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Homomorphisms between Solomon's Descent Algebras

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In a previous paper (see A. Garsia and C. Reutenauer (*Adv. in Math.* 77, 1989, 189–262)), we have studied algebraic properties of the descent algebras Σ_n, and shown how these are related to the canonical decomposition of the free Lie algebra corresponding to a version of the Poincaré–Birkhoff–Witt theorem. In the present paper, we study homomorphisms between these algebras Σ_n. The existence of these homomorphisms was suggested by properties of some directed graphs that we constructed in the previous paper (reference above) describing the structure of the descent algebras. More precisely, examination of the graphs suggested the existence of homomorphisms $\Sigma_n \to \Sigma_{n-s}$ and $\Sigma_n \to \Sigma_{n+s}$. We were then able to construct, for any s $(0<s<n)$, a surjective homomorphism $\Delta_s: \Sigma_n \to \Sigma_{n-s}$ and an embedding $\Gamma_s: \Sigma_{n-s} \to \Sigma_n$, which reflects these observations. The homomorphisms Δ_s may also be defined as derivations of the free associative algebra $\mathbb{Q}\langle t_1, t_2, ...\rangle$ which sends t_i on t_{i-s}, if one identifies the basis element $D_{\subseteq S}$ of Σ_n with some word (coding S) on the alphabet $T=\{t_1, t_2, ...\}$. We show that this mapping is indeed a homomorphism, using the combinatorial description of the multiplication table of Σ_n given in the previous paper (reference above).

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INTRODUCTION

In the algebra of the symmetric group $\mathbb{Q}[S_n]$, let us define the element

$$D_{\subseteq S}=\sum_{\mathrm{Des}(\sigma)\subseteq S}\sigma,$$

where $\mathrm{Des}(\sigma)$ denotes the descent set of σ and $S\subseteq\{1,...,n-1\}$. In [7], Solomon shows that the linear span Σ_n of these 2^{n-1} elements forms a subalgebra of $\mathbb{Q}[S_n]$. In fact, Solomon shows that this is the case for any finite Coxeter group.

In a previous paper [2], we have studied algebraic properties of Σ_n, and shown how these are related to the canonical decomposition of the free Lie algebra corresponding to a version of the Poincaré–Birkhoff–Witt theorem. In particular, we were able to compute a complete family of orthogonal primitive indempotents E_λ of Σ_n (indexed by partitions of n). We also computed the dimensions of the quasi-ideals $E_\lambda \Sigma_n E_\mu$.

In the present paper, we study homomorphisms between these algebras Σ_n. The existence of these homomorphisms was suggested by properties of the directed graphs (see [2]) describing the structure of these descent algebras. More precisely, examination of these graphs suggested the existence of homomorphisms $\Sigma_n\to\Sigma_{n-s}$ and $\Sigma_n\to\Sigma_{n+s}$ which send the idempotent E_λ on to $E_{\lambda\setminus s}$ (resp. $E_{\lambda\cup s}$). As we shall see, for any s, $0\leqslant s\leqslant n$, one can define a surjective homomorphism $\Delta_s\colon\Sigma_n\to\Sigma_{n-s}$ and an embedding $\Gamma_s\colon\Sigma_{n-s}\to\Sigma_n$, which reflects these observations. The homomorphisms Δ_s may also be defined as derivations of the free associative algebra $\mathbb{Q}\langle t_1, t_2, ...\rangle$ which sends t_i on t_{i-s}, if one identifies the basis element $D_{\subseteq S}$ of Σ_n with some word (coding S) on the alphabet $T=\{t_1, t_2, ...\}$. We show that this mapping is indeed an homomorphism (Theorem 1.1), using the combinatorial description of the multiplication table of Σ_n given in [1].

In Theorem 2.1, we show that Δ_s has the expected behaviour with respect to the idempotents E_λ. This is to say that $\Delta_s(E_\lambda)=0$ if λ does not contain the part s, and $\Delta_s(E_\lambda)=E_{\lambda\setminus s}$ if λ contains the part s. This follows from an argument involving noncommutative logarithms and exponentials, and using generating series for the E_λ's. We deduce from Theorem 2.1 the surjectivity of Δ_s (Corollary 2.2). We further give a direct description of Δ_s in terms of permutations in Theorem 2.3. For the simplest case $s=1$, Δ_1 corresponds to erasing in each permutation (considered as a word) the digit n. For the general case, the Lie polynomials step in again, by acting as derivations on the shuffle algebra. In fact, Theorem 2.3 involves an operation introduced by Ree in [6].

In Section 3, we establish the existence of the embeddings $\Gamma_s\colon\Sigma_{n-s}\to\Sigma_n$ such that $\Delta_s\circ\Gamma_s=Id$, and with the expected behavior on the idempotents E_λ (Theorem 3.1). Actually, the precise description of Γ_s makes use of another basis J_P of Σ_n introduced in [2].

In Section 4, we show that Δ_s corresponds to the derivation $s(\partial/\partial p_s)$ under the natural homomorphism from Σ_n onto the algebra Sym_n of homogeneous symmetric functions of degree n with inner product.

1. Homomorphisms

Let Σ_n denote the subspace of the symmetric group algebra $\mathbb{Q}[S_n]$ spanned by the elements $D_{=S}$, $S \subseteq \{1, ..., n-1\}$, where $D_{=S}$ is the sum of all permutations σ whose descent set

$$\mathrm{Des}(\sigma) = \{i \mid 1 \leqslant i \leqslant n-1, \sigma(i) \geqslant \sigma(i+1)\}$$

is equal to S. It has been shown, first by Solomon [7], that Σ_n is a subalgebra of $\mathbb{Q}[S_n]$. Clearly, Σ_n admits as another basis the family $D_{\subseteq S}$, $S \subseteq \{1, ..., n-1\}$, defined by

$$D_{\subseteq S} = \sum_{T \subseteq S} D_{=T}$$

(that is, $D_{\subseteq S}$ is the sum of permutations having a descent set *contained* in S). The multiplication table of Σ_n is easily described in term of this last basis (see Garsia and Remmel [1]). First, note that there is a natural bijection between subsets S of $\{1, ..., n-1\}$ and compositions of n (a sequence of positive integers whose sum is n). For $S = \{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k\}$, one defines the composition $p(S) = (s_1 - s_0, s_2 - s_1, ..., s_{k+1} - s_k)$ where $s_0 = 0$ and $s_{k+1} = n$. From now on, $D_{\subseteq S}$ will be denoted by $B_{p(S)}$.

Let us call the *pseudo-composition* of n any sequence v of nonnegative integers whose sum is n. To each pseudo-composition v, one naturally associates the composition $p(v)$ of n obtained by omitting the zeros. Now, to each matrix

$$M = (n_{ij})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l}$$

with entries in $\mathbb{N}$, there corresponds naturally the pseudo-composition

$$w(M) = (n_{11}, n_{21}, ..., n_{k1}, n_{12}, n_{22}, ..., n_{k2}, ..., n_{kl})$$

obtained by reading the entries of M starting from the upper left corner, down the first column, then the second, etc. Similarly, one defines the *row sum* (resp. *column sum*) of M as the pseudo-composition

$$r(M) = (n_{11} + n_{12} + \cdots + n_{1l}, n_{21} + n_{22} + \cdots + n_{2l}, ..., n_{k1} + n_{k2} + \cdots + n_{kl})$$

$$c(M) = (n_{11} + n_{21} + \cdots + n_{k1}, n_{12} + n_{22} + \cdots + n_{k2}, ..., n_{1l} + n_{2l} + \cdots + n_{kl}).$$

Then, for two compositions p and q of n, one has

$$B_p B_q = \sum_M B_{p(w(M))}, \tag{1.1}$$

where the sum is extended to all matrices (with entries in $\mathbb{N}$) whose row sum is p and column sum is q (see [1]).

EXAMPLE 1. For $p = 22$ and $q = 31$ the possible matrices are $M = \binom{1\ 1}{2\ 0}$ or $M = \binom{2\ 0}{1\ 1}$. Hence $B_{22} B_{31} = B_{121} + B_{211}$.

We now define for each s, $1 \leqslant s \leqslant n$, a linear mapping $\Delta_s : \Sigma_n \to \Sigma_{n-s}$. For each composition $p = p_1 p_2 \cdots p_k$ of n and each part $p_i \geqslant s$, define the pseudo-composition u_i by replacing p_i in p by $p_i - s$. Then define

$$\Delta_s(B_p) = \sum_{p_i \geqslant s} B_{p(u_i)}. \tag{1.2}$$

EXAMPLE 2. $\Delta_2(B_{213}) = B_{13} + B_{211}$.

1.1. THEOREM. *The mapping Δ_s is an algebra homomorphism $\Sigma_n \to \Sigma_{n-s}$.*

Proof. We have to show that $\Delta_s(B_p B_q) = \Delta_s(B_p)\, \Delta_s(B_q)$. Now, we have Eq. (1.2), and similarly

$$\Delta_s(B_q) = \sum_{q_j \geqslant s} B_{p(v_j)}, \tag{1.3}$$

where v_j is obtained from q by replacing q_j by $q_j - s$. Note that Eq. (1.1) admits the following easy generalization: for any two pseudo-compositions u and v, one has

$$B_{p(u)} B_{p(v)} = \sum_M D_{p(w(M))}, \tag{1.4}$$

where the sum is extended to all matrices M having row sum u and column sum v. Equations (1.1), (1.2), and (1.3) imply that

$$\Delta_s(B_p)\, \Delta_s(B_q) = \sum_{p_i, q_j \geqslant s} \sum_{M_{ij}} B_{p(w(M_{ij}))}, \tag{1.5}$$

where the second sum is extended to all matrices whose row sum is u_i and column sum is v_j. On the other hand, we have by Eq. (1.1)

$$\begin{aligned}\Delta_s(B_p B_q) &= \Delta_s\left(\sum_M B_{p(w(M))}\right)\\ &= \sum_M \sum_{m_{ij}\geqslant s} B_{p(w(M_{ij}))},\end{aligned} \tag{1.6}$$

where the sum extends to all matrices $M=(m_{ij})$ with row sum p and column sum q, and where M_{ij} is obtained by replacing in M the entry m_{ij} by $m_{ij}-s$. Note that M_{ij} has row sum u_i and column sum v_j, which shows that the sum (1.6) is contained in the sum (1.5). Conversely, if M_{ij} has row sum u_i and column sum v_j, then by adding s to its ij-entry, one obtains a matrix of row sum p and column sum q, which shows that the reverse inclusion also holds. This concludes the proof. ∎

2. Derivations

Let $T=\{t_1, t_2, ...\}$ be an infinite alphabet and T^* the free monoid it generates. There is a natural linear isomorphism δ from the free associative algebra $\mathbb{Q}\langle T\rangle$ onto $\Sigma=\bigoplus_{n\geqslant 0}\Sigma_n$ defined by

$$\delta(t_{p_1}t_{p_2}\cdots t_{p_k})=B_p,$$

where $p=p_1p_2\cdots p_k$ is a composition of n. Now, the homomorphism $\Delta_s\colon \Sigma_n\to\Sigma_{n-s}$ of the previous section clearly defines a linear mapping $\Sigma\to\Sigma$, which we also denote by Δ_s (if $n<s$, $\Delta_s(\Sigma_n)=0$). We shall now introduce a derivation D_s on $\mathbb{Q}\langle T\rangle$ such that $\delta\circ\Delta_s=D_s\circ\delta$. This derivation of the algebra $\mathbb{Q}\langle T\rangle$ is defined by

$$D_s(t_i)=\begin{cases} t_{i-s}, & \text{if } i>s;\\ 1, & \text{if } i=s;\\ 0, & \text{if } i<s.\end{cases}$$

It is convenient to write $t_0=1$, and $t_i=0$ when $i<0$. So we can now write $D_s(t_i)=t_{i-s}$.

In [2], we have defined special elements e_λ of $\mathbb{Q}\langle T\rangle$, for each partition λ, and shown that the elements $E_\lambda=\delta(e_\lambda)$ are mutually orthogonal idempotents decomposing 1 and generating a complement of the radical of Σ_n (see [2, Theorems 3.3, 3.1, and 1.1]). We shall establish now the effect of the homomorphisms Δ_s on these idempotents. To this end we make use of the isomorphism δ, and work with D_s over $\mathbb{Q}\langle T\rangle$. The following formulas (see [2, Theorem 3.3]) define the e_λ's. Let y, x_1, x_2, x_3, ... be commuting variables, which also commute with the t_i's: Then one sets

$$\sum_{n\geq 1} e_n y^n = \log\left(\sum_{i\geq 0} t_i y^i\right) \tag{2.1}$$

(recall that $t_0 = 1$) and

$$\sum_\lambda e_\lambda X^\lambda = \exp\left(\sum_{j\geq 1} e_j x_j\right), \tag{2.2}$$

where $X^\lambda = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, if $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$.

2.1. THEOREM. *The image of E_λ under Δ_s is 0 if s is not a part of λ. If s is a part of λ, let $\lambda\backslash s$ denote the partition obtained by deleting one part s in λ; then $\Delta_s(E_\lambda) = E_{\lambda\backslash s}$.*

Proof. Recall that in a topological algebra with a derivation $a \mapsto a'$, if a commutes with a', then one has the usual formulas $\log(a)' = a'a^{-1}$ and $\exp(a)' = a' \exp(a)$ (log and exp are defined by the usual series). Of course, there are some convergence hypotheses (which in our case are trivial). Extend D_s to the variables y, x_1, x_2, x_3, ... by $D_s(y) = D_s(x_1) = D_s(x_2) = \cdots = 0$. Note that

$$D_s\left(\sum_{i\geq 0} t_i y^i\right) = \sum_{i\geq 0} D_s(t_i)\, y^i = \sum_{i\geq 0} t_{i-s} y^i = y^s \sum_{i\geq 0} t_i y^i,$$

hence $\sum t_i y^i$ and its D_s-derivative commute. Thus we obtain by (2.1)

$$\sum_{n\geq 1} D_s(e_n)\, y^n = D_s\left(\sum t_i y^i\right)\left(\sum t_i y^i\right)^{-1} = y^s.$$

This shows that

$$D_s(e_n) = \begin{cases} 1, & \text{if } n = s \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Similarly, $D_s(\sum_{i\geq 1} e_j x_j) = \sum_{j\geq 1} D_s(e_j)\, x_j = x_s$ commutes with $\sum e_j x_j$; it follows from (2.2) that

$$\begin{aligned} \sum D_s(e_\lambda)\, X^\lambda &= D_S\left(\sum_\lambda e_\lambda X^\lambda\right) \\ &= D_s\left(\sum_{j\geq 1} e_j x_j\right) \exp\left(\sum_{j\geq 1} e_j x_j\right) \end{aligned}$$

$$= x_s \exp\left(\sum_{j \geqslant 1} e_j x_j\right)$$

$$= x_s \left(\sum_{\lambda} e_\lambda X^\lambda\right).$$

Hence we deduce

$$D_s(e_\lambda) = \begin{cases} 0, & \text{if } s \text{ is not a part of } \lambda \\ e_{\lambda \backslash s}, & \text{if } s \text{ is a part of } \lambda. \end{cases} \tag{2.4}$$

Translating with δ equations (2.3) and (2.4), we obtain the theorem. ∎

2.2. Corollary. *The mappings Δ_s and D_s are surjective.*

Proof. It is enough to show that D_s is surjective. Now, $D_s(e_n) = 1$ if $n = s$; and is 0 otherwise. Moreover, the e_n's freely generate the algebra $\mathbb{Q}\langle T \rangle$, since

$$e_1 = t_1,$$
$$e_2 = t_2 - \tfrac{1}{2} t_1^2,$$
$$e_3 = t_3 - \tfrac{1}{2} t_1 t_2 - \tfrac{1}{2} t_2 t_1 + \tfrac{1}{3} t_1^3,$$
$$\vdots$$

which are triangular algebraic relations between the e_n's and the t_n's. We show that each product of the form

$$x = x_0 e_s^{i_1} x_1 \cdots e_s^{i_k} x_k, \tag{$*$}$$

where $k \geqslant 1$ and where each x_j is a product of e_i's distinct from e_s, is in the image of D_s. In view of the above observation, this will imply the corollary. We prove the claim by induction on $i = i_2 + \cdots + i_k$. Observe that x is equal to

$$D_s\left(\frac{1}{i_1 + 1} x_0 e_s^{i_1 + 1} x_1 \cdots e_s^{i_k} x_k\right) - y,$$

where either $k = 1$ and $y = 0$ hence x is in $\mathrm{Im}(D_s)$, or $k \geqslant 2$ and y is a linear combination of terms of the form $(*)$ with a smaller i; in the latter case, we know by induction that $y \in \mathrm{Im}(D_s)$, hence $x \in \mathrm{Im}(D_s)$. ∎

We have described Δ_s by its effect on the basis B_p. Recall that Σ_n is a subalgebra of $\mathbb{Q}[S_n]$. So, it is natural to ask for a description of Δ_s directly in terms of permutations. We shall now show that this is possible. As in [2], we shall see that the free Lie algebra plays a crucial role.

Let A be the finite alphabet $\{1, 2, ..., n\}$ and consider each permutation as a word on A. Thus $\mathbb{Q}[S_n]$ is a subspace of the free associative algebra $\mathbb{Q}\langle A\rangle$. In particular, Σ_n is a subspace of $\mathbb{Q}\langle A\rangle$. For any word u in the free monoid A^*, define a linear mapping

$$\pi_u\colon \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$$

by the formula

$$\pi_u(w) = \sum_{w = xuy} xy,$$

if w is a word in A^*. This is to say that π_u erases occurences of u in the words. Actually, we shall extend linearly π to all of $\mathbb{Q}\langle A\rangle$ as

$$\pi_R = \sum_u \langle R, u\rangle\, \pi_u,$$

where R is in $\mathbb{Q}\langle A\rangle$, and $\langle R, u\rangle$ is the coefficient of u in R. We also define a linear mapping $\rho\colon \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$ (first introduced by Ree [6]) recursively by the formula $\rho(\varepsilon) = 0$, where ε denotes the empty word, $\rho(a) = a$ if $a \in A$, and $\rho(aub) = \rho(au)\, b - \rho(ub)\, a$, for any word u and letters a, b in A.

2.3. THEOREM. *For $1 \leqslant s \leqslant n$, let w_s be the word $(n-s+1)\cdots n$ in A^*. Then for any P in Σ_n, one has*

$$\Delta_s(P) = \pi_{\rho(w_s)}(P).$$

EXAMPLE 3. (1) $s = 1$. Then $\rho(w_1) = n$, hence $\Delta_1(P)$ is obtained by erasing the digit n in each permutation appearing in P.

(2) $s = 2$. Then $\rho(w_2) = (n-1)\, n - n(n-1)$, hence $\Delta_2(P)$ is the sum of the permutations obtained by erasing in each permutation appearing in P the factor $(n-1)\, n$ (if it appears), minus the sum of those obtained by erasing the factor $n(n-1)$. Thus

$$\begin{aligned} \Delta_2(B_{31}) &= \Delta_2(1234 + 1243 + 1342 + 2341) \\ &= 12 - 12 + 12 + 21 = 12 + 21 \\ &= B_{11}. \end{aligned}$$

(3) $s = 3$. Then

$$\begin{aligned} \rho(w_3) &= \rho((n-2)(n-1))\, n - \rho((n-1)\, n)(n-2) \\ &= (n-2)(n-1)\, n - (n-1)(n-2)\, n \\ &\quad - (n-1)\, n(n-2) + n(n-1)(n-2), \end{aligned}$$

and one has to erase all these words when they appear in a permutation, with the sign according to its sign in $\rho(w_3)$.

Let us recall the definition of the following left action of $\mathbb{Q}\langle A\rangle$ on itself. For words w and u, set $wu^{-1}=v$ when $w=vu$, and $wu^{-1}=0$ when u is not a right factor of w. We denote $wu^{-1}=u \lhd w$. Then $u \lhd w$ extends bilinearly to a left action of $\mathbb{Q}\langle A\rangle$ on itself by the formula

$$\begin{aligned} P \lhd Q &= \sum_{u, w \in A^*} \langle P, u\rangle\langle Q, w\rangle\, u \lhd w \\ &= \sum_{w=uv} \langle P, u\rangle\langle Q, w\rangle\, v, \end{aligned}$$

where $\langle P, u\rangle$ is (as above) the coefficient of u in P.

Recall that we consider each permutation in S_n to be a word on A. We define for a permutation w, its *dual* $\downarrow w$ to be the inverse (in S_n) of w; this mapping is then linearly extended to $\mathbb{Q}[S_n]$. We shall further need the following normalization mapping. For a word w on A, of length k and without repeated letter, define $N(w)$ to be the following permutation: suppose $w=i_1 i_2 \cdots i_k$ and let τ be the unique increasing mapping

$$\tau\colon \{i_1, i_2, ..., i_k\} \to \{1, 2, ..., k\}.$$

Then $N(w)=\tau(i_1)\,\tau(i_2)\cdots\tau(i_k)$. For example, $N(7831)=3421$. Then N is also extended linearly to the linear span of words with no repeated letter.

2.4. LEMMA. *For s, $1 \leqslant s \leqslant n$, let α be a permutation in S_s. Define the word $u=(n-s+\alpha(1))\cdots(n-s+\alpha(s))$. Then for any permutation w in S_n, one has*

$$\downarrow \pi_u(w) = \sum_{j=0}^{n-s} N\{(\downarrow w)[(j+\alpha^{-1}(1))\cdots(j+\alpha^{-1}(s))]^{-1}\}. \qquad (2.5)$$

The lemma and its proof are rather technical, so we shall first illustrate it with an example. Set α as the permutation $231 \in S_3$. Then for $n=7$, we have $u=675$. Let $w=2367514$. Then $\downarrow \pi_u(w) = \downarrow 2314 = 3124$. Now, $\downarrow w = 6127534$, and $\downarrow \alpha = 312$, so the words $(j+\alpha^{-1}(1))\cdots(j+\alpha^{-1}(s))$ of the right-hand side of (2.5) are 312, 423, 534, 645, and 756. Only one of these, i.e., 534, is a right factor of $\downarrow w$, so that the right-hand side of (2.5) is $N(6127)=3124$.

Proof. Suppose $\pi_u(w) \neq 0$, then $w=xuy$. Let j be the length of x, thus $0 \leqslant j \leqslant n-s$. Hence the permutation w sends $j+t$ onto $n-s+\alpha(t)$ for

$1 \leqslant t \leqslant s$. We deduce from this that $\downarrow w$ has the right factor $a_1 a_2 \cdots a_s$ $(a_i \in A)$ with

$$a_r = j + t \Leftrightarrow n - s + r = n - s + \alpha(t) \Leftrightarrow t = \alpha^{-1}(r).$$

And finally $a_1 a_2 \cdots a_s = (j + \alpha^{-1}(1)) \cdots (j + \alpha^{-1}(s))$.

Thus the right hand side of (2.5) is equal to $N(v)$ with $\downarrow w = v a_1 a_2 \cdots a_s$. Since $\{a_1, a_2, ..., a_s\} = \{j+1, j+2, ..., j+s\}$, the permutation $N(v) \in S_{n-s}$ is obtained by replacing in v each digit $d \geqslant j$ (or equivalently $\geqslant j+s$) by $d-s$. Hence $N(v)$ sends each i onto $\downarrow w(i)$ when $\downarrow w(i) \leqslant j$, and onto $\downarrow w(i) - s$ otherwise. We conclude that $\downarrow N(v)$ is the permutation which sends all $k \in \{1, 2, ..., j\}$ onto $w(k)$, and all $k \in \{j+1, j+2, ..., n-s\}$ onto $w(k+s)$. In other words, $\downarrow N(v) = xy = \pi_u(w)$.

Suppose now that the right-hand side of (2.5) is not zero. Then for some unique $j \in \{0, ..., n-s\}$, the word $\downarrow w$ admits the right factor $(j + \alpha^{-1}(1)) \cdots (j + \alpha^{-1}(s))$. This implies that w has the factor $(n - s + \alpha(1)) \cdots (n - s + \alpha(s))$, which further implies that $\pi_u(w) \neq 0$, which concludes the proof. ∎

We have already come across particular instances of the scalar product $\langle P, Q \rangle$ on $\mathbb{Q}\langle A \rangle$ for which A^* is an orthonormal basis. Now, define a linear mapping $\lambda: \mathbb{Q}\langle A \rangle \to \mathbb{Q}\langle A \rangle$ by $\lambda(a_1 a_2 a_3 \cdots a_n) = [\cdots[[a_1, a_2], a_3], ..., a_n]$ (this is the Lie bracketing from left to right; and we implicitly suppose that the a_i's are in A). Then it has been shown in [6] (see also [4, Example 5.3.2]) that λ is the adjoint of ρ for the above scalar product.

2.5. LEMMA. *There exist rational integers θ_α $(\alpha \in S_s)$ such that*

$$\rho(w_s) = \sum_{\alpha \in S_s} \theta_\alpha [(n - s + \alpha(1)) \cdots (n - s + \alpha(s))], \tag{2.6}$$

where w_s is as in Theorem 2.3. Moreover, for any $j \geqslant 0$, one has

$$\lambda[(j+1) \cdots (j+s)] = \sum_{\alpha \in S_s} \theta_\alpha [(j + \alpha^{-1}(1)) \cdots (j + \alpha^{-1}(s))]. \tag{2.7}$$

Proof. It is clear, by definition of ρ, that $\rho(12 \cdots s)$ is a linear combination of permutations in S_s; hence for some $\theta_\alpha \in \mathbb{Q}_s$,

$$\rho(12 \cdots s) = \sum_{\alpha \in S_s} \theta_\alpha \alpha.$$

Denoting by $P \cdot x$ the result of the right action by position of $x \in \mathbb{Q}[S_s]$ on $P \in \mathbb{Q}\langle A \rangle$ (P homogeneous of degree s), we have $\rho(w) = w \cdot \rho(12 \cdots s)$, for any word of length s. Now, it is clear that the adjoint of $w \mapsto w \cdot \alpha$ is $w \mapsto w \cdot \alpha^{-1}$. Thus we obtain that

$$\lambda(12 \cdots s) = \sum_{\alpha \in S_s} \theta_\alpha \alpha^{-1}.$$

Hence

$$\begin{aligned} \rho(w_s) &= [(n-s+1) \cdots n] \cdot \rho(12 \cdots s) \\ &= \sum_\alpha \theta_\alpha [(n-s+1) \cdots n] \cdot \alpha \\ &= \sum_{\alpha \in S_s} \theta_\alpha [(n-s+\alpha(1)) \cdots (n-s+\alpha(s))] \end{aligned}$$

and similarly

$$\begin{aligned} \lambda((j+1) \cdots (j+s)) &= [(j+1) \cdots (j+s)] \cdot \lambda(12 \cdots s) \\ &= \sum_\alpha \theta_\alpha [(j+1) \cdots (j+s)] \cdot \alpha^{-1} \\ &= \sum_{\alpha \in S_s} \theta_\alpha [(j+\alpha^{-1}(1)) \cdots (j+\alpha^{-1}(s))]. \end{aligned}$$

This concludes the proof. ∎

Proof of Theorem 2.3. Recall that $\sigma \mapsto \sigma^{-1}$ defines an anti-isomorphism of the algebra $\mathbb{Q}[S_n]$. We denote $\downarrow \Sigma_n$ the image of Σ_n under this anti-isomorphism, and $\downarrow \Delta_s$ the algebra homomorphism $\downarrow \Sigma_n \to \downarrow \Sigma_{n-s}$ induced by Δ_s, that is, $\downarrow \Delta_s(\downarrow B_q) = \downarrow(\Delta_s(B_p))$. A fundamental remark (see [1, 2]) is that $\downarrow B_p$ has a very special form, using the shuffle product $ш$ in $\mathbb{Q}\langle A \rangle$. For a composition $p = p_1 p_2 \cdots p_k$ of n, define the factorization of the word $12 \cdots n = E_1 E_2 \cdots E_k$ to be the only factorization such that for all i's, $length(E_i) = p_i$. Then

$$\downarrow B_p = E_1 ш E_2 ш \cdots ш E_k. \tag{2.8}$$

It follows from Eqs. (2.5) and (2.6) that

$$\begin{aligned} \downarrow \pi_{\rho(w_s)}(B_p) &= \downarrow \sum_{\alpha \in S_s} \theta_\alpha \pi_{(n-s+\alpha(1)) \cdots (n-s+\alpha(s))}(B_p) \\ &= \sum_{\alpha \in S_s} \theta_\alpha \sum_{j=0}^{n-s} N((\downarrow B_p)\, [(j+\alpha^{-1}(1)) \cdots (j+\alpha^{-1}(s))]^{-1}) \\ &= \sum_{j=0}^{n-s} N\Bigg(\bigg(\sum_{\alpha \in S_s} \theta_\alpha [(j+\alpha^{-1}(1)) \\ &\qquad \cdots (j+\alpha^{-1}(s))]\bigg) \lhd (\downarrow B_p)\Bigg). \end{aligned}$$

Using Eq. (2.7), we obtain

$$\downarrow \pi_{p(w_s)}(B_p)=\sum_{j=0}^{n-s} N(\lambda((j+1)\cdots(j+s)) \lhd (\downarrow B_p)).$$

Now it is well known that the mapping $\omega_R: \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$ defined as $\omega_R(a)=a \lhd R$, is a derivation for the shuffle product. Moreover, $(P \lhd Q) \lhd R = P \lhd (Q \lhd R)$, hence $R \mapsto \omega_R$ is an algebra homomorphism. Since the Lie bracket of two derivations is also a derivation, it follows that $R \mapsto P \lhd R$ is a derivation for the shuffle of any *Lie polynomial* (element of the free Lie algebra generated by A) P, and in particular for $P=\lambda((j+1)\cdots(j+s))$. Hence by (2.8)

$$\begin{aligned}\downarrow \pi_{p(w_s)}(B_p)&=\sum_{j=0}^{n-s} N(\lambda((j+1)\cdots(j+s))\\ &\quad \lhd (E_1 \sqcup\!\sqcup E_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_k))\\ &=\sum_{j=0}^{n-s} N\Bigg(\sum_{i=1}^{k} E_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_{i-1} \sqcup\!\sqcup (\lambda((j+1)\\ &\quad \cdots(j+s)) \lhd E_i) \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_k\Bigg).\end{aligned}$$

Observe that $\lambda((j+1)\cdots(j+s))$ is the sum of $(j+1)\cdots(j+s)$ and of $\pm$permutations of this word. Moreover, E_i is an increasing word, thus

$$\begin{aligned}\lambda((j+1)\cdots(j+s)) \lhd E_i &= [(j+1)\cdots(j+s)] \lhd E_i\\ &= E_i[(j+1)\cdots(j+s)]^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\downarrow \pi_{p(w_s)}(B_p)=\sum_{i=1}^{k} N\Bigg(\sum_{j=0}^{n-s} E_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_{i-1}\\ \sqcup\!\sqcup E_i[(j+1)\cdots(j+s)]^{-1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_k\Bigg).\end{aligned}$$

In the second sum of the right-hand side of this last formula, at most one term is nonzero. Let $q_i=p_1+\cdots+p_i$, then $E_i=(q_{i-1}+1)\cdots(q_{i-1}+p_i)$; then this term is $(q_{i-1}+1)\cdots(q_{i-1}+p_i-s)$ (which is 0 if $s \geqslant p_i$). We conclude that

$$\begin{aligned}\downarrow \pi_{p(w_s)}(B_P)=\sum_{i=1}^{k} N\Bigg(\sum_{j=0}^{n-s} E_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_{i-1}\\ \sqcup\!\sqcup [(q_{i-1}+1)\cdots(q_{i-1}+p_i-s)] \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_k\Bigg).\end{aligned}$$

Now, it is easy to see that $N(\cdots)$ is equal to $E_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_{i-1} \sqcup\!\sqcup E'_i \sqcup\!\sqcup E'_{i+1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup E'_k$ where $12 \cdots n-s = E_1 \cdots E_{i-1} E'_i E'_{i+1} \cdots E'_k$ is the factorization of $12 \cdots (n-s)$ in words having respective lengths $p_1, ..., p_{i-1}, p_i - s, p_{i+1}, ..., p_k$. In other words, the shuffle in question is precisely $\downarrow B_{p(u_i)}$ where u_i is obtained by replacing p_i by $p_i - s$ in p. Finally,

$$\downarrow \pi_{p(w_s)}(B_p) = \sum_{p_i \geqslant s} \downarrow B_{p(u_i)}$$

$$= \downarrow \left(\sum_{p_i \geqslant s} B_{p(u_i)} \right) = \downarrow (\Delta_s(B_p))$$

which proves the theorem. ∎

3. Embeddings

We establish now that the surjective homomorphism Δ_s has a right inverse. More precisely, we show the existence of injective homomorphisms $\Gamma_s : \Sigma_n \to \Sigma_{n+s}$ such that $\Delta_s \circ \Gamma_s = Id$.

First of all, we recall a fundamental result of [2], in a slightly different language. Recall that two words u and v in T^* are said to be *conjugate* if for some words x and y one has $u = xy$ and $v = yx$. This is an equivalence relation, and an equivalence class is called a *circular word*. A word is *primitive* if it is not a nontrivial power of another word. Conjugation preserves primitivity, so we may speak of *primitive circular words*. Given a multiset of primitive circular words M on T, the *shape* $\lambda(M)$ is the partition $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ if M has α_i occurrences of the letter t_i. The *type* $\tau(M)$ of M is $(|C_1|, |C_2|, ..., |C_k|)$ if $M = \{C_1, C_2, ..., C_k\}$ (as a multiset) and $|C_i|$ is the *weight* of the circular word C_i, which is $i_1 + i_2 + \cdots + i_p$ if $w = t_{i_1} t_{i_2} \cdots t_{i_p}$ is a word in the equivalence class C_i. For example, the shape of the multiset M in Fig. 1 is 3322211 and its type is 6422.

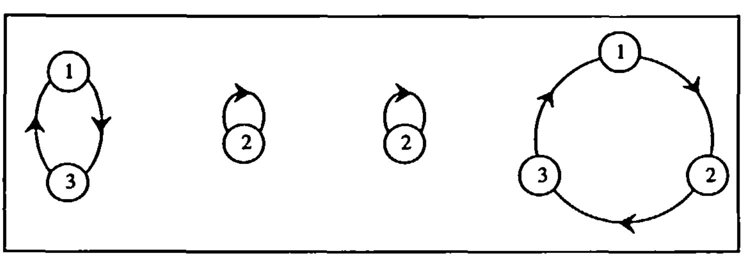

FIGURE 1

Recall the orthogonal idempotents E_λ (with λ a partition of n) of sum 1 in Σ_n that were defined in Section 2. The following result is proved in [2, Theorem 5.4]: *for two partitions λ and μ of n, the dimension of the subspace $E_\lambda \Sigma_n E_\mu$ of Σ_n is equal to the number of multisets of primitive circular words of shape λ and type μ.*

Actually, the latter result is a consequence of a more precise one. In [2], a basis $(J_p)_p$ of Σ_n, indexed by compositions of n, is defined. There is a weight preserving bijection between compositions and multisets of primitive circular words: to each composition p, considered as word on 1, 2, 3, ..., one associates the unique decreasing factorization of p in Lyndon words. Moreover Lyndon words are naturally in bijection with primitive circular words. Define the shape and type of p to be the corresponding shape and type of this multiset of primitive circular words. It is shown in [2, Theorem 5.4] that the J_p's, with p of shape λ and type μ, form a basis of $E_\lambda \Sigma_n E_\mu$.

The bijection above induces, through the mapping $M \mapsto M \cup \{s\}$, defined on multisets of primitive circular words (this mapping adds to M the circular word formed of the single letter s), a mapping $p \mapsto p \cup \{s\}$ on compositions. For any composition p, with associated multiset M, denote by $\alpha_s(p)$ the *multiplicity* of s in M (or equivalently, the number of times the word s appears in the Lyndon decomposition of p).

3.1. THEOREM. *Let n and s be fixed. Define $E = \sum_{s \in \lambda} E_\lambda$ in Σ_{n+s}, where the sum is extended to all partitions λ of $n+s$ having the part s. Then $E\Sigma_{n+s}E$ is a subalgebra Σ'_n of Σ_{n+s} with neutral element E and the restriction $\Delta_s | \Sigma'_n$ is an isomorphism $\Sigma'_n \to \Sigma_n$ sending each idempotent E_λ ($s \in \lambda$) onto $E_{\lambda \backslash s}$. In particular, the inverse of $\Delta_s | \Sigma'_n$ defines an embedding $\Gamma_s : \Sigma_n \to \Sigma_{n+s}$ such that $\Delta_s \circ \Gamma_s = Id$. This embedding is also defined by*

$$\Gamma_s(J_p) = J_{p \cup s} \tag{3.1}$$

for any composition p of n.

Proof. Since the E'_λ are orthogonal idempotents, E is clearly an idempotent and $\Sigma'_n = E\Sigma_n E$ is a subalgebra having E as neutral element. By Theorem 2.1, we have

$$\Delta_s(E) = \sum_{|\lambda| = n} E_\lambda = 1 \qquad (\text{in } \Sigma_n)$$

so that $\Delta_s(\Sigma'_n) = \Delta_s(E\Sigma_{n+s}E) = \Delta_s(\Sigma_{n+s}) = \Sigma_n$ (by Corollary 2.2). So $\Delta_s | \Sigma'_n$ is a surjective homomorphism $\Sigma'_n \to \Sigma_n$ and we have only to establish that $\dim \Sigma'_n = \dim \Sigma_n$. But this is a consequence of the result of [2] mentioned above. Indeed, $M \mapsto M \cup \{s\}$ defines a bijection between multisets of total weight n and multisets of total weight $n+s$ which have s as one of their circular words. Moreover, since the E_λ's are orthogonal, we have the direct sum decompositions

$$\Sigma_n = \bigoplus_{|\lambda| = |\mu| = n} E_\lambda \Sigma_n E_\mu$$

and

$$\Sigma'_n = \bigoplus_{|\lambda| = |\mu| = n} E_{\lambda \cup s} \Sigma_n E_{\mu \cup s};$$

To show Eq. (3.1), we only need to show that $(*)$ $\Delta_s(J_{p \cup \{s\}}) = J_p$, because $J_{p \cup \{s\}}$ is in Σ'_n. Indeed, by what we have just said, Σ'_n is a complement of $\ker \Delta_s$ in Σ_n, so that for any x in Σ_n, there is a unique y in Σ_{n+s} with $\Delta_s(y) = x$, and this y is precisely $\Gamma_s(x)$.

In order to prove $(*)$, we first recall some results of [2]. The morphism $t_i \mapsto e_i$ defines a weight preserving automorphism of the algebra $\mathbb{Q}\langle T \rangle$ (see the proof of Corollary 2.2). Now, take any composition p of n, considered as a word in the letters $e_1, e_2, \ldots$ (this is to say that $p = p_1 p_2 \cdots p_k$ is coded as $e_{p_1} e_{p_2} \cdots e_{p_k}$), then decompose it as a decreasing product of Lyndon words $p = L_1 \cdots L_m$ and define

$$K_p = \frac{1}{\tau(p)!} (b[L_1], \ldots, b[L_m])$$

where

$$(P_1, \ldots, P_m) = \frac{1}{m!} \sum_{\sigma \in S_m} P_{\sigma(1)} \cdots P_{\sigma(m)},$$

where $b[L]$ is the Lie polynomial in the variables e_i corresponding to the Lyndon word L, where $\tau(p) = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ is the type of p, and finally where

$$\tau(p)! = \alpha_1! \, \alpha_2! \cdots \alpha_n!.$$

Then $J_p = \delta(K_p)$, for the morphism $\delta: \mathbb{Q}\langle T \rangle \to \bigoplus \Sigma_n$ defined in Section 2.

Observe that $(P_1, \ldots, P_m)$ does not depend on the order of the P_i's, and that

$$(1, P_2, \ldots, P_m) = (P_2, \ldots, P_m).$$

Observe also that $\tau(p \cup \{s\})! = (a_s + 1)\, \tau(p)!$, and that $a_s + 1$ is the number of times the Lyndon word e_s appears in the decreasing factorization of $p \cup \{s\}$. Moreover, $D_s(e_s) = 1$ and $D_s(b[L]) = 0$ when $L \neq e_s$: indeed, either $b[L]=e_i$ with $i\neq s$ hence $D_s(b[L])=0$ (cf. Theorem 2.1)), or $b[L]=[b[L'], b[L'']]$ and in that case one obtains

$$D_s(b[L])=[D_s(b[L']), b[L'']]+[b[L'], D_s(b[L''])]=0$$

by induction (because $D_s(b[L'])=0$ or 1, and similarly for $D_s(b[L''])$). Note that

$$D_s(P_0, ..., P_m)=\sum_{j=0}^{m} (P_0, ..., P_{j-1}, D_s(P_j), P_{j+1}, ..., P_m)$$

so that

$$\begin{aligned} D_s(K_{p\cup\{s\}}) &= \frac{1}{\tau(p\cup\{s\})!} D_s(e_s, b[L_1], ..., b[L_m]) \\ &= \frac{1}{\tau(p)!}\frac{1}{\alpha_s+1}\Bigg\{(1, b[L_1], ..., b[L_m]) \\ &\quad + \sum_{j=1}^{m} (e_s, b[L_1], ..., b[L_{j-1}], D_s(b[L_j]), \\ &\quad b[L_{j+1}], ..., b[L_m])\Bigg\}. \end{aligned}$$

But $D_s(b[L_j])=0$ unless $L_j=e_s$ in which case it is 1, and as the latter case occurs for α_s values of j, we obtain

$$\begin{aligned} D_s(K_{p\cup\{s\}}) &= \frac{1}{\tau(p)!} (b[L_1], ..., b[L_m]) \\ &= K_p. \end{aligned}$$

Thus finally

$$\begin{aligned} \Delta_s(J_{p\cup\{s\}}) &= \Delta_s\circ\delta(K_{p\cup\{s\}}) \\ &= \delta\circ D_s(K_{p\cup\{s\}}) \\ &= \delta(K_p)=J_p \end{aligned}$$

which ends the proof. ▮

4. Symmetric Functions

Define a linear mapping $\Phi: \Sigma=\bigoplus_{n\geq 0}\Sigma_n\to \mathrm{Sym}$, the algebra of symmetric functions, by setting

$$\Phi(B_p)=h_{p_1}\cdots h_{p_k},$$

where the h's are the usual homogeneous symmetric functions (see [5]). It is shown implicitly in [7] and directly in [2] that Φ restricted to Σ_n is a surjective algebra homomorphism from Σ_n onto Sym_n, the algebra of homogeneous symmetric functions of degree n with inner product. Moreover, $\Phi(E_\lambda) = z_\lambda^{-1} p_\lambda$ where the p_λ are the power symmetric functions, and $z_\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \, \alpha_2! \cdots \alpha_n!$ when $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$.

Now define $\delta_s \colon \mathrm{Sym} \to \mathrm{Sym}$ to be the derivation which sends each h_n onto h_{n-s}; this is possible because Sym is freely generated by the h_n's. Then it is clear by Eq. (1.2) that $\Phi \circ \Delta_s = \delta_s \circ \Phi$. Moreover, by the inner multiplication table of the homogeneous symmetric functions (see [3, Lemma 2.9.16]), δ_s restricted to Sym_{n+s} is an algebra homomorphism $\mathrm{Sym}_{n+s} \to \mathrm{Sym}_n$ for the inner product. By a previous formula and Theorem 2.1, we conclude that

$$\begin{aligned} \delta_s(p_n) &= n\delta_s \circ \Phi(E_n) \\ &= n\Phi \circ \Delta_s(E_n) \\ &= \begin{cases} 0 & \text{if } n \neq s \\ s & \text{if } n = s. \end{cases} \end{aligned}$$

Hence δ_s is the derivation of Sym which sends each p_n onto 0, except p_s which is sent onto s (cf. [5, Example I.5.3]).

Note added in proof. The embeddings of Sect. 3 exist only if $s = 1$ or 2.

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