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Schur Indices and Splitting Fields of the Unitary Reflection Groups

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The concept of a reflection in Euclidean space was generalized by Shephard [16]. A reflection in unitary space is a linear transformation of finite period with the property that all but one of its characteristic values are equal to 1. While reflections in Euclidean space must have period 2, a reflection in unitary space may have period m for any integer $m > 1$. A finite group generated by unitary reflections (called simply a *reflection group* throughout this paper) can be decomposed as a product of irreducible groups. Shephard and Todd [18] classified the irreducible groups, as listed in Table I. In this paper the representations (finite-dimensional over the field of complex numbers) of the reflection groups are studied and the following theorem is proved.

THEOREM 1. *Let G be a reflection group and let F be the field generated over $\mathbf{Q}$ by the values of the characters of G. Then each representation of G is similar to an F-representation.*

In other words, it is shown that F is a splitting field for G. The approach to this theorem is by way of the Schur index. Clearly, it is sufficient to assume that G is an irreducible reflection group and to show that the Schur index $m_F(\chi) = 1$ for each irreducible character χ of G. In fact, it is shown that $m_{\mathbf{Q}}(\chi) = 1$ except for the 24 characters listed in Theorem 2. The notation G_n refers to the order of listing of groups in Table I. The notation Z_m denotes a cyclic group of order m.

THEOREM 2. *If G is an irreducible reflection group and χ is an irreducible*

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TABLE I

The Irreducible Reflection Groups

	Symbol	Dimension	Order	Splitting field	Nontrivial division algebras
1.	$[3^{n-1}]$	n	$(n+1)!$	Q	
2.	$G(m, p, n)$	n	$m^n \cdot n!/p$	$Q(\epsilon_m)$	
3.	$[\ ]^m$	1	m	$Q(\epsilon_m)$	
4.	3[3]3	2	$2^3 \cdot 3$	$Q(-3^{1/2})$	$\Delta(2, \infty)$
5.	3[4]3	2	$2^3 \cdot 3^2$	$Q(-3^{1/2})$	$\Delta(2, \infty)$
6.	3[6]2	2	$2^4 \cdot 3$	$Q(-1^{1/2}, -3^{1/2})$	
7.	$\langle 3, 3, 2\rangle_6$	2	$2^4 \cdot 3^2$	$Q(-1^{1/2}, -3^{1/2})$	
8.	4[3]4	2	$2^5 \cdot 3$	$Q(-1^{1/2})$	$\Delta(3, \infty)$
9.	4[6]2	2	$2^6 \cdot 3$	$Q(\epsilon_8)$	
10.	4[4]3	2	$2^5 \cdot 3^2$	$Q(-1^{1/2}, -3^{1/2})$	$\Delta(3, \infty)$
11.	$\langle 4, 3, 2\rangle_{12}$	2	$2^6 \cdot 3^2$	$Q(\epsilon_8, -3^{1/2})$	
12.	$GL(2, 3)$	2	$2^4 \cdot 3$	$Q(-2^{1/2})$	
13.	$\langle 4, 3, 2\rangle_2$	2	$2^5 \cdot 3$	$Q(\epsilon_8)$	
14.	3[8]2	2	$2^4 \cdot 3^2$	$Q(-2^{1/2}, -3^{1/2})$	
15.	$\langle 4, 3, 2\rangle_6$	2	$2^5 \cdot 3^2$	$Q(\epsilon_8, -3^{1/2})$	
16.	5[3]5	2	$2^3 \cdot 3 \cdot 5^2$	$Q(\epsilon_5)$	$\Delta(2, \infty)$, $\Delta(3, \infty)$, $\Gamma(\infty, \infty)$
17.	5[6]2	2	$2^4 \cdot 3 \cdot 5^2$	$Q(\epsilon_5, -1^{1/2})$	
18.	5[4]3	2	$2^3 \cdot 3^2 \cdot 5^2$	$Q(\epsilon_5, -3^{1/2})$	$\Delta(2, \infty)$, $\Delta(3, \infty)$, $\Gamma(\infty, \infty)$
19.	$\langle 5, 3, 2\rangle_{30}$	2	$2^4 \cdot 3^2 \cdot 5^2$	$Q(\epsilon_5, -1^{1/2}, -3^{1/2})$	
20.	3[5]3	2	$2^3 \cdot 3^2 \cdot 5$	$Q(5^{1/2}, -3^{1/2})$	$\Delta(2, \infty)$, $\Delta(3, \infty)$, $\Gamma(\infty, \infty)$
21.	3[10]2	2	$2^4 \cdot 3^2 \cdot 5$	$Q(5^{1/2}, -1^{1/2}, -3^{1/2})$	
22.	$\langle 5, 3, 2\rangle_2$	2	$2^4 \cdot 3 \cdot 5$	$Q(5^{1/2}, -1^{1/2})$	
23.	[3, 5]	3	$2^3 \cdot 3 \cdot 5$	$Q(5^{1/2})$	
24.	$[1\ 1\ 1^4]^4$	3	$2^4 \cdot 3 \cdot 7$	$Q(-7^{1/2})$	
25.	3[3]3[3]3	3	$2^3 \cdot 3^4$	$Q(-3^{1/2})$	$\Delta(2, \infty)$
26.	3[3]3[4]2	3	$2^4 \cdot 3^4$	$Q(-3^{1/2})$	$\Delta(2, \infty)$
27.	$[1\ 1\ 1^4]^5$	3	$2^4 \cdot 3^3 \cdot 5$	$Q(-3^{1/2}, 5^{1/2})$	
28.	[3, 4, 3]	4	$2^7 \cdot 3^2$	Q	
29.	$[2\ 1\ 1]^4$	4	$2^9 \cdot 3 \cdot 5$	$Q(-1^{1/2})$	
30.	[3, 3, 5]	4	$2^6 \cdot 3^2 \cdot 5^2$	$Q(5^{1/2})$	$\Delta(2, 3)$
31.		4	$2^{10} \cdot 3^2 \cdot 5$	$Q(-1^{1/2})$	
32.	3[3]3[3]3[3]3	4	$2^7 \cdot 3^5 \cdot 5$	$Q(-3^{1/2})$	$\Delta(2, \infty)$, $\Delta(5, \infty)$
33.	$[2\ 2\ 1]^3$	5	$2^7 \cdot 3^4 \cdot 5$	$Q(-3^{1/2})$	
34.	$[3\ 2\ 1]^3$	6	$2^9 \cdot 3^7 \cdot 5 \cdot 7$	$Q(-3^{1/2})$	
35.	$[3^{2,2,1}]$	6	$2^7 \cdot 3^4 \cdot 5$	Q	
36.	$[3^{3,2,1}]$	7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	Q	
37.	$[3^{4,2,1}]$	8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	Q	

character of G, then $m_{\mathbf{Q}}(\chi) = 1$ *unless* $m_{\mathbf{Q}}(\chi) = 2$ *and one of the following cases occur.*

(i) $G = G_4, G_5, G_{25}$, *or* G_{26}, *and* χ *is the faithful rational-valued character of* $SL(2, 3)$ *or* $SL(2, 3) \times Z_2$.

(ii) $G = G_8$ *or* G_{10}, *and* χ *is the faithful rational-valued character of the dicyclic group of order* 12.

(iii) $G = G_{16}, G_{18}$, *or* G_{20}, *and* χ *is a faithful character of* $SL(2, 5)$. *There are* 2 *rational-valued characters with degrees* 4 *and* 6, *and* 2 *characters with degree* 2 *and values that generate* $\mathbf{Q}(5^{1/2})$.

(iv) $G = G_{30}$ *and* χ *is the faithful rational-valued character of* G *with degree* 48.

(v) $G = G_{32}$ *and* χ *is a faithful rational-valued character of* $Sp(4, 3)$. *There are* 4 *such characters, with degrees* 20, 60, 64, *and* 80.

The result stated in Theorem 1 generalizes what is known about the Euclidean reflection groups that are Weyl groups. The Weyl groups $W(A_n)$ are the symmetric groups Sym$(n + 1)$. As is well known, Young [21] showed that each representation of Sym(n) is similar to a rational representation. It was later shown by a number of authors (see [1] for references) that each representation of an irreducible Weyl group is similar to a rational representation.

2. The Schur Index

A number of propositions concerning the Schur index are listed in this section for easy reference later. Elementary properties of the Schur index can be found in [9, Chap. II]. Throughout this paper "character" means an absolutely irreducible complex-valued character, unless reducible character or modular character is specified. If χ is a character of G and F is a field (of characteristic 0), then $F(\chi)$ denotes the field generated over F by the values of χ. If a field is not specified in reference to the Schur index, then the field is assumed to be $\mathbf{Q}$. In the following, $\mathbf{Q}_p$ denotes the p-adic completion of $\mathbf{Q}$ and (χ, ϕ) denotes the ordinary character inner product.

Corresponding to each character χ of G with $F = \mathbf{Q}(\chi)$, there is a simple component A of the group algebra FG such that χ does not vanish everywhere on A. The algebra A is a matrix algebra over a division algebra D with center F. The index of D is $m_{\mathbf{Q}}(\chi)$. Furthermore, for each extension field E of F, the index of $D \otimes_F E$ is $m_E(\chi)$. Since F is an algebraic number field, then the sum of the local invariants of D must be congruent to 0(mod 1). In particular, if $m_{\mathbf{Q}}(\chi) = 2$, then D must have invariant $\frac{1}{2}$ at an even number of primes of

F and $m_E(\chi) = 2$ for an even number of $\mathfrak{P}$-adic completions E of F. The division algebras corresponding to the characters listed in Theorems 2 are determined in this paper. Although the following notation is not standard, it is useful for identifying these division algebras. Let $\Delta(p, q)$ denote the quaternion division algebra with center $\mathbf{Q}$ and nonzero invariants $\frac{1}{2}$ at the distinct rational primes p and q. Let $\Gamma(\infty, \infty)$ denote the quaternion division algebra with center $\mathbf{Q}(5^{1/2})$ and nonzero invariants $\frac{1}{2}$ at the 2 infinite primes of $\mathbf{Q}(5^{1/2})$. The division algebras that occur in correspondence to characters of the irreducible reflection groups are listed in Table I.

In the following, χ is a character of G.

(2.1) (Brauer–Speiser theorem). If χ is real-valued, then $m_{\mathbf{Q}}(\chi) \leqslant 2$. In particular, if $\chi(1)$ is odd, then $m_{\mathbf{Q}}(\chi) = 1$.

(2.2) Suppose ϕ is a character of a subgroup H of G and $\mathbf{Q}(\phi) = \mathbf{Q}(\chi)$. If (χ, ϕ) is relatively prime to $m_{\mathbf{Q}}(\chi)$ and to $m_{\mathbf{Q}}(\phi)$, then $m_F(\phi) = m_F(\chi)$ for all fields F.

(2.3) If $p \nmid |G|$ for a finite prime p, then $m_{\mathbf{Q}_p}(\chi) = 1$.

(2.4) (Witt [20]) $m_{\mathbf{Q}_p}(\chi) \mid p - 1$ if p is odd, and $m_{\mathbf{Q}_p}(\chi) \leqslant 2$ if $p = 2$ or ∞.

(2.5) If F is an algebraic number field, and $|\mathbf{Q}_p F(\chi) : \mathbf{Q}_p(\chi)|$ is divisible by $m_{\mathbf{Q}_p}(\chi)$ for all primes p, then $m_F(\chi) = 1$.

(2.6) (Berman [5]) Suppose that G is q-hyperelementary for a prime q; i.e., that G contains a cyclic normal subgroup and the quotient group is a q-group. If ψ is an irreducible p-modular constituent of χ for $p \neq q$, then $m_{\mathbf{Q}_p}(\chi) = |\mathbf{Q}_p(\chi, \psi) : \mathbf{Q}_p(\chi)|$.

The next two results are special cases of [3, Theorem 4].

(2.7) If $\mathbf{Q}(\chi) = \mathbf{Q}(-1^{1/2})$, then $m_{\mathbf{Q}_p}(\chi) = 1$ if $p \not\equiv 1 \pmod 4$.

(2.8) If $\mathbf{Q}(\chi) = \mathbf{Q}(-3^{1/2})$, then $m_{\mathbf{Q}_p}(\chi) = 1$ if $p \not\equiv 1 \pmod 3$.

The problem of determining Schur indices over the real field $\mathbf{R}$ was solved by Frobenius and Schur (see [10, Sect. 3]). They defined

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

and proved the following two statements.

(2.9) If χ is not real valued, then $\nu(\chi) = 0$. If χ is real-valued, then $\nu(\chi) = 1$ if $m_{\mathbf{R}}(\chi) = 1$ and $\nu(\chi) = -1$ if $m_{\mathbf{R}}(\chi) = 2$.

(2.10) If t is the number of involutions in G and $\chi_1, \ldots, \chi_n$ are all characters of G, then $t + 1 = \sum_{i=1}^{n} \nu(\chi_i)\, \chi_i(1)$.

3. Some Linear Groups

To facilitate the case-by-case study of the irreducible reflection groups, this section is devoted to the calculation of Schur indices for some linear groups that occur as quotient groups of the reflection groups.

The group $SL(2, 3)$ has one character with Schur index 2. This group has a center of order 2 and the central quotient group is Alt(4), so the nonfaithful characters have Schur index 1. Each of the faithful characters is an extension of the faithful character ϕ of the quaternion Sylow 2-subgroup. The character ϕ is rational-valued and $m_{\mathbf{Q}}(\phi) = 2$. Two extensions of ϕ have values that generate $\mathbf{Q}(-3^{1/2})$, so these characters have Schur index 1. The other extension χ is rational-valued so $m_{\mathbf{Q}}(\chi) = 2$. The division algebra corresponding to χ is $\Delta(2, \infty)$. Furthermore, if $F = \mathbf{Q}(-1^{1/2})$, $\mathbf{Q}(-2^{1/2})$, or $\mathbf{Q}(-3^{1/2})$, then $m_F(\chi) = m_F(\phi) = 1$. In particular, $\mathbf{Q}(-3^{1/2})$ is a splitting field for $SL(2, 3)$.

The characters of $GL(2, 3)$ can be rationally represented with 2 exceptions. The character χ of $SL(2, 3)$ extends to a pair of characters θ_1 and θ_2 of $GL(2, 3)$ with $\mathbf{Q}(\theta_i) = \mathbf{Q}(-2^{1/2})$. Since $m_F(\chi) = 1$ for $F = \mathbf{Q}(-2^{1/2})$, then $m_{\mathbf{Q}}(\theta_i) = 1$. Thus, each character of $GL(2, 3)$ has Schur index 1.

The characters of $SL(2, 5)$ were known to Schur, and the Schur indices of these characters were probably also known to him. Janusz [12] has computed the Schur indices and division algebras for $SL(2, q)$ for all q. The five nonfaithful characters of $SL(2, 5)$ are the characters of Alt(5), so each has Schur index 1. Each of the four faithful characters has Schur index 2. The characters with degrees 4 and 6 are rational-valued and correspond to the division algebras $\Delta(3, \infty)$ and $\Delta(2, \infty)$, respectively. The other two characters have degree 2 and their values generate the field $\mathbf{Q}(5^{1/2})$. Each of these corresponds to the division algebra $\Gamma(\infty, \infty)$. Let $F = \mathbf{Q}(5^{1/2}, \epsilon_n)$, where ϵ_n denotes a primitive nth root of unity for $n = 3$, 4, or 5. Since $| \mathbf{Q}_p F : \mathbf{Q}_p | = 2$ for $p = 2, 3$, or ∞, then, by (2.5), $m_F(\chi) = 1$ if χ is one of the above mentioned characters with degree 4 or 6. Since $\mathbf{Q}_\infty = \mathbf{R}$ and $\mathbf{R}F = \mathbf{C}$, then $m_F(\chi) = 1$ if χ is one of the characters of degree 2. Thus, F is a splitting field for $SL(2, 5)$.

The group $SL^{\pm}(2, 5)$ consists of the elements of $GL(2, 5)$ with determinant ± 1 and it contains a central element of order 4. Each character of $SL^{\pm}(2, 5)$ is an extension of a character of $SL(2, 5)$. The nonfaithful characters clearly have Schur index 1. If χ is a faithful character of $SL^{\pm}(2, 5)$, then $\mathbf{Q}(\chi) = \mathbf{Q}(-1^{1/2})$ or $\mathbf{Q}(-1^{1/2}, 5^{1/2})$. If $F = \mathbf{Q}(\chi)$ and ϕ is the character of $SL(2, 5)$ that extends to χ, then $m_F(\phi) = 1$ by the last paragraph. Hence, $m_{\mathbf{Q}}(\chi) = 1$. Thus, each character of $SL^{\pm}(2, 5)$ has Schur index 1.

The group $PSp(4, 3)$ is the unique simple group of order 25920. The characters of $PSp(4, 3)$ were given by Frame [11]. This group is isomorphic to the commutator subgroup G^+ of the Weyl group $W(E_6)$. Since G^+ has index

2 in $W(E_6)$, then, each character of $W(E_6)$ either remains irreducible or decomposes into 2 distinct characters when restricted to G^+. Since each character of $W(E_6)$ can be rationally represented [1], then each character of G^+ has Schur index 1.

The group $Sp(4, 3)$ is the twofold central extension of $PSp(4, 3)$. The character tables for $Sp(4, q)$, q odd, were given by Srinivasan [19]. The characters for $Sp(4, 3)$ are given in Table II, and the notation used coincides with that used by Srinivasan. Ten characters are not listed explicitly in the table; these are the complex conjugates of characters that are listed. The 18 conjugacy classes that are not represented in the table can be obtained from listed classes by taking inverses or by multiplying by the central involution z. Since the characters of $PSp(4, 3)$ have Schur index 1, there are nine characters to consider. Five characters have values that generate the field $\mathbf{Q}(-3^{1/2})$. Since $|Sp(4, 3)| = 2^7 \cdot 3^4 \cdot 5$, then each of these characters has Schur index 1 by (2.3) and (2.8). The remaining four characters are $\chi_1^{(1)}$, χ_2, ξ_1, and ξ_1'. These are rational-valued. By the Frobenius–Schur formula (2.10) for counting involutions, these four characters have no real splitting fields. Hence, each of these characters has Schur index 2.

Let

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{in} \quad Sp(4, 3).$$

Then, x belongs to the class A_{32}, its centralizer has order $2^3 \cdot 3^3$, and its normalizer has order $2^4 \cdot 3^3$. The Sylow 2-subgroup S of $\mathbf{N}(\langle x \rangle)$ is semidihedral and $S_0 = S \cap \mathbf{C}(x)$ is dihedral. One can take $S = \langle a, b \rangle$ and $S_0 = \langle a^2, b \rangle$, where

$$a = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then, $a^2 = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, $b^2 = 1$, and $bab = a^3$. Let $H = \langle x \rangle S$. Then, H has a faithful rational-valued character ψ of degree 4 formed by inducing a faithful character of $\langle x \rangle S_0$. The character ψ vanishes outside of $\langle x, a^4 \rangle$. Furthermore, $\psi(x) = \psi(x^{-1}) = -2$, $\psi(a^4) = -4$, and $\psi(xa^4) = \psi(x^{-1}a^4) = 2$. By the Frobenius–Schur formula (2.9), ψ has no real splitting field. Thus, $m_{\mathbf{Q}}(\psi) = 2$. Since $|H| = 2^4 \cdot 3$ and $m_{\mathbf{R}}(\psi) = 2$, then $m_F(\psi) = 2$ for $F = \mathbf{Q}_2$ or $\mathbf{Q}_3$. H is 2-hyperelementary, and an irreducible 3-modular constituent α of ψ is a faithful ordinary character of S with degree 2. Thus, $\mathbf{Q}(\alpha) =$

TABLE II

Character Table for $Sp(4, 3)$

	1	z	A_{21}	A_{31}	A_{32}	A_{41}	B_1	B_2	B_6	B_7	C_1	C_{21}	D_1	D_{21}	D_{32}	D_{34}
Order	1	2	3	3	3	9	5	8	4	12	4	12	2	6	6	6
Centralizer	1	1	1296	108	216	18	10	8	96	12	96	24	576	144	36	36
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_1^{(2)}$	64	64	-8	-2	4	1	-1	0	0	0	0	0	0	0	0	0
χ_6	20	20	2	-1	5	-1	0	0	4	1	0	0	4	-2	1	1
χ_7	60	60	6	-3	-3	0	0	0	4	1	0	0	-4	2	-1	-1
ξ_{21}	40	40	$-8-6\omega$	1	-2	ω	0	0	0	0	0	0	-8	-2ω	1	-2ω
ϕ_1	10	10	$-2+3\omega$	1	1	ω	0	0	-2	1	2	$-\omega$	2	$2+3\omega$	-1	-1
ϕ_3	30	30	$3+9\omega$	0	-3	0	0	0	2	-1	2	$1+\omega$	6	$-1+\omega$	0	$-1-2\omega$
ϕ_9	30	30	3	3	3	0	0	0	2	-1	-2	1	-10	-1	-1	-1
θ_1	45	45	-9ω	0	0	0	0	1	-3	0	1	ω	-3	3ω	0	0
θ_3	5	5	$2+3\omega$	2	-1	$1+\omega$	0	-1	1	1	1	ω	-3	$-2-\omega$	0	$1+2\omega$
θ_9	24	24	6	3	0	0	-1	0	0	0	0	0	8	2	-1	2
θ_{10}	6	6	-3	0	3	0	1	0	2	-1	2	-1	-2	1	-2	1
θ_{11}	15	15	-3	3	0	0	0	1	3	0	-1	-1	7	1	1	-2
θ_{12}	15	15	6	0	3	0	0	-1	-1	-1	3	0	-1	2	2	-1
θ_{13}	81	81	0	0	0	0	1	-1	-3	0	-3	0	9	0	0	0
$\chi_1^{(1)}$	64	-64	-8	-2	4	1	-1	0	0	0	0	0	0	0	0	0
χ_2	80	-80	8	-4	2	-1	0	0	0	0	0	0	0	0	0	0
ξ_1	20	-20	-7	2	2	-1	0	0	0	0	2	-1	0	3	0	0
ξ_1'	60	-60	-3	0	-6	0	0	0	0	0	-2	1	0	3	0	0
ξ_{21}'	20	-20	$-1-6\omega$	-1	-4	$1+\omega$	0	0	0	0	2	-1	0	$-1+2\omega$	$1+2\omega$	0
ϕ_5	20	-20	$5-3\omega$	2	2	$-\omega$	0	0	0	0	2	$1+\omega$	0	$3+3\omega$	0	0
ϕ_7	60	-60	$-3-9\omega$	3	0	0	0	0	0	0	-2	$-1-\omega$	0	$-1+\omega$	$-1-2\omega$	0
θ_5	36	-36	9ω	0	0	0	1	0	0	0	2	$-\omega$	0	3ω	0	0
θ_7	4	-4	$1+3\omega$	1	-2	ω	-1	0	0	0	2	$1+\omega$	0	$-1+\omega$	$-1-2\omega$	0

$\mathbf{Q}(-2^{1/2})$, since S is semidihedral. Hence, by (2.6), $m_{\mathbf{Q}_3}(\psi) = | \mathbf{Q}_3(-2^{1/2}) : \mathbf{Q}_3 | = 1$. Therefore, $m_{\mathbf{Q}_2}(\psi) = 2$. Computations show that $(\chi_{2H}, \psi)_H = 13$, $(\xi_{1H}, \psi)_H = 3$, and $(\xi'_{1H}, \psi)_H = 11$. Hence, by (2.2), $m_F(\chi_2) = m_F(\xi_1) = m_F(\xi_1') = m_F(\psi)$ for all fields F. In particular, since $| \mathbf{Q}_p(-3^{1/2}) : \mathbf{Q}_p | = 2$ for $p = 2$ or ∞, then, by (2.5), each of these characters has Schur index 1 over $\mathbf{Q}(-3^{1/2})$. Furthermore, the division algebra corresponding to each of these characters is the ordinary quaternion algebra $\Delta(2, \infty)$.

Now, let y be an element of order 5 in $Sp(4, 3)$. Then, $| \mathbf{C}(y)| = 10$ and $| \mathbf{N}(\langle y \rangle)| = 40$. Thus, $N = \mathbf{N}(\langle y \rangle)$ has a faithful rational-valued character β of degree 4 that is induced from a faithful linear character of $\mathbf{C}(y) = \langle y, z \rangle$. Clearly, $\beta(y^i) = -1$ and $\beta(y^i z) = 1$ for $i = 1,..., 4$. Let $\chi = \chi_1^{(1)}$. Computations show that $(\chi_N, \beta)_N = 13$. Thus, $m_F(\beta) = m_F(\chi)$ for all fields F. In particular, β has no real splitting field. Thus, by the Frobenius–Schur formula (2.9), N must have a cyclic Sylow 2-subgroup T. An irreducible 5-modular constituent ρ of β is a faithful ordinary character of T, so $\mathbf{Q}(\rho) = \mathbf{Q}(\epsilon_8)$. Since N is 2-hyperelementary, then, by (2.6), $m_{\mathbf{Q}_5}(\chi) = m_{\mathbf{Q}_5}(\beta) = | \mathbf{Q}_5(\epsilon_8) : \mathbf{Q}_5 | = 2$. The corresponding division algebra is $\Delta(5, \infty)$. Furthermore, since $| \mathbf{Q}_p(-3^{1/2}) : \mathbf{Q}_p | = 2$ for $p = 5$ or ∞, then χ has Schur index 1 over $\mathbf{Q}(-3^{1/2})$. Hence, $\mathbf{Q}(-3^{1/2})$ is a splitting field for $Sp(4, 3)$.

4. Reflection Groups

The first three entries in Shephard and Todd's list of irreducible reflection groups represent three infinite families of groups. The first family consists of the symmetric groups. The third consists of the cyclic groups; all characters here are linear. The other family consists of the groups $G(m, p, n)$. This class of groups includes the Weyl groups $W(B_l)$ and $W(D_l)$, and also the dihedral groups. In this section, it is shown that each character of the groups $G(m, p, n)$ has Schur index 1. The remaining 34 irreducible reflection groups are treated in successive sections.

Let m and n be integers > 1 and let $p \mid m$, where p is not necessarily prime. Let $\{u_1, ..., u_n\}$ be a basis for unitary n-space U and let $\epsilon = \epsilon_m$ be a primitive mth root of unity. Let

$$B = \{(b_1, ..., b_n) : 1 \leqslant b_i \leqslant m\}$$

and let

$$A = \{(a_1, ..., a_n) \in B : a_1 + \cdots + a_n \equiv 0 \pmod{p}\}.$$

For $a \in A$ and $\sigma \in \mathrm{Sym}(n)$, define the transformation $T_{a,\sigma}$ on U by $T_{a,\sigma}(u_i) = \epsilon^{a_i} u_{\sigma(i)}$. The group $G = G(m, p, n)$ consists of all such $T_{a,\sigma}$ and has order $m^n \cdot n!/p$. The subgroup $\{T_{a,1} : a \in A\}$ can be identified with A. The subgroup

$\{T_{1,\sigma} : \sigma \in \mathrm{Sym}(n)\} \cong \mathrm{Sym}(n)$ and is denoted by S. A typical element of S is denoted simply by σ. Clearly, $G = AS$, with $A \lhd G$ and $A \cap S = \langle 1 \rangle$. For $a \in A$ and $b \in B$, let $a \circ b = \sum_{i=1}^{n} a_i b_i$. The linear characters of A are of the form λ_b, where $\lambda_b(a) = \epsilon^{a \circ b}$. Obviously, $\lambda_b = \lambda_{b'}$ if and only if $a \circ b \equiv a \circ b' \pmod{m}$ for all $a \in A$. The following proposition and corollary are essential in showing that each character of G has Schur index 1. In the corollary, wr means wreath product.

PROPOSITION. *Suppose that* $H = RC$ *where* $R \lhd H$, $R \cap C = \langle 1 \rangle$, *and that* $R = D_1 \times \cdots \times D_s$. *Suppose that the factors* D_i *are permuted by the elements of* C *in a manner such that if* $x \in C$ *and* $x^{-1} D_i x = D_i$, *then* $x \in \mathbf{C}(D_i)$. *Let* F *be a field of characteristic* 0. *For each* D_i, *let* β_i *be a character of* D_i *such that* β_i *can be afforded by an* F*-representation. Suppose that* $\rho = \beta_1 \cdots \beta_s$ *is invariant in* H. *Then, there exists a character* ϕ *of* H *that extends* ρ *and that can be afforded by an* F*-representation.*

Proof. Since ρ is invariant in H, then, for $x \in H$, $x^{-1} D_i x = D_j$ implies that $\beta_i^x = \beta_j$.

Let e_1 be a primitive idempotent of FD_1 such that $FD_1 e_1$ is a minimal left ideal affording β_1. For each D_i that is C-conjugate to D_1, let $e_i = x^{-1} e_1 x$ for some $x \in C$ such that $D_i = x^{-1} D_1 x$. The hypothesis guarantees that e_i does not depend on the choice of x; if $x, y \in C$ and $x^{-1} D_1 x = y^{-1} D_1 y$, then $xy^{-1} \in \mathbf{C}(D_1)$, so $x^{-1} e_1 x = y^{-1} e_1 y$. Repeat this process for each C-orbit of $\{D_1, ..., D_s\}$, selecting one idempotent for each orbit and using it to determine others. The result of this process is a system of primitive idempotents $e_1, ..., e_s$ such that $FD_i e_i$ is a minimal left ideal of FD_i affording β_i for each i and such that if $x \in C$ and $x^{-1} D_i x = D_j$, then $x^{-1} e_i x = e_j$. Let $e_0 = e_1 \cdots e_s$. Then, FRe_0 is a minimal left ideal of FR affording ρ and $x^{-1} e_0 x = e_0$ for all $x \in C$.

Set $f = (1/|G|) \sum_{x \in C} x$. Then, f is an idempotent in FC such that $FCf = Ff$. Set $e = e_0 f = f e_0$. Then, $FHe = FRCe_0 f = FRe_0 FCf = FRe_0 Ff = FRe_0 f$. Hence, $\dim_F FHe \leqslant \dim_F FRe_0$.

Let ϕ be the (possibly reducible) character of H afforded by FHe. Since FHe_0 affords ρ^H and $FHe \subseteq FHe_0$, then $\phi \subseteq \rho^H$. Since $\rho(1) = \dim_F FRe_0 \geqslant \dim_F FHe = \phi(1)$, then $\rho(1) = \phi(1)$. Hence, ϕ extends ρ and ϕ is irreducible.

COROLLARY. *Each character of* $\mathrm{Sym}(r) \operatorname{wr} \mathrm{Sym}(l)$ *can be afforded by a rational representation.*

Proof. Let $M = \mathrm{Sym}(r) \operatorname{wr} \mathrm{Sym}(l)$ and let N be the normal subgroup of M that is a direct product $N_1 \times \cdots \times N_l$ of l copies of $\mathrm{Sym}(r)$. Let ψ be a character of M and let μ be an irreducible constituent of ψ_N. Let ξ be a character of the inertial group $I(\mu)$ such that $\mu \subseteq \xi_N$ and $\xi^M = \psi$. Since

$\mathrm{Sym}(l)$ permutes the N_i, then $I(\mu) = I_1 \times \cdots \times I_t$, where $I_i \cong \mathrm{Sym}(r)$ wr $\mathrm{Sym}(l_i)$ and $l_1 + \cdots + l_t = l$. Let $R_i = I_i \cap N$, so that R_i is the direct product of l_i copies of $\mathrm{Sym}(r)$. Then, $N = R_1 \times \cdots \times R_t$ and $\mu = \rho_1 \cdots \rho_t$, where ρ_i is a character of R_i. Furthermore, $I_i = R_i C_i$, where $C_i \cong \mathrm{Sym}(l_i)$, $R_i \cap C_i = \langle 1 \rangle$, and C_i permutes the copies of $\mathrm{Sym}(r)$ lying in R_i in the manner described in the proposition. The hypotheses of the proposition are satisfied, so ρ_i can be extended to a rationally represented character ϕ_i of I_i. Let $\phi = \phi_1 \cdots \phi_t$. Then, ϕ extends μ and $\xi = \phi\alpha$ for some character α of $I(\mu)/N$. Since $I(\mu)/N \cong \mathrm{Sym}(l_1) \times \cdots \times \mathrm{Sym}(l_t)$, then α can be rationally represented. Hence, ξ and $\psi = \xi^M$ can be affordered by rational representations.

Now, let χ be a character of $G = G(m, p, n)$ and let $\lambda = \lambda_b$ be a fixed linear character of A with $\lambda \subseteq \chi_A$. Let $I = I(\lambda)$ and let $I_0 = I \cap S$. Then, $I = AI_0$ and

$$I_0 = \{\sigma \in S : a_1b_1 + \cdots + a_nb_n \equiv a_1b_{\sigma(1)} + \cdots + a_nb_{\sigma(n)} (\mathrm{mod}\ m) \text{ for all } a \in A\}.$$

Let $X = \{1,\ldots, n\}$. Define the equivalence relation "$\sim$" on X by $i \sim j$ if and only if $b_i = b_j$. Let $X_1, \ldots, X_k$ be the equivalence classes. By taking $a = (\ldots, 0, 1, 0,\ldots, 0, m-1, 0,\ldots)$, it is clear that when $\sigma \in I_0$, then $b_i = b_j$ if and only if $b_{\sigma(i)} = b_{\sigma(j)}$. Hence, I_0 permutes the sets $X_1, \ldots, X_k$. Let $Y = \{X_1, \ldots, X_k\}$ and let $Y_1, \ldots, Y_t$ be the orbits under action of I_0. If $\sigma \in S$, then $\sigma \in I_0$ if and only if σ permutes the X_i's and fixes the Y_j's setwise. For each j, let $M_j = \{\sigma \in I_0 : \sigma \text{ fixes } Y_j \text{ setwise and fixes all elements of } X \text{ not involved in } Y_j\}$. All X_i lying in Y_j have a common size r_j and M_j is naturally isomorphic to $\mathrm{Sym}(r_j)$ wr $\mathrm{Sym}(l_j)$, where l_j is the length of the orbit Y_j. Furthermore, $I_0 = M_1 \times \cdots \times M_t$. In particular, by the corollary, each character of I_0 can be rationally represented.

Let θ be the character of I such that $\lambda \subseteq \theta_A$ and $\theta^G = \chi$. Since λ is a linear character and $I(\lambda)$ is a semidirect product AI_0, then λ can be extended to a character ψ of I by $\psi(ay) = \lambda(a)$ for $a \in A$, $y \in I_0$. Thus, the character θ of I is of the form $\psi\eta$, where η is a character of I_0. Let $E = \mathbf{Q}(\lambda) = \mathbf{Q}(\theta)$ and let $F = \mathbf{Q}(\chi)$. Then $\chi_I \supseteq \Sigma\theta^\tau$, where τ ranges over $\mathrm{Gal}(E/F)$. By Mackey decomposition of $(\theta^G)_I = \chi_I$, there exists $x \in G$ such that

$$\theta^\tau \subseteq ((\theta^x)_{x^{-1}Ix \cap I})^I.$$

But for $a \in A$, $\theta(a) = \lambda(a)\,\eta(1)$, so $\lambda^\tau = \lambda^x$. Thus, if $y \in I$,

$$\lambda^{xyx^{-1}} = (\lambda^\tau)^{yx^{-1}} = (\lambda^{yx^{-1}})^\tau = (\lambda^{x^{-1}})^\tau = \lambda.$$

Thus, $x \in \mathbf{N}(I)$. Hence, $\theta^\tau = \theta^x$. Therefore, if $J = \{x \in G : \theta^x \text{ is algebraically conjugate to } \theta\}$, then $I \lhd J$ and $\mathbf{Q}(\theta^J) = F$. Hence, $m_{\mathbf{Q}}(\chi) = m_{\mathbf{Q}}(\theta^J)$.

Obviously, $J = AJ_0$, where $I_0 \lhd J_0 \subseteq S$. Since η is rational-valued, then $J_0 \subseteq I(\eta)$. Since $I_0 = M_1 \times \cdots \times M_t$, then $\eta = \eta_1 \cdots \eta_t$ for characters η_i of M_i. Each element of J_0 must permute the M_i's and the η_i's, and hence, must permute the X_i's and Y_j's. Put an ordering on the elements of each X_i and each Y_j. Let $\sigma_1 \in J_0$ such that $\sigma_1^{-1} M_i \sigma_1 = M_j$. Then, there exists $\sigma_2 \in M_j$ such that $\sigma_2\sigma_1 : Y_i \to Y_j$ preserves the ordering and $\sigma_2\sigma_1 : X_{i'} \to X_{j'}$ preserves the ordering. Let $T = \{\sigma \in J_0 : \sigma$ preserves the orderings of the Y_j's and the X_i's$\}$. Then, $I_0 \cap T = \langle 1 \rangle$ and the argument above shows that T contains enough elements so that $J_0 = I_0 T$. Furthermore, the conditions of the proposition are satisfied so that η can be extended to a character ϕ of J_0 that can be rationally represented. Since $J = IT$, then, for

$$x \in J_0,$$

$$\theta^J(x) = \sum_{\sigma \in T} \theta(\sigma^{-1} x \sigma) = \sum_{\sigma \in T} \psi(\sigma^{-1} x \sigma)\, \eta(\sigma^{-1} x \sigma) = \sum_{\sigma \in T} \eta(\sigma^{-1} x \sigma) = \eta^{J_0}(x).$$

Hence, $(\theta^J)_{J_0} = \eta^{J_0}$. Therefore, $(\theta^J, \phi^J)_J = ((\theta^J)_{J_0}, \phi)_{J_0} = (\eta^{J_0}, \phi) = 1$. Since ϕ can be rationally represented, then $m_{\mathbf{Q}}(\theta^J) = 1$. Therefore, $m_{\mathbf{Q}}(\chi) = 1$ for all characters χ of $G(m, p, n)$.

5. Two-Dimensional Groups

There are 19 two-dimensional irreducible reflection groups other than the groups that belong to the infinite families mentioned in the preceding section. The Schur indices of these groups are studied in this section. The generators and relations given by Shephard and Todd make it very easy to identify the two-dimensional groups with other known groups. The collineation groups of these groups are the tetrahedral, octahedral, and icosahedral groups.

The two-dimensional reflection groups related to the tetrahedral group Alt (4) are G_4, G_5, G_6, and G_7. The group G_4 is isomorphic to $SL(2, 3)$, the binary tetrahedral group. As observed in Section 3, this group has one rational-valued character with Schur index 2. The group G_5 is isomorphic to $SL(2, 3) \times Z_3$, so this group also has one character with Schur index 2. Each of these two groups has splitting field $\mathbf{Q}(-3^{1/2})$, the field generated by the values of the characters. The group G_6 is an extension of $SL(2, 3)$ by a central element of order 4 whose square is the central involution z of $SL(2, 3)$. The quotient group $G_6/\langle z \rangle$ is isomorphic to $\text{Alt}(4) \times Z_2$, so the nonfaithful characters of G_6 have Schur index 1. Each faithful character is an extension of a character of $SL(2, 3)$ and has $-1^{1/2}$ in the field generated by its values. Since all characters of $SL(2, 3)$ have Schur index 1 over $\mathbf{Q}(-1^{1/2})$, then all characters of G_6 have Schur index 1. Since G_7 is isomorphic to $G_6 \times Z_3$, then each character of G_7 has Schur index 1.

The groups G_8 through G_{15} have collineation groups isomorphic to the octahedral group Sym(4). The group G_8 has center of order 4 and its Sylow 2-subgroups are nonabelian with no elements of order 8. There is a normal subgroup of G_8 whose quotient group is dicyclic of order 12. The dicyclic group has four nonfaithful characters with Schur index 1. The faithful character χ of the dicyclic group is rational valued with degree 2. By the Frobenius–Schur formula (2.9), χ has no real splitting field. Furthermore, its 3-modular character is the faithful ordinary character of a cyclic group of order 4, so $M_{\mathbf{Q}_3}(\chi) = 2$ by (2.6). Hence, $m_{\mathbf{Q}}(\chi) = 2$ and the corresponding division algebra is $\Delta(3, \infty)$. Each of the remaining characters of G_8 has values that generate $\mathbf{Q}(-1^{1/2})$, so by (2.3) and (2.7), each of these has Schur index 1. Since $| \mathbf{Q}_p(-1^{1/2}) : \mathbf{Q}_p | = 2$ for $p = 3$ or ∞, then, by (2.5), $m_F(\chi) = 1$ if $F = \mathbf{Q}(-1^{1/2})$. Hence, $\mathbf{Q}(-1^{1/2})$ is a splitting field for G_8.

The group G_9 is an extension of G_8 by a central element of order 8 whose square z is in G_8. Since $G_9/\langle z \rangle$ is isomorphic to Sym (4) $\times Z_2$, then the characters with kernel containing z can be rationally represented. Each of the remaining characters has values that generate a field containing $-1^{1/2}$ and is an extension of a character of G_8. Since the characters of G_8 have Schur index 1 over $\mathbf{Q}(-1^{1/2})$, then all characters of G_9 have Schur index 1. The groups G_{10} and G_{11} are isomorphic to $G_8 \times Z_3$ and $G_9 \times Z_3$, respectively. Hence, G_{10} has one character with Schur index 2 while each character of G_{11} has Schur index 1.

The group G_{12} is isomorphic to $GL(2, 3)$. As noted in Section 3, each character of this group has Schur index 1. The group G_{13} is an extension of $GL(2, 3)$ by a central element whose square is in $GL(2, 3)$. Thus, each character of G_{13} is an extension of a character of $GL(2, 3)$ and has Schur index 1. The groups G_{14} and G_{15} are isomorphic to $G_{12} \times Z_3$ and $G_{13} \times Z_3$, respectively, so each character has Schur index 1.

The remaining two-dimensional groups have collineation groups that are isomorphic to the icosahedral group Alt(5). The binary icosahedral group is $SL(2, 5)$. The groups G_{16}, G_{18}, and G_{20} are isomorphic to $SL(2, 5) \times Z_5$, $SL(2, 5) \times Z_{15}$, and $SL(2, 5) \times Z_3$, respectively. Hence, as noted in Section 3, each of these groups has four rational-valued characters with Schur index 2. Since each of these characters has Schur index 1 over $\mathbf{Q}(-3^{1/2})$ and over $\mathbf{Q}(\epsilon_5)$, then Theorem 1 holds for these groups. The groups G_{17}, G_{19}, G_{21}, and G_{22} are isomorphic to $SL^{\pm}(2, 5) \times Z_5$, $SL^{\pm}(2, 5) \times Z_{15}$, $SL^{\pm}(2, 5) \times Z_3$, and $SL^{\pm}(2, 5)$, respectively. By the results of Section 3, each character of these groups has Schur index 1. This completes the treatment of the two-dimensional irreducible reflection groups.

6. Some Higher-Dimensional Groups

The remaining groups, with four exceptions, are treated in this section. The other four groups are discussed in the following sections.

The group G_{23} is the Euclidean reflection group [3, 5]. It is isomorphic to $\mathrm{Alt}(5) \times Z_2$, so each character has Schur index 1. The group G_{24} is denoted by $[111^4]^4$ in the notation of Coxeter [8], who showed that it is isomorphic to $PSL(2, 7) \times Z_2$. Janusz [12] showed that each character of $PSL(2, 7)$ has Schur index 1, so the same is true of G_{24}. The groups G_{25} and G_{26} are discussed in the next section.

The group G_{27} has center of order 6 and its collineation group is isomorphic to Alt(6). It contains a subgroup H that is identified by Shephard and Todd as the group $(3, 3 \mid 4, 5)$ of Coxeter [6], also denoted by (3, 4, 5; 3). The group H of order 1080 was studied by Miller [14]. Miller showed that H has a center of order 3 and contains no subgroup of order 360. Hence, H must be a nonsplit central extension of Alt(6). The universal central extension $\tilde{A}_6$ of Alt(6) has order $6 \cdot 360$ and H is a homomorphic image. Clearly

$$G_{27} \cong H \times Z_2 .$$

Schur [15] first gave the characters of $\tilde{A}_6$. There are 17 characters of H, including the seven characters of Alt(6). There are six faithful characters of H whose values generate $\mathbf{Q}(-3^{1/2})$, so each of these has Schur index 1 by (2.3) and (2.8). Each of the remaining four characters has degree 3 and its values generate $\mathbf{Q}(-3^{1/2}, 5^{1/2})$. Since $m_{\mathbf{Q}}(\chi) \mid \chi(1)$, then, by (2.3) and (2.4), each of these characters has Schur index 1.

The group G_{28} is the Weyl group $W(F_4)$, also denoted by [3, 4, 3]. Kondo [13] showed that all characters of $W(F_4)$ can be rationally represented. The group G_{29} is discussed in Section 8.

The group $G = G_{30}$ is the Euclidean reflection group [3, 3, 5]. This group has exactly one character with Schur index 2, as shown by Benson and Grove [4]. The commutator subgroup G^+ is the even subgroup and G^+ is isomorphic to the central product $SL(2, 5) * SL(2, 5)$. Hence, the characters of G^+ are central products of characters of $SL(2, 5)$. Let α and β be the characters of $SL(2, 5)$ with $\alpha(1) = 4$, $\beta(1) = 6$, and $m_{\mathbf{Q}}(\alpha) = m_{\mathbf{Q}}(\beta) = 2$. Then, $\chi = (\alpha * \beta)^G$ is a rational-valued character with degree 48 and $m_{\mathbf{Q}}(\chi) = 2$. The corresponding division algebra is $\Delta(2, 3)$. Since $\mid \mathbf{Q}_p(5^{1/2}) : \mathbf{Q}_p \mid = 2$ for $p = 2$ or 3, then $m_F(\chi) = 1$ for $F = \mathbf{Q}(5^{1/2})$ by (2.5). Thus, $\mathbf{Q}(5^{1/2})$, the field generated by the characters of G, is a splitting field for G.

The group G_{31} is treated in Section 9.

The group $G = G_{32}$ has a center of order 2 and its collineation group is isomorphic to $PSp(4, 3)$, the simple group of order 25920. Since $Sp(4, 3)$ is the universal central extension of $PSp(4, 3)$, then G is isomorphic to either

$PSp(4, 3) \times Z_6$ or $Sp(4, 3) \times Z_3$. Since G is a four-dimensional reflection group, then it has a character of degree 4. Since $PSp(4, 3)$ does not have a character of degree 4, then the first case is impossible. Thus,

$$G \cong Sp(4, 3) \times Z_3 .$$

By the results of Section 3, G has exactly four characters with Schur index 2. Furthermore, the field $\mathbf{Q}(-3^{1/2})$ generated by the values of the characters of G is a splitting field for G.

The group G_{33} is $[211]^3$ in the notation of Coxeter [8]. Coxeter showed that $G_{33} \cong PSp(4, 3) \times Z_2$, so each character of G_{33} has Schur index 1. The group G_{34} was considered by the author [2]; each character has Schur index 1. The groups G_{35}, G_{36}, and G_{37} are the Weyl groups $W(E_6)$, $W(E_7)$, and $W(E_8)$, respectively. It was shown by the author [1] that each character can be afforded by a rational representation.

7. The Hessian Group

The group $G = G_{25}$ has a center of order 3 and its collineation group $\bar{G}$ is the Hessian group of order 216. The generators for G given by Shephard and Todd are reflections r_1, r_2, and r_3 of order 3.

$$r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad r_2 = \frac{1}{-3^{1/2}} \begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ω is a primitive cube root of unity. The center is generated by $z = (r_1 r_2^{-1} r_3)^4 = \omega I$. The subgroup N generated by z, $c = (r_1 r_2)^2 (r_2 r_3)^2$, and $r_1 r_3^{-1}$ is normal and elementary abelian of order 27. The subgroup T generated by $a = r_1 r_2$ and $c^{-1} r_2 r_3 c$ is quaternion of order 8 and is a Sylow 2-subgroup of G. This subgroup is normalized by r_1 and $H = T\langle r_1 \rangle \cong SL(2, 3)$. Furthermore, $H \cap N = \langle 1 \rangle$ so $G/N \cong H \cong SL(2, 3)$. It is easy to check that $\mathbf{C}_N(a^2) = \langle z \rangle$. Since $\mathbf{Z}(H) = \langle a^2 \rangle$, then $\mathbf{N}_N(T) = \langle z \rangle$ and $\mathbf{N}_G(T) = H \times \langle z \rangle$. Thus, G contains nine Sylow 2-subgroups and the intersection of any two is trivial.

Coxeter [7] gave a transitive permutation representation of degree 9 for the Hessian group. A permutation representation of G can be constructed using the mappings $r_1 \to (456)(798)$, $r_2 \to (249)(375)$, and $r_3 \to (123)(465)$. The kernel of this permutation representation is $\langle z \rangle$ and the collineation group $\bar{G}$ can be identified with the resulting permutation group. In this representation, the subgroup H is mapped isomorphically onto a one-point stabilizer. The elements of $\bar{T}$ fix precisely one point, while $\bar{r_1}$ fixes three points. Thus, r_1 is contained in the normalizer of precisely three Sylow 2-subgroups of G. Also, $\bar{G}$ contains exactly 160 elements fixing a point. Since $\bar{c} = (168)(249)(357)$ and $\bar{G}$ is transitive, then $\bar{c}$ has at least eight conjugates. Since $\bar{c} \in \bar{N}$ and $\bar{N}$ is a normal subgroup of $\bar{G}$ of order 9, then $\bar{c}$ has exactly eight conjugates. The elements $\bar{c}\bar{r}_1$ and $\bar{c}\bar{r}_1^{-1}$ each have order 3, fix no points, and are not conjugate to $\bar{c}$, since they do not lie in $\bar{N}$. They are not conjugate to each other since r_1 and r_1^{-1} are not conjugate in G/N. Furthermore, a Sylow 3-subgroup of $\bar{G}$ is nonabelian of order 27 and its center is contained in $\bar{N}$, so $\bar{c}\bar{r}_1$ and $\bar{c}\bar{r}_1^{-1}$ each have centralizers of order 9 and lie in classes of size 24. Thus, $\bar{G}$ has 10 conjugacy classes represented by $\bar{1}$, $\bar{a}$, $\bar{a}^2$, $\bar{r}_1$, $\bar{r}_1^{-1}$, $\bar{r}_1\bar{a}^2$, $\bar{r}_1\bar{a}^{-2}$, $\bar{c}$, $\bar{c}\bar{r}_1$, and $\bar{c}\bar{r}_1^{-1}$. $\bar{G}$ has seven nonfaithful characters that are characters of $\bar{G}/\bar{N} \cong G/N \cong SL(2, 3)$. The permutation representation affords the reducible character $1 + \theta$, where θ is a faithful rational-valued character of $\bar{G}$ with degree 8. The two remaining characters of $\bar{G}$ are $\theta\lambda$ and its complex conjugate, where λ is a nonprincipal linear character of G/N.

The natural three-dimensional unitary representation of G given by Shephard and Todd affords a faithful character ϕ of G that vanishes on no elements of H. Thus, if $x \in H$, then the elements x, xz, and xz^{-1} are mutually nonconjugate. Hence, for each $x \in H$, the classes containing x, xz, and xz^{-1} each contain the same number of elements as the class of $\bar{x}$ in $\bar{G}$. This accounts for 21 conjugacy classes. The character ϕ vanishes on each of the elements c, cr_1, and cr_1^{-1}. It is shown below that all faithful characters of G vanish on these three elements. Thus, for example, c is conjugate to cz and cz^{-1}, and the class of c contains 24 elements. In particular, it follows that there are 24 conjugacy classes.

More faithful characters can be formed by multiplying ϕ by each of the characters of G/N and by taking complex conjugates. This yields 14 faithful characters of G, of degrees 3, 6, and 9. The sum of the squares of the degrees of all 24 characters equals $|G|$, so these are all the characters of G. In particular, all faithful characters vanish on c, cr_1, and cr_1^{-1}.

The character table is given in Table III. For each pair of complex-conjugate characters, only one is listed in the table. Also, for the three classes containing x, xz, and xz^{-1} for each $x \in H$, only one class is listed.

Since $G/N \cong SL(2, 3)$, then G has a rational-valued character χ with Schur index 2. Each of the other characters of G with kernel N has Schur index 1. The rational-valued character θ occurs with multiplicity 1 in a permutation character of degree 9, so $m_{\mathbf{Q}}(\theta) = 1$. Each of the remaining characters has values that generate $\mathbf{Q}(-3^{1/2})$. Since $|G| = 2^3 \cdot 3^4$, then, by (2.3) and (2.8), each of these characters has Schur index 1. Furthermore, since $m_F(\chi) = 1$ for $F = \mathbf{Q}(-3^{1/2})$, then $\mathbf{Q}(-3^{1/2})$ is a splitting field for G.

The group G_{26} is unique among the higher-dimensional reflection groups in that it contains reflections of order 2 and order 3. As indicated by Shephard

TABLE III

Character Table for G_{25}

Representative	1	z	c	a^2	a	r_1	r_1^{-1}	a^2r_1	$a^2r_1^{-1}$	cr_1	cr_1^{-1}
Order	1	3	3	2	4	3	3	6	6	4	3
Class size	1	1	24	9	54	12	12	12	36	72	72
	1	1	1	1	1	1	1	1	1	1	1
λ	1	1	1	1	1	ω	$-1-\omega$	ω	$-1-\omega$	ω	$-1-\omega$
	3	3	3	3	-1	0	0	0	0	0	0
χ	2	2	2	-2	0	-1	-1	1	1	-1	-1
	2	2	2	-2	0	$-\omega$	$1+\omega$	ω	$-1-\omega$	$-\omega$	$1+\omega$
θ	8	8	-1	0	0	2	2	0	0	-1	-1
	8	8	-1	0	0	2ω	$2-2\omega$	0	0	$-\omega$	$1+\omega$
ϕ	3	3ω	0	-1	1	$1-\omega$	$2+\omega$	$1+\omega$	$-\omega$	0	0
	3	3ω	0	-1	1	$1+2\omega$	$-1-2\omega$	-1	-1	0	0
	3	3ω	0	-1	1	$-2-\omega$	$-1+\omega$	$-\omega$	$1+\omega$	0	0
	9	9ω	0	-3	-1	0	0	0	0	0	0
	6	6ω	0	2	0	$-1+\omega$	$-2-\omega$	$1+\omega$	$-\omega$	0	0
	6	6ω	0	2	0	$-1-2\omega$	$1+2\omega$	-1	-1	0	0
	6	6ω	0	2	0	$2+\omega$	$1-\omega$	$-\omega$	$1+\omega$	0	0

and Todd, G_{25} is contained in G_{26} as a subgroup of index 2. Since G_{26} contains an element of order 2 in its center, then $G_{26} \cong G_{25} \times Z_2$. Each of the 2 characters of G_{26} induced from the character χ of G_{25} is rational-valued and has Schur index 2. Each of the other characters has Schur index 1 and $\mathbf{Q}(-3^{1/2})$ is a splitting field for G_{26}.

8. The Group $[2\ 1\ 1]^4$

The group $G = G_{29}$ is denoted by $[2\ 1\ 1]^4$ in the notation of Coxeter [8]. Shephard and Todd gave the following four generators.

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The central element is $z = (r_1r_2r_3r_4)^5 = iI$, where $i = -1^{1/2}$. The commutator subgroup is the even subgroup G^+ and has index 2 in G. If $a = r_2r_3$, then the normal subgroup N generated by a^2 and z has order 64 and $G/N \cong \mathrm{Sym}(5)$. This isomorphism can be realized by using the mappings $r_1 \to (12)$, $r_2 \to (23)$, $r_3 \to (45)$, and $r_4 \to (34)$. The subgroup $H = \langle r_1, r_2, r_3\rangle$, is a reflection subgroup isomorphic to $G(2, 1, 3)$ (otherwise known as $[3, 4]$), and $H \times \langle z\rangle$ is the centralizer of a reflection. Furthermore, $\mathbf{N}_G(H) = H \times \langle z\rangle$ and H has 40 conjugates corresponding to the 40 reflections. The subgroup $K = \langle r_2, r_3, r_4\rangle$ is a reflection subgroup isomorphic to $G(4, 4, 3)$ ($[1\ 1\ 1]^4$ in the notation of Coxeter [8]). This subgroup has normalizer $K \times \langle z\rangle$ and has 20 conjugates in G.

The conjugacy classes of G can be obtained by using information about the classes of G/N, H, and K with techniques similar to those used in [2] for the group G_{34}. The details are omitted, but the 37 conjugacy classes are given in Table IV. Representatives are listed for one conjugacy class corresponding to each of the 12 classes of G/N. Representatives of the other 25 classes can be obtained by multiplying the given representatives by z, z^{-1}, or z^2 where appropriate. In the table, the even classes are listed first.

The character table for G is summarized in Table V. In addition to the 18 characters listed, there are five complex conjugates and 14 other characters of the form $\phi_n\phi_2$. The characters can be derived as follows. Let ϕ_{15} be the

TABLE IV

Conjugacy Classes of G_{29} [a]

		Representative	Order	Class Size	Centralizer
	Even classes				
I	(4)	1	1	1	$2^9 \cdot 3 \cdot 5$
II	(2)	$(r_2r_3)^2$	2	30	2^8
III	(2)	r_2r_3	2	120	2^6
IV	(4)	r_1r_3	4	60	2^7
V	(1)	c	8	480	2^4
VI	(4)	r_3r_4	3	320	$2^3 \cdot 3$
VII	(4)	$(r_1r_3r_2r_4)^4$	5	384	$2^2 \cdot 5$
	Odd classes				
XII	(4)	r_1	2	40	$2^6 \cdot 3$
XIII	(2)	$(r_2r_3r_4)^2r_4$	4	240	2^5
XIV	(4)	$r_2r_3r_4$	8	240	2^5
XV	(2)	b	4	480	2^4
XVI	(4)	$r_1r_2r_3$	6	320	$2^3 \cdot 3$

[a] $b = \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$

character of the natural four-dimensional representation given by Shephard and Todd. Let α, α_1, and β denote the reducible characters induced from the principal characters of $K \times \langle z \rangle$, $K \times \langle z^2 \rangle$, and H, respectively. The characters $\phi_1, \ldots, \phi_5$ are characters of Sym(5). The remaining characters of the collineation group $\bar{G}$ are given by the following equations

$$\phi_6 = \alpha - \phi_1 - \phi_3,$$
$$\phi_7 = \beta - \phi_1 - \phi_2 - \phi_4 - \phi_6,$$
$$\phi_8 = \phi_6\phi_3 - \phi_6 - \phi_7.$$

The characters ϕ_9 and ϕ_{10} are obtained by applying Schur's method of partitioning Kronecker powers of characters to ϕ_{15}^2 as in [2]. Thus,

$$\phi_9(g) = \tfrac{1}{2}(\phi_{15}(g)^2 - \phi_{15}(g^2)),$$
$$\phi_{10}(g) = \tfrac{1}{2}(\phi_{15}(g)^2 + \phi_{15}(g^2)),$$

TABLE V

Character Table for G_{29}

	1	z^2	z	II	III	IV	V	VI	VII	XII	XIII	XIV	XV	XVI
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
ϕ_3	4	4	4	4	0	0	0	1	-1	2	2	0	0	-1
ϕ_4	5	5	5	5	1	1	1	-1	0	1	1	-1	-1	1
ϕ_5	6	6	6	6	-2	-2	-2	0	1	0	0	0	0	0
ϕ_6	15	15	15	-1	-1	3	-1	0	0	3	-1	-1	1	0
ϕ_7	15	15	15	-1	3	-1	-1	0	0	3	-1	1	-1	0
ϕ_8	30	30	30	-2	-2	-2	2	0	0	0	0	0	0	0
ϕ_9	6	6	-6	-2	-2	2	0	0	1	0	0	0	$-2i$	0
ϕ_{10}	10	10	-10	2	2	2	0	1	0	4	0	0	0	1
ϕ_{11}	10	10	-10	2	-2	-2	0	1	0	2	2	0	0	-1
ϕ_{12}	24	24	-24	-8	0	0	0	0	-1	0	0	0	0	0
ϕ_{13}	20	20	-20	4	0	0	0	-1	0	2	-2	0	0	-1
ϕ_{14}	6	6	-6	-2	2	-2	0	0	1	0	0	2	0	0
ϕ_{15}	4	-4	$4i$	0	0	2	0	1	-1	2	0	0	$1-i$	1
ϕ_{16}	16	-16	$16i$	0	0	0	0	1	1	4	0	0	0	-1
ϕ_{17}	20	-20	$20i$	0	0	2	0	-1	0	2	0	0	$-1+i$	1
ϕ_{18}	24	-24	$24i$	0	0	-4	0	0	-1	0	0	0	0	0

for all $g \in G$. Then

$$\phi_{11} = \alpha_1 - \alpha - \phi_{10},$$
$$\phi_{12} = \phi_9\phi_3,$$
$$\phi_{13} = \phi_{10}\phi_3 - \phi_9 - \phi_{10}.$$

There remains two characters of $G/\langle z^2 \rangle$, each of degree 6 since the sum of the squares of the degrees of all characters must equal the order of the group. By the orthogonality relations, at least one of these must not vanish on the odd element b (see Table IV), so these two remaining characters are associates: ϕ_{14} and $\phi_{14}\phi_2$. Their values are obtained by use of the orthogonality relations. The faithful characters in the list are ϕ_{15}, $\phi_{16} = \phi_{15}\phi_3$, $\phi_{17} = \phi_{15}\phi_4$, and $\phi_{18} = \phi_{15}\phi_5$.

The only characters that are not rational-valued are $\phi_9, \phi_{15}, \ldots, \phi_{18}$. The characters $\phi_1, \ldots, \phi_5$ of Sym(5) are known to have Schur index 1. Since ϕ_6 and ϕ_7 have odd degrees, then each has Schur index 1 by (2.1). The characters ϕ_{10} and ϕ_{11} occur in the permutation character α_1 with multiplicity 1, so each of these has Schur index 1. Thus, $\phi_6\phi_3$ and $\phi_{10}\phi_3$ are afforded by rational representations, so the above equations show that ϕ_8 and ϕ_{13} have Schur index 1. Calculations show that $(\phi_{14}, \phi_{10}\phi_8) = 1$, so $m_{\mathbf{Q}}(\phi_{14}) = 1$. For the last rational-valued character, $\phi_{12} = \phi_{14}\phi_3$, so $m_{\mathbf{Q}}(\phi_{12}) = 1$. Since ϕ_{13} is afforded by a $\mathbf{Q}(-1^{1/2})$-representation and each of the remaining characters has values that generate $\mathbf{Q}(-1^{1/2})$, then the above equations show that each of these has Schur index 1. Hence, each character of G_{29} has Schur index 1.

9. The Final Case

The group $G = G_{31}$ has center of order 4 and contains G_{29} as a subgroup of index 6. This group is a four-dimensional reflection group that cannot be generated by four reflections. The generators given by Shephard [17] are the reflections r_1, r_2, r_3, and r_4 of the subgroup G_{29} together with

$$r_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The subgroup N of G_{29} is also normal in G and G/N is isomorphic to Sym(6). This isomorphism can be realized by mapping $r_5 \to (16)$ and extending the map from G_{29} onto Sym(5). The subgroup $M = \langle r_2, r_3, r_4, r_5 \rangle$ is a reflection group isomorphic to $G(4, 2, 3)$. The normalizer of M is $M \times \langle z \rangle$, and $M \times \langle z \rangle$ is the centralizer of a reflection in G. There are 60 conjugates of M corresponding to the 60 reflections.

The conjugacy classes of G can be obtained in a manner similar to that mentioned in the previous section and details are omitted. The classes are listed in Table VI. The numbering of the classes is consistent with the numbering of the classes of the subgroup G_{29} as they are listed in Table IV.

The character table for G is summarized in Table VII, with the same conventions as used in Table V. The characters can be derived as follows. Let θ_{21} be the character of the natural four-dimensional representation of G given by Shephard [17]. Let ρ be the reducible character induced from the

TABLE VI

Conjugacy Classes of G_{31}

		Order	Class size	Centralizer
	Even classes			
I	(4)	1	1	$2^{10} \cdot 3^2 \cdot 5$
II	(2)	2	30	$2^9 \cdot 3$
III	(2)	2	360	2^7
IV	(4)	4	180	2^8
V	(1)	8	1440	2^5
VI	(4)	3	640	$2^3 \cdot 3^2$
VII	(4)	5	2304	$2^5 \cdot 5$
VIII	(4)	3	160	$2^5 \cdot 3^2$
IX	(2)	6	960	$2^4 \cdot 3$
X	(4)	8	720	2^6
XI	(1)	8	2880	2^4
	Odd classes			
XII	(4)	2	60	$2^8 \cdot 3$
XIII	(1)	4	720	2^6
XIV	(4)	8	720	2^6
XV	(1)	4	2880	2^4
XVI	(4)	6	1920	$2^3 \cdot 3$
XVII	(4)	12	960	$2^4 \cdot 3$
XVIII	(2)	24	1920	$2^3 \cdot 3$
XIX	(1)	4	360	2^7
XX	(4)	4	30	$2^9 \cdot 3$
XXI	(2)	8	240	$2^6 \cdot 3$

principal character of $M \times \langle z \rangle$. The characters $\theta_1, \ldots, \theta_7$ are characters of Sym(6). The remaining characters of the collineation group $\bar{G}$ are given by the following equations.

$$\begin{aligned}
\theta_8 &= \theta_{21}\bar{\theta}_{21} - \theta_1, \\
\theta_9 &= \rho - \theta_1 - \theta_3 - \theta_5 - \theta_8, \\
\theta_{10} &= \theta_8\theta_3 - \theta_9, \\
\theta_{11} &= \theta_8\theta_4 - \theta_8\theta_2 - \theta_{10}, \\
\theta_{12} &= \theta_8\theta_5 - \theta_8 - \theta_9 - \theta_{10}\theta_2 .
\end{aligned}$$

The characters θ_{13} and θ_{14} are found by partitioning θ_{21}^2. Thus,

$$\begin{aligned}
\theta_{13}(g) &= \tfrac{1}{2}(\theta_{21}(g)^2 - \theta_{21}(g^2)), \\
\theta_{14}(g) &= \tfrac{1}{2}(\theta_{21}(g)^2 + \theta_{21}(g^2)),
\end{aligned}$$

for all $g \in G$. The remaining characters are given by the following equations.

$$\begin{aligned}
\theta_{15} &= \theta_{13}\theta_3, \\
\theta_{16} &= \theta_{13}\theta_4 - \bar{\theta}_{13}, \\
\theta_{17} &= \theta_{14}\theta_3 - \bar{\theta}_{14}, \\
\theta_{18} &= \theta_{14}\theta_4 - \bar{\theta}_{14}, \\
\theta_{19} &= \theta_{14}\theta_6 - \theta_{17} - \theta_{18}, \\
\theta_{20} &= \theta_{13}\theta_6 - \theta_{16}, \\
\theta_{22} &= \theta_{21}\theta_3, \\
\theta_{23} &= \theta_{21}\theta_4, \\
\theta_{24} &= \theta_{21}\theta_5, \\
\theta_{25} &= \theta_{21}\theta_6, \\
\theta_{26} &= \theta_{21}\theta_7 .
\end{aligned}$$

The characters $\theta_1, \ldots, \theta_{12}, \theta_{17}, \theta_{19}$, and θ_{20} are rational-valued. The characters $\theta_1, \ldots, \theta_7$ of Sym(6) are known to have Schur index 1. Since θ_8, θ_{10}, θ_{11}, and θ_{12} have odd degrees, then each of these has Schur index 1 by (2.1). The character θ_9 occurs with multiplicity 1 in the permutation character ρ, so it has Schur index 1. A different technique must be applied to the three remaining rational-valued characters since they are derived from characters that are not rational-valued. When these three characters are restricted to the subgroup $S = G_{29}$, the following decompositions occur.

$$\begin{aligned}
(\theta_{17})_S &= \phi_{10} + \phi_{11} + \phi_{20}, \\
(\theta_{19})_S &= \phi_{11} + \phi_{11}\phi_2, \\
(\theta_{20})_S &= \phi_{12} + \phi_{14} + \phi_{14}\phi_2 .
\end{aligned}$$

TABLE VII

Character Table for G_{31}

	I	z^2	z	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI
θ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
θ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
θ_3	5	5	5	5	1	1	1	2	0	-1	-1	-1	-1	3	3	1	1	0	-1	-1	-1	-1	-1
θ_4	5	5	5	5	1	1	1	-1	0	2	2	-1	-1	1	1	-1	-1	1	0	0	-3	-3	-3
θ_5	9	9	9	9	1	1	1	0	-1	0	0	1	1	3	3	-1	-1	0	0	0	3	3	3
θ_6	10	10	10	10	-2	-2	-2	1	0	1	1	0	0	2	2	0	0	-1	1	1	-2	-2	-2
θ_7	16	16	16	16	0	0	0	-2	1	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0
θ_8	15	15	15	-1	-1	3	-1	0	0	3	-1	1	-1	3	-1	-1	1	0	1	-1	-1	7	-1
θ_9	30	30	30	-2	2	2	-2	0	0	-3	1	0	0	6	-2	0	0	0	-1	1	2	2	-2
θ_{10}	45	45	45	-3	-3	1	1	0	0	0	0	-1	1	3	-1	-1	1	0	0	0	-1	-9	3
θ_{11}	15	15	15	-1	3	-1	-1	0	0	3	-1	-1	1	3	-1	1	-1	0	1	-1	3	-5	-1
θ_{12}	45	45	45	-3	1	-3	1	0	0	0	0	1	-1	3	-1	1	-1	0	0	0	-5	3	3
θ_{13}	6	6	-6	-2	-2	2	0	0	1	3	1	0	0	0	0	0	$-2i$	0	i	$-i$	0	$4i$	$-2i$
θ_{14}	10	10	-10	2	2	2	0	1	0	1	-1	$2i$	0	4	0	0	0	1	i	i	0	$4i$	$2i$
θ_{15}	30	30	-30	-10	-2	2	0	0	0	-3	-1	0	0	0	0	0	$-2i$	0	$-i$	i	0	$-4i$	$2i$
θ_{16}	24	24	-24	-8	0	0	0	0	-1	3	1	0	0	0	0	0	0	0	i	$-i$	0	$-8i$	$4i$
θ_{17}	40	40	-40	8	0	0	0	1	0	-2	2	0	0	8	0	0	0	-1	0	0	0	0	0
θ_{18}	40	40	-40	8	0	0	0	-2	0	1	-1	0	0	0	0	0	0	0	i	i	0	$-8i$	$-4i$
θ_{19}	20	20	-20	4	-4	-4	0	2	0	2	-2	0	0	0	0	0	0	0	0	0	0	0	0
θ_{20}	36	36	-36	-12	4	-4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
θ_{21}	4	-4	$4i$	0	0	2	0	1	-1	-2	0	$1+i$	0	2	0	0	$1-i$	1	$1+i$	0	0	$2+2i$	0
θ_{22}	20	-20	$20i$	0	0	2	0	2	0	2	0	$-1-i$	0	6	0	0	$1-i$	0	$-1-i$	0	0	$-2-2i$	0
θ_{23}	20	-20	$20i$	0	0	2	0	-1	0	-4	0	$-1-i$	0	2	0	0	$i-1$	1	0	0	0	$-6-6i$	0
θ_{24}	36	-36	$36i$	0	0	2	0	0	1	0	0	$1+i$	0	6	0	0	$i-1$	0	0	0	0	$6+6i$	0
θ_{25}	40	-40	$40i$	0	0	-4	0	1	0	-2	0	0	0	4	0	0	0	-1	$1+i$	0	0	$-4-4i$	0
θ_{26}	64	-64	$64i$	0	0	0	0	-2	-1	4	0	0	0	0	0	0	0	0	0	0	0	0	0

Since θ_{11} and θ_{12} can be afforded by rational representations, then, by (2.1) and (2.2), each of the three characters has Schur index 1. Each of the remaining characters has values that generate $Q(-1^{1/2})$. Since θ_{21} is afforded by the $Q(-1^{1/2})$-representation given by Shephard, then the defining equations show that each of the remaining characters has Schur index 1. Hence, each character of G_{31} has Schur index 1.

This completes the proof of Theorems 1 and 2.

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