions of (1.1) are the critical points of the functional \( J : W^{1,p}_0(\Omega) \to \mathbb{R} \), defined by
\[
J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\mu}{p} \int_\Omega |u|^{p'} \, dx - \int_\Omega F(u) \, dx.
\]
Let \( k_1 \) be the smallest integer such that \( \mathbb{R}^s \) can be covered by a sequence \( \{B_i\} \) of open balls so that each point of \( \mathbb{R}^s \) belongs to at most \( k_1 \) balls. This integer is known to be \( s \) for \( s \leq 8 \) [5].

The main result of the paper is

**Theorem 1.1.** Assume that \( f \) satisfies (1.2)–(1.6) with \( \mu > (N/p)(q - p) \). Then (1.1) has a nontrivial solution \( u \in W^{1,p}_0(\Omega) \) provided \( M < \lambda_1/k_1 \).

The norm \( \| \cdot \|_{1,p} \) on \( W^{1,p}_0(\Omega) \) is defined as \( \|u\|_{1,p} = (\int_\Omega |\nabla u|^p \, dx)^{1/p} \).
2. Preliminary results

The following lemmas are needed for our further discussion.

**Lemma 2.1** (Hardy–Sobolev inequality). Let \( \Omega = \Omega' \times \mathbb{R}^{N-k} \) with \( \Omega' \) bounded domain with Lipschitz boundary and let \( u \in W^{1,p}_0(\Omega) \); then
\[
\int_{\Omega} \frac{|u|^p}{d^p(\xi)} \, dx \leq \left( \frac{p-1}{p} \right)^p \int_{\Omega} |\nabla u|^p \, dx.
\]

**Proof.** Since \( \Omega' \) is bounded domain with smooth boundary, using the results in [6] we have that
\[
\int_{\Omega'} |u|^p \, d\xi, \partial \Omega' \leq \mu_p \int_{\Omega'} |\nabla u|^p \, d\xi, \forall u \in C^\infty_c(\Omega'),
\]
where \( \mu_p = ((p-1)/p)^p \).

Let \( u \in C^\infty_c(\Omega) \), \( x_1, x_2, \ldots, x_n \). Then \( u(\xi, x_{k+1}, \ldots, x_n) \in C^\infty_c(\Omega') \).

Now integrating in \( x_{k+1}, x_{k+2}, \ldots, x_n \) both sides, we get
\[
\int_{\Omega} \frac{|u|^p}{d^p(\xi, \partial \Omega')} \, dx \leq \mu_p \int_{\Omega} |\nabla u|^p \, dx.
\]

Since dist(\( (\xi, x') \)) = dist(\( x, \partial \Omega' \)) for all \( x \) such that \( x = (\xi, x') \) we have
\[
\int_{\Omega} \frac{|u(x)|^p}{d(x, \partial \Omega')^p} \, dx \leq \mu_p \int_{\Omega} |\nabla u|^p \, dx.
\]

**Lemma 2.2.** Let \( \Omega' \) be a bounded domain in \( \mathbb{R}^k \) and let
\[
\lambda'_1 = \inf_{0 \neq u \in W^{1,p}_0(\Omega')} \frac{\int_{\Omega'} |\nabla u|^p \, dx - \mu \int_{\Omega'} |u|^p \, dx}{\int_{\Omega'} |u|^p \, dx}.
\]
Then there exists a \( \phi_1(x) > 0 \) in \( W^{1,p}_0(\Omega') \) such that \( \lambda'_1 \) is achieved at \( \phi_1 \).

**Proof.** Follows from the Ekeland variational principle and Theorem 2.5 stated below. For a proof we refer to [1].

**Lemma 2.3.** Let \( \Omega = \Omega' \times \mathbb{R}^{N-k}, \Omega' \) is bounded in \( \mathbb{R}^k \), and let \( \lambda'_1 \) is the first eigenvalue of \( L_\mu \) on \( W^{1,p}_0(\Omega') \). Then
\[
\int_{\Omega} |\nabla u|^p - \int_{\Omega} \frac{\mu}{d^p} |u|^p \, dx \geq \lambda'_1 \int_{\Omega} |u|^p \, dx, \forall u \in W^{1,p}_0(\Omega).
\]
Proof. The proof given here is adopted form [3]. Let \( u \in C_0^\infty (\Omega) \). Fix \( x_{k+1}, x_{k+2}, \ldots, x_n \), then \( u(., x_{k+1}, \ldots, x_n) \in C_0^\infty (\Omega') \). By Lemma 2.2, we get
\[
\lambda_1 \int_\Omega |u|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx - \int_\Omega \frac{\mu}{d_p(\xi, \partial \Omega')} |u|^p \, dx.
\]
Now integrating in \( x_{k+1}, x_{k+2}, \ldots, x_n \) on both sides, we get
\[
\lambda_1 \int_\Omega |u|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx - \int_\Omega \frac{\mu}{d_p(\xi, \partial \Omega')} |u|^p \, dx.
\]
(2.2)
Since \( \text{dist}(\xi, \partial \Omega') = \text{dist}(x, \partial \Omega) \) for all \( x \) such that \( x = (\xi, x_{k+1}, \ldots, x_n) \), (2.2) gives the required inequality.

Theorem 2.4 (Mountain-pass lemma [8]). Let \( E \) be a real Banach space and suppose that \( I \in C^1(E, \mathbb{R}) \) satisfies the condition
\[
\max \{ I(0), I(u_1) \} \leq \alpha < \beta \leq \inf_{\|u\|_E = \eta} I(u)
\]
for some \( \eta > 0 \) and \( u_1 \in E \) with \( \|u\|_E > \eta \). Then there exists a sequence \( \{u_n\} \) such that
\[ (1) \quad u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega), \]
\[ (ii) \quad u_n \to u \text{ in } L^p(\Omega), \]
\[ (iii) \quad f_n \to f \text{ in } W^{-1,p'}, \]
\( gn \) is a bounded sequence of Radon measures. Then there exists a subsequence \( \{u_n\} \) of \( \{u_n\} \) such that \( \nabla u_n \rightharpoonup \nabla u \text{ a.e. in } \Omega \).

Theorem 2.5 [2]. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Suppose \( u_n \in W^{1,p}(\Omega) \) satisfies
\[ -\Delta_p u_n = f_n + g_n \text{ in } D'(\Omega), \]
and (i) \( u_n \to u \text{ weakly in } W^{1,p}(\Omega), \) (ii) \( u_n \to u \text{ in } L^p(\Omega), \) (iii) \( f_n \to f \text{ in } W^{-1,p'}, \)
(IV) \( gn \) is a bounded sequence of Radon measures. Then there exists a subsequence \( \{u_n\} \) of \( \{u_n\} \) such that \( \nabla u_n \rightharpoonup \nabla u \text{ a.e. in } \Omega \).

Theorem 2.6 (Concentration of compactness principle [8]). Let \( \{\mu_n\} \) be a sequence of probability measures on \( \mathbb{R}^N \). Then there exists a subsequence \( \{\mu_n\} \) for which one of the following three possibilities occurs:

1. Vanishing: For every \( R > 0 \), one has
\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \mu_n(B_R(z)) = 0.
\]
(2) **Concentration:** There exists a sequence \( \{z_n\} \subset \mathbb{R}^N \) such that \( \forall \epsilon > 0 \) there exists \( R > 0 \) with \[
\mu_n(\mathbb{R}^N \setminus B_R(z_n)) < \epsilon, \quad \forall n \in \mathbb{N}.
\]

(3) **Dichotomy:** There exist sequences \( \{z_n\} \subset \mathbb{R}^N, \{R^1_n\}, \{R^2_n\} \subset \mathbb{R}, \) and \( \lambda \in (0, 1) \) such that

\(\text{(a)}\) \( R^1_n, R^2_n \to \infty \) and \( R^1_n/R^2_n \to 0, \)

\(\text{(b)}\) \( \mu_n(B_{R^1_n}(z_n)) \to \lambda \) as \( n \to \infty, \)

\(\text{(c)}\) \( \mu_n(B_{R^2_n}(z_n) \setminus B_{R^1_n}(z_n)) \to 0, \)

\(\text{(d)}\) For any \( \epsilon > 0 \) there exists \( R > 0 \) such that \( \mu_n(B_R(z_n)) \geq \lambda - \epsilon, \forall n \in \mathbb{N}. \)

**Lemma 2.7.** Suppose that \( (u_n) \) is bounded in \( L^p(\mathbb{R}^N) \). In addition, assume that

\[
\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^p \, dx \to 0 \quad \text{as} \quad n \to \infty,
\]

for some \( R > 0. \) Then \( u_n \to 0 \) in \( L^q(\mathbb{R}^N) \) for all \( q \in (p, p^*). \)

### 3. Proof of Theorem 1.1

We need following results to prove Theorem 1.1.

**Lemma 3.1.** Suppose (1.2) and (1.3) hold. Then functional \( J \) satisfies the following conditions:

(i) There exist \( \eta, \gamma \) such that \( J(u) \geq \gamma \) for \( \|u\|_{1,p} = \eta; \)

(ii) There exists \( u_1 \in E \) with \( \|u_1\|_{1,p} > \eta \) such that \( J(u) < 0. \)

**Proof.** From (1.2) and (1.3) it follows that for any \( \epsilon > 0 \) there exist \( A(\epsilon) \) such that

\[
F(s) \leq \frac{\lambda_1 - \epsilon}{p} |s|^p + A(\epsilon) |s|^q, \quad \forall s \in \mathbb{R}.
\]

Therefore

\[
J(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\mu}{p} \int_{\Omega} |u|^p - \frac{\lambda_1 - \epsilon}{p} \int_{\Omega} |u|^p - A(\epsilon) \|u\|^q_{1,p}
\]

\[
\geq (1/p) \left( 1 - \frac{\lambda_1 - \epsilon}{\lambda_1} \right) \left( 1 - \frac{\mu}{\mu_p} \right) \|u\|^p_{1,p} - A(\epsilon) \|u\|^q_{1,p},
\]

i.e.,

\[
J(u) \geq \alpha \|u\|^p_{1,p} - A(\epsilon) \|u\|^q_{1,p},
\]

for some \( \alpha > 0. \) Hence (ii) follows for small \( \eta > 0. \)
Let $\epsilon > 0$ such that $\lambda_1 + \epsilon < \beta$. Then there exists a $\phi \not\equiv 0$ in $W^{1,p}_0(\Omega)$ such that
\[
\int_{\Omega} |\nabla \phi|^p \, dx - \mu \int_{\Omega} \frac{1}{d^p} |\phi|^p \, dx \leq \left( \lambda_1 + \frac{\epsilon}{2} \right) \int_{\Omega} |\phi|^p \, dx.
\]
Now by (1.3) there exists $M = M(\epsilon)$ such that
\[
F(s) \geq \lambda_1 + \frac{\epsilon}{2} |s|^p, \quad \forall |s| \geq M,
\]
and
\[
\int_{\Omega} F(t\phi) \, dx = \int_{S \cap \{|t\phi| > M\}} F(t\phi) + \int_{S \cap \{|t\phi| \leq M\}} F(t\phi) \, dx \geq \lambda_1 + \frac{\epsilon}{2} t^p G(t) + |S| \inf_{s \in [-M,M]} F(s),
\]
where $G(t) = \int_{S \cap \{|t\phi| > M(\epsilon)\}} |\phi|^p \, dx$.
Therefore
\[
\int_{\Omega} F(t\phi) \geq \frac{\lambda_1 + \epsilon/2}{p} t^p G(t) - C_\epsilon \quad \forall t,
\]
and
\[
J(t\phi) \leq (1/p) \int_{\Omega} |\nabla t\phi|^p - \frac{\mu}{p} \int_{\Omega} \frac{1}{d^p} |t\phi|^p - \frac{\lambda_1 + \epsilon/2}{p} t^p G(t) + C_\epsilon
\]
\[
= (1/p) \int_{\Omega} |\nabla t\phi|^p - \frac{\mu}{p} \int_{\Omega} \frac{1}{d^p} |t\phi|^p - \frac{\lambda_1 + \epsilon/2}{p} t^p G(t) + C_\epsilon
\]
\[
= (1/p) \int_{\Omega} |\nabla t\phi|^p - \frac{\mu}{p} \int_{\Omega} \frac{1}{d^p} |t\phi|^p - \frac{\lambda_1 + \epsilon/2}{p} \int_{\Omega} |\phi|^p + \frac{t^p}{p} \epsilon(t) + C_\epsilon,
\]
where $\epsilon(t) \to 0$ as $t \to \infty$.
\[
J(t\phi) \leq -\frac{\epsilon}{4} t^p \int_{\Omega} |\phi|^p + C_\epsilon.
\]
Hence $J(t\phi) \to -\infty$ as $t \to \infty$. \qed

By Lemma 3.1 and Theorem 2.4, we obtain a sequence $\{u_n\}$ such that
\[
J(u_n) \to c > 0, \quad \left(1 + \|u_n\|_{1,p}\right) \left\|J'(u_n)\right\|_{(W^{1,p}_0(\Omega))^*} \to 0.
\] (3.1)

**Lemma 3.2.** There exists $C > 0$ such that $\|u_n\|_{1,p} \leq C$, $\forall n \in \mathbb{N}$.

**Proof.** If $0 < q < p^*$ and $t \in (0,1)$ are such that $1/q = (1-t)/v + t/p^*$, then the following interpolation inequality holds:
\[
\|u\|_{0,q} \leq \|u\|_{0,v}^{1-v} \|u\|_{0,p^*}^v, \quad \forall u \in L^v \cap L^{p^*},
\]
where \( \|u\|_{0,s}^v = \int_{\Omega} |u|^s \, dx \). On the other hand, using (1.5) and (1.1), we obtain

\[
a \int_{\Omega} |u_n|^v \, dx \leq \int_{\Omega} \left[ f(u_n)u_n \, dx - pF(u) \right] \, dx = pJ(u_n) - \langle J'(u_n), u_n \rangle \leq C_0,
\]

for some constant \( C_0 > 0 \), so that

\[
|u|_{0,v} \leq D_0, \quad \forall n \in \mathbb{N}. \tag{3.2}
\]

Now given \( \epsilon > 0 \), (1.3) and (1.4) imply that

\[
F(s) \leq \frac{\alpha + \epsilon}{p} |s|^p + C\epsilon |s|^q, \quad \forall s \in \mathbb{R},
\]

so that we obtain

\[
\left( 1 - \frac{\mu}{\mu_p} \right) \int_{\Omega} |\nabla u_n|^p \, dx \leq pJ(u_n) + p \int_{\Omega} F(u_n) \, dx
\]

\[
\leq C + (\alpha + \epsilon) \int_{\Omega} |u_n|^p \, dx + C' \int_{\Omega} |u_n|^q \, dx
\]

\[
\leq C + (\alpha + \epsilon) \int_{\Omega} |u_n|^p \, dx + C' \|u_n\|_{0,v}^{(1-\theta)q} \|u_n\|_{0,p}^q.
\]

Now using Poincare's inequality and (3.2), we get

\[
\int_{\Omega} |\nabla u_n|^p \, dx \leq A + B \left( \int_{\Omega} |\nabla u_n|^p \, dx \right)^{tq/p}, \quad \forall n \in \mathbb{N}. \tag{3.3}
\]

Since \( v > (N/p)(q - p) \) we have \( tq < p \). By (3.3) we have

\[
\int_{\Omega} |\nabla u_n|^p \, dx \leq C, \quad \forall n \in \mathbb{N},
\]

for some \( C > 0 \). \( \square \)

**Definition 3.3.** Let \( v \in W_{0}^{1,p}(\Omega) \). For any measurable set \( S \subset \mathbb{R}^{N-k} \)

\[
\mu_v(S) = \frac{q_v(v)}{q(v)}, \quad q_v(v) = \frac{1}{p} \int_{\Omega} \left( |\nabla u|^p + |v|^p \right) \, dx, \quad q(v) = \frac{1}{p} \|v\|_{1,p}^p.
\]

**Lemma 3.4.** Suppose there exists a sequence \( \{v_n\} \subset W_{0}^{1,p}(\Omega) \) such that

(i) \( \|v_n\|_{1,p} \leq C \),

(ii) \( \|J'(v_n)\|_{W_{0}^{1,p}(\Omega)} \to 0 \) as \( n \to \infty \),

(iii) \( \forall \epsilon > 0 \) there exists \( B \subset \mathbb{R}^{N-k} \) such that \( \mu_v(B^C) \leq \epsilon, \forall n \in \mathbb{N} \).
Then there exists a subsequence \( \{v_n\} \) of \( \{v_n\} \) and \( v \in W^{1,p}_0(\Omega) \) such that \( \nabla v_n \rightharpoonup \nabla v \) a.e. in \( \Omega \), \( v_n \rightharpoonup v \) weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^p(\Omega) \).

**Proof.** Since \( v_n \) is bounded, there exists a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) and \( v \in W^{1,p}_0(\Omega) \) such that \( v_{n_k} \rightharpoonup v \) weakly in \( W^{1,p}_0(\Omega) \) and \( v_{n_k} \rightarrow v \) a.e. in \( \Omega \). By (iii), for any \( \epsilon > 0 \), there exists a ball \( B \subset \mathbb{R}^{N-k} \) such that

\[
q_{B^c}(v_{n_k}) \leq \frac{\epsilon}{p} \|v_{n_k}\|^p \leq \epsilon C_1, \quad \forall n \in \mathbb{N},
\]

i.e.,

\[
\int_{\Omega' \times B^c} |\nabla v_{n_k}|^p + |v_{n_k}|^p \, dx \leq \epsilon C_1,
\]

and \( v_{n_k} \rightarrow v \) in \( L^q(B) \) \( \forall q < p^* \). Therefore \( v_{n_k} \rightarrow v \) in \( L^q(\Omega) \).

Now \( v_n \in W^{1,p}(\Omega) \) satisfies

\[
-\Delta_p v_n - \frac{\mu}{d_p} |v_n|^{p-2} v_n = f(x, v_n) + o(1) \quad \text{in } D'(\Omega' \times B).
\]

By Theorem 2.5, there exists a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) such that \( \nabla v_{n_k}(x) \rightharpoonup \nabla v(x) \) a.e. in \( \Omega' \times B \). Therefore by choosing a sequence of balls \( \{B_i\} \) we may assume the existence of a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) such that \( v_{n_k} \rightharpoonup v \) weakly in \( W^{1,p}(\Omega) \), strongly in \( L^p(\Omega) \) and \( \nabla v_{n_k}(x) \rightharpoonup v(x) \) a.e. in \( \Omega \). \( \square \)

**Proof of Theorem 1.1.** Let \( \{u_n\} \) be a sequence as in (3.1). We use Theorem 2.6 and check each one of the possibilities:

1. **Vanishing:** Suppose vanishing occurs.

For \( R > 0 \) and \( \epsilon > 0 \) given, there exists \( n_0 = n_0(R, \epsilon) \) such that

\[
q_{B_R(z)}(u_n) \leq \epsilon q(u_n), \quad \forall n \geq n_0, \quad \forall z \in \mathbb{R}^{N-k},
\]

i.e.,

\[
\int_{\Omega' \times B_R(z)} (|u_n|^p + |\nabla u_n|^p) \, dx \leq \epsilon \int_{\Omega' \times \mathbb{R}^{N-k}} |u_n|^p + |\nabla u_n|^p \, dx \leq \epsilon C_1.
\]

By (1.2) and (1.6), for a given \( \epsilon' > 0, \exists M_{\epsilon'} \) such that

\[
f(s)s \leq \epsilon'|s|^{p^*} + M_{\epsilon'} |s|^q + (M + \epsilon)|s|^p, \quad \forall s \in \mathbb{R}.
\]

Therefore,

\[
\int_{\Omega' \times B_R(z)} f(u_n)u_n \, dx \leq \epsilon' \int_{\Omega' \times B_R(z)} |u_n|^p \, dx + M_{\epsilon'} \int_{\Omega' \times B_R(z)} |u_n|^q \, dx + (M + \epsilon) \int_{\Omega' \times B_R(z)} |u_n|^p \, dx.
\]
By using Gagliardo–Nirenberg inequality, there exists $C_1 > 0$ independent of $R$ such that

$$
\int_{\Omega' \times B_R(z)} f(u_n)u_n \, dx \leq \epsilon' C_1 \left( \int_{\Omega'} |u_n|^p + |\nabla u_n|^p \, dx \right)^{p'/p - 1}
+ M \epsilon' \int_{\Omega'} |u_n|^q + \int_{\Omega' \times B_R(z)} |u_n|^p \, dx.
$$

Using (3.4), we get

$$
\int_{\Omega' \times B_R(z)} f(u_n)u_n \, dx \leq \epsilon' \tilde{C} + M \epsilon' \int_{\Omega} |u_n|^q + \int_{\Omega' \times B_R(z)} |u_n|^p \, dx,
$$

(3.5)

for some $\tilde{C} > 0$. Let

$$
k_1 = \min \{ n \in \mathbb{N} : R^k \text{ can be covered by a family of balls } \{ B_R \} \text{ such that each point of } R^k \text{ belongs to at most } n \text{ balls} \}.
$$

Then summing (3.5) over these balls,

$$
\int_{\Omega} f(u_n)u_n \leq k_1 \epsilon' \tilde{C} + k_1 M \epsilon' \int_{\Omega} |u_n|^q + (M + \epsilon') k_1 \int_{\Omega} |u_n|^p
$$

and

$$
\langle J'(u_n), u_n \rangle \geq \left( 1 - \frac{k_1 (M + \epsilon')}{\lambda_1} \right) \left( 1 - \frac{\mu}{\mu_p} \right) \| u_n \|_{1,p}^p - \epsilon' \tilde{C} - k_1 M \epsilon' \int_{\Omega} |u_n|^q \, dx.
$$

Since $M < \lambda_1/k_1$, choose $\epsilon' > 0$ small such that $1 - k (M + \epsilon')/\lambda_1 > 0$. We have

$$
C' \int_{\Omega} |\nabla u_n|^p \, dx \leq \langle J'(u_n), u_n \rangle + \epsilon' C' + k_1 M \epsilon' \int_{\Omega} |u_n|^q \, dx.
$$

Now we observe by Lemma 2.7 that $u_n \to 0$ in $L^q(\Omega)$. Hence $u_n \to 0$ in $W_0^{1,p}(\Omega)$ so that $J(u_n) \to 0$, contradiction to (3.1).

(2) Concentration: Suppose concentration occurs.

Then there exists a sequence $\{ z_n \} \subset \mathbb{R}^{N-k}$ such that $\forall \epsilon > 0$ there is $R > 0$ satisfying

$$
\int_{\Omega' \times B_R(z_n)} |\nabla u_n|^p + |u_n|^p \, dx \leq \epsilon \int_{\Omega' \times \mathbb{R}^{N-k}} |\nabla u_n|^p + |u_n|^p \, dx.
$$

Let $x = y + z_n$, $v_n(y) = u_n(y + z_n)$, $\mu_n(S) = \mu_{v_n}(S)$. Then

$$
\mu_n \left( B_R(0)^c \right) \leq \epsilon, \quad \forall n \in \mathbb{N}.
$$

Choosing subsequences, if necessary, it follows from Lemma 3.4 that there exists a subsequence $\{ v_n \}$ of $\{ v_n \}$ and $v \in W_0^{1,p}(\Omega)$ such that $\nabla v_n \to \nabla v$ a.e. in $\Omega$. Now by observing that $J'(u_n) = J'(v_n)$ and letting $n \to \infty$ in $J'(v_n)$ we get $J'(v) = 0$, also
\[
c = \lim_{n \to \infty} J'(v_n)v_n - J(v_n) = \lim_{n \to \infty} \int_\Omega f(v_n)v_n - \int_\Omega F(v_n) = \int_\Omega f(v)v - \int_\Omega F(v) = J(v) - J'(v)v = J(v).
\]

Hence \( v \) is a nontrivial solution of (1.1).

(3) Dichotomy: Assume that there exist sequences \( \{z_n\} \subset \mathbb{R}^N \), \( \{R_1^n\}, \{R_2^n\} \subset \mathbb{R} \), and \( \lambda \in (0, 1) \) with \( \tilde{u}_n(y) = u_n(y + z_n) \) and \( \tilde{\mu}_n(S) = \mu_{\tilde{u}_n} \). We have

(a) \( R_1^n, R_2^n \to \infty \) and \( R_1^n / R_2^n \to 0 \),
(b) \( \tilde{\mu}_n(B_{R_1^n}) \to \lambda \) as \( n \to \infty \),
(c) \( \mu_n(B_{R_2^n} \setminus B_{R_1^n}) \to 0 \),
(d) for any \( \epsilon > 0 \), there exists \( R > 0 \) such that \( \mu_n(B_R) \geq \lambda - \epsilon, \forall n \in \mathbb{N} \).

Let us define \( \tilde{w}_n = \psi_n \tilde{u}_n \), where \( \psi_n \in C_0^\infty(\mathbb{R}^n) \) is a cutoff function with \( 0 \leq \psi_n \leq 1 \) and \( \psi_n = 1 \) on \( z_n + B_{R_1^n} \) and \( 0 \) on \( (z_n + B_{R_2^n})^c \).

**Claim 1.** There exist \( C_1, C_2 > 0 \) such that the sequence \( \{\tilde{w}_n\} \) satisfies

(i) \( 0 < C_1 \leq \|\tilde{w}_n\| \leq C_2, \forall n \in \mathbb{N} \),
(ii) \( \forall \epsilon > 0 \) there exists \( R > 0 \) such that
\[
\int_{\Omega \times B_R} (|\tilde{w}_n|^p + |
abla \tilde{w}_n|^p) \leq \epsilon, \quad \forall n \in \mathbb{N},
\]
(iii) \( \nabla J(\tilde{w}_n) \to 0 \) in \( W^{-1,p'} \).

If Claim 1 is proved, it follows from Lemma 3.4 that there exists \( \tilde{w} \in W^{1,p}_0(\Omega) \) such that \( \nabla \tilde{w}_n \to \nabla \tilde{w} \) a.e. in \( \Omega \). Therefore letting \( n \to \infty \) in (iii) above, we get \( J'(\tilde{w}) = 0 \). Suppose \( \tilde{w} = 0 \). Then testing (iii) against \( \tilde{w}_n \) and letting \( n \to \infty \) we get \( \|\tilde{w}_n\|_{1,p} \to 0 \), which is contradiction to (i) above. Therefore \( \tilde{w} \) is a nontrivial critical point of \( J \).

**Proof of Claim 1.** (b) implies that
\[
\frac{qB_{R_1^n}(\tilde{u}_n)}{q(\tilde{u}_n)} \to \lambda \quad \text{as} \quad n \to \infty
\]
and, since \( \lambda \in (0,1) \), we clearly have
\[
\frac{qB_{R_1^n}(\tilde{u}_n)}{q(\tilde{u}_n)} \geq \frac{\lambda}{2},
\]
for all \( n \in \mathbb{N} \) sufficiently large. On the other hand, there exists a constant \( \delta > 0 \) such that
\[
q(\tilde{u}_n) \geq \delta, \quad \forall n \in \mathbb{N},
\]
(3.6)
since otherwise we would obtain $q(\tilde{u}_n) \to 0$, which would imply $\tilde{u}_n \to 0$ and contradict the fact that $J(u_n) \to c > 0$. Therefore using Lemmas 3.2 and 3.5 together with the fact that $|\psi_n| \leq 1$ and $|\nabla \psi_n| \leq D$, for some $D > 0$, we obtain

$$0 < C_1 \leq \lambda \delta / 2 \leq (\delta / 2) q(\tilde{u}_n) \leq q_{B_{r_n}}(\tilde{u}_n) = q_{B_{r_n}}(\tilde{w}_n) \leq q(\tilde{w}_n) \leq Dq(u_n) \leq C_2,$$

and hence (i) is true. Next we prove (ii). (a)–(d) above implies that for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $R > 0$ such that

$$\tilde{\mu}_n(B_{r_n} \setminus B_R) = \tilde{\mu}_n(B_{r_n}) - \tilde{\mu}_n(B_{R}) \leq \lambda + \epsilon - \lambda + \epsilon = 2\epsilon, \quad \forall n \geq n_0.$$ 

In other words, we have

$$\tilde{\mu}_n(B_{r_n} \setminus B_R) \leq 2\epsilon, \quad \forall n \geq n_0.$$ 

Therefore, recalling that $\tilde{w}_n = 0$ outside $z_n + B_{r_n}$ and $|\psi_n| \leq 1$ and $|\nabla \psi_n| \leq D$, we obtain

$$q_{B_{r_n}}(\tilde{w}_n) = q_{B_{r_n} - B_R}(\tilde{w}_n) \leq Dq_{B_{r_n} - B_R}(\tilde{u}_n) \leq 2D\epsilon, \quad \forall n \geq n_0.$$ 

Hence (ii) is true. It remains to prove (iii). Let $\epsilon > 0$ and $R > 0$ be given as in (ii). Note that any $\phi \in C_0^\infty(\Omega)$ can be written as $\phi = \phi_1^1 + \phi_2^1$, with $\text{supp}(\phi_1^1) \subset \Omega' \times B_{R+1}$ and $\text{supp}(\phi_2) \subset \Omega' \times B_R$.

**Claim 2.** For some $K_0, \tilde{K}_0 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\phi_i^j\|_{1,p} \leq K_0 \|\phi\|_{1,p}, \quad i = 1, 2, \quad n \geq n_0,$$

$$\|\nabla J(\tilde{u}_n), \phi_i^j\| \leq \epsilon (p-1)/p \tilde{K}_0 \|\phi\|_{1,p}, \quad i = 1, 2, \quad n \geq n_0.$$ 

If Claim 2 is proved, we obtain

$$\nabla J(\tilde{u}_n) \to 0 \quad \text{in } W^{-1,p'},$$

so that (c) also holds true.

**Proof of Claim 2.** It is clear that there exists a constant $K_0 > 0$ such that $\|\phi_1^j\|_{1,p} \leq K_0 \|\phi\|_{1,p}$. Since $J'(\tilde{u}_n) = J'(u_n) \to 0$, we have, for all $n \in \mathbb{N}$ sufficiently large,

$$\|\nabla J(\tilde{u}_n), \phi_1^j\| = \|\nabla J(\tilde{u}_n), \phi_1^1\| \leq \epsilon \|\phi_1^1\|_{1,p} \leq \epsilon K_0 \|\phi\|_{1,p}.$$ 

By (1.2) and Holder’s inequality, we get

$$\|\nabla J(\tilde{u}_n), \phi_2^j\| \leq \left( \int_{\Omega' \times B_R} |\nabla \tilde{u}_n|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega' \times B_R} |\nabla \phi_2^j|^p dx \right)^{\frac{1}{p}}$$

$$+ K_1 \left( \int_{\Omega' \times B_R} |\tilde{u}_n|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega' \times B_R} |\phi_1^j|^p dx \right)^{\frac{1}{p}}$$

$$+ K_2 \left( \int_{\Omega' \times B_R} |\tilde{u}_n|^{p^*} dx \right)^{\frac{p+1}{p^*}} \left( \int_{\Omega' \times B_R} |\phi_2^j|^{p^*} dx \right)^{\frac{1}{p^*}},$$
so that we can obtain

\[ \left| \langle J(\tilde{u}_n), \phi_R^2 \rangle \right| \leq K_3 (q B^c_R (\tilde{u}_n))^{(p-1)/p} \| \phi_R^2 \|_{1,p} + K_4 (q B^c_R (\tilde{u}_n))^{(p^*-1)/p^*} \| \phi_R^2 \|_{1,p} \]

\[ \leq \epsilon^{(p-1)/p} K_0 \| \phi \|_{1,p}, \]

for some \( K_0 \). Hence the proof of (iii) is complete. \qed

References