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A note on portfolios with risk-free internal gains

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Abstract

In this paper we show that if a not-necessarily-self-financing portfolio has instantaneously riskless internal gains, then on an infinitesimal time-interval, the increase in the internal gains on the portfolio is the same as the change in the price of that amount of bonds which has the same wealth as the portfolio has. As an application of this result, we derive the Black–Scholes PDE by using the original derivation of Black and Scholes, and we show that it can be made completely rigorous.

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1. Introduction

1.1. Instantaneously risk-free internal gains and risk-free interest

We consider a complete, frictionless, continuous-time market model with a single bond of constant interest rate, where no arbitrage opportunities are allowed. It is well-known that if a self-financing portfolio is risk-free in such a model, then the wealth of the portfolio must appreciate at the bond's risk-free interest rate.

Our aim here is to study whether it is possible to relate *instantaneous risk-freeness* of a portfolio to the risk-free interest rate, even when the portfolio is not self-financing.

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We remind the reader that an adapted continuous process $(t,\omega)\mapsto X_t(\omega)$ is called *instantaneously risk-free*, if the increase in X_t from t_1 until t_2 can be expressed as an ordinary (non-stochastic) integral

$$X_{t_2}-X_{t_1}=\int_{t_1}^{t_2} Z_t\,\mathrm{d}t \tag{1}$$

for some adapted process $(t,\omega)\mapsto Z_t(\omega)$, or which is the same

$$\mathrm{d}X_t=Z_t\,\mathrm{d}t.$$

The answer to the question above is "Yes," if it is not the total wealth of the portfolio which is riskless: if the process of *internal gains* on a portfolio is instantaneously riskless, then, instantaneously (that is, over an infinitesimal time-interval), the internal gain must be the same as the change in the price of the amount of bonds that has the same wealth as the portfolio. The crucial element in the proofs is the lack of arbitrage opportunities, for which the precise mathematical formulation is given in Definition 1.

The notion of *internal gains* and of the *influx of external funds* into the portfolio in case of a continuous-time model were investigated by Merton in [1]. Let V_t^{p} denote the price-process of the portfolio V_t; the change in the price over a time-interval, $V_{t_2}^{\mathrm{p}}-V_{t_1}^{\mathrm{p}}$, originates from two factors: the market prices of the instruments in the portfolio change, and money might flow in or out of the portfolio. The sum of these two parts is the total change in the wealth of the portfolio. The part that is due to the market is called *internal gains*. The other part, the influx of external funds into the portfolio, is in a sense the "*cost*" of maintaining the portfolio over the time interval in question: if the investor wants to maintain certain positions in the various assets, then it may be necessary for him to invest in the portfolio in order to do this. (We would like to emphasize that this "cost" has nothing to do with transaction costs in a market with friction.) Let us denote the internal gains over the time period $[t_1,t_2]$ by $\mathscr{G}_{t_1}^{t_2}(V_.)$. The influx of external funds will be denoted by $\mathscr{C}_{t_1}^{t_2}(V_.)$. It is not immediate how to identify these quantities for a portfolio in a continuous-time model. According to Merton in [1], they should be identified as in formulas (3) and (4) of the present paper. Merton arrives at these results through a discrete-time approximation; it seems not to be possible to derive the correct formulas purely by continuous-time considerations.

All that said, we have

$$V_{t_2}^{\mathrm{p}}-V_{t_1}^{\mathrm{p}}=\mathscr{G}_{t_1}^{t_2}(V_.)+\mathscr{C}_{t_1}^{t_2}(V_.).$$

Our aim is to show that if the process $\mathscr{G}_0^t(V_.)$ is instantaneously risk-free, that is, if there is a continuous adapted process, Y_t, such that

$$\mathrm{d}\mathscr{G}_0^t(V_.)=Y_t\,\mathrm{d}t,$$

then for all t

$$Y_t=V_t^{\mathrm{p}}\cdot r$$

almost everywhere, where r is the risk-free interest rate. This is our main result, and it is the conclusion of Theorem 3.

It is then natural to ask, what happens if instead of the gain, it is the inflow of external funds that is instantaneously risk-free. We state those results in Theorem 5, although we can say much less in that case.

As an application of Theorem 3, we shall revisit the original δ-hedge argument (sometimes called *the risk-free portfolio method*) of Black and Scholes, which they give to derive their famous formula.

2. Risk-free internal gains and risk-free interest

Suppose that our complete, frictionless market has, besides a single, riskless bond (with price β_t) of constant interest rate r, n other market instruments with price processes $S_{1,t},\ S_{2,t},\ldots,S_{n,t}$. A portfolio in such a market can be described by an $n+1$-tuple

$$V_t = (L_t, M_{1,t}, M_{2,t}, \ldots, M_{n,t})$$

where $t \in [0,T]$ represents time, and $L_t, M_{1,t}, \ldots$ are adapted processes (semimartingales would be more appropriate here, but for the mathematical content of this paper, plain adapted processes are enough); L_t stands for the number of bonds in the portfolio, $M_{1,t}$ the number of the first instrument, $M_{2,t}$ the number of the second, and so forth. The reader should note that instead of independent instruments, we rather have very correlated ones in mind (such as a stock and an option on this stock), because that is the case when the theorems below are non-trivial.

The value (or price) of such a portfolio is of course

$$V_t^{\mathrm{p}} = L_t\beta_t + \sum_{j=1}^{n} M_{j,t} S_{j,t}.$$

Following Itô, the change in the price of this portfolio from time t_1 to t_2 can be written as

$$V_{t_2}^{\mathrm{p}} - V_{t_1}^{\mathrm{p}} = \mathscr{G}_{t_1}^{t_2}(V_\cdot) + \mathscr{C}_{t_1}^{t_2}(V_\cdot), \tag{2}$$

where

$$\mathscr{G}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2} \left[L_t\,\mathrm{d}\beta_t + \sum_{j=1}^{n} M_{j,t}\,\mathrm{d}S_{j,t} \right] \tag{3}$$

and

$$\mathscr{C}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2} \left[\beta_t\,\mathrm{d}L_t + \sum_{j=1}^{n} S_{j,t}\,\mathrm{d}M_{j,t} \right] + \int_{t_1}^{t_2} \left[\sum_{j=1}^{n} \mathrm{d}\langle S_j, M_j\rangle_t \right]. \tag{4}$$

As we mentioned in the introduction, according to Merton's analysis in [1], $\mathscr{G}_{t_1}^{t_2}(V_\cdot)$ should be interpreted as that part of the change in the price of the portfolio which arises from changes in the market prices only, the interpretation of $\mathscr{C}_{t_1}^{t_2}(V_\cdot)$ is the influx of external funds into our portfolio over this time interval (i.e. the "cost" of ensuring the right amount of instruments in the portfolio).

In this terminology, a portfolio V_t is self-financing if and only if

$$\mathscr{C}_{t_1}^{t_2}(V_\cdot) = 0 \tag{5}$$

almost everywhere for any time interval $[t_1, t_2] \subset [0, T]$.

It goes without saying that all the above stochastic processes are defined over and event space Ω with a filtration $\mathscr{F}_t$ and a probability measure P.

In order to avoid the pathologies of doubling portfolio-strategies, it is common to require that the losses on portfolios are bounded. We will not explicitly require this, which is only a matter of convenience from our part. Everything in the paper can be done so that this requirement is imposed on all portfolios, making the reasoning somewhat more cumbersome, but not at all more lucid.

We first recall a precise formulation of arbitrage opportunity from [2].

Definition 1. A self-financing portfolio W_t, is called a risk-free bond arbitrage on $[t_1, t_2]$ if there is a $\lambda \in \mathbf{R}$ such that $P(W_{t_1}^{\mathrm{p}}/\beta_{t_1} \leqslant \lambda) = 1$, $P(W_{t_2}^{\mathrm{p}}/\beta_{t_2} \geqslant \lambda) = 1$, and $P(W_{t_2}^{\mathrm{p}}/\beta_{t_2} > \lambda) > 0$.

That is, the portfolio makes a profit with some nonzero probability, but it surely does not create a loss by the end of the interval.

Theorem 3 is our main result. It is the non-self-financing analogue of the well known fact that the price of an (instantaneously) risk-free self-financing portfolio must follow the price of an appropriate amount of the bond. The meaning of our result is that if the process of internal gains is instantaneously risk-free, then, the increase in the internal gains locally follows the increase in the price of that amount of the bond that initially has the same value as the value of the portfolio. We first prove the following lemma.

Lemma 2. *Suppose we have two portfolios V_t and $\tilde{V}_t$, such that for each $[t_1, t_2] \subset [0, T]$,*

$$\mathscr{G}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2} Y_t \,\mathrm{d}t \tag{6}$$

and

$$\mathscr{G}_{t_1}^{t_2}(\tilde{V}_\cdot) = \int_{t_1}^{t_2} \tilde{Y}_t \,\mathrm{d}t \tag{7}$$

hold, where Y_t and $\tilde{Y}_t$ are continuous adapted processes. If $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ for all $t \in [0, T]$ and if the market model contains no risk-free bond arbitrage, then for each $t \in [0, T]$,

$$Y_t = \tilde{Y}_t \quad \text{a.e. on } \Omega. \tag{8}$$

Proof. The main idea is to reason from the lack of arbitrage opportunities in the following way: if at a certain moment, t_0, we recognize that $Y_{t_0}(\omega) > \tilde{Y}_{t_0}(\omega)$, then we start maintaining a long position in V_t, and a short position in $\tilde{V}_t$. The initial transaction at t_0, namely, creating the long position of V_{t_0} and the short position of $\tilde{V}_{t_0}$, has no cost, since we assumed that the two portfolios have equal prices. We maintain these positions until $Y_t(\omega)$ and $\tilde{Y}_t(\omega)$ become equal. Before that happens, the inflow of external funds into V_t is less than inflow into $\tilde{V}_t$ (since $Y_t(\omega) > \tilde{Y}_t(\omega)$), therefore, maintaining $V_t - \tilde{V}_t$ produces money surplus (see formula (19)), which we continually invest in the bond (the amount of the bond that piles up until time t this way will be denoted by L_t below, see (9)). $V_t - \tilde{V}_t$ together with these bonds constitute a portfolio W_t, which is a risk-free bond arbitrage: it is self-financing, it has zero wealth at the beginning, and as large a wealth at the end, as it is the value of the bonds that pile up this way. But we assumed that the market model accommodates no risk-free bond arbitrage, therefore $Y_{t_0}(\omega) \leqslant \tilde{Y}_{t_0}(\omega)$. $Y_{t_0}(\omega) \geqslant \tilde{Y}_{t_0}(\omega)$ is shown in a similar manner, so $Y_{t_0}(\omega) = \tilde{Y}_{t_0}(\omega)$. We now present the details of this argument.

Suppose there is a $t_0 \in [0, T]$ with $P(Y_{t_0} > \tilde{Y}_{t_0}) > 0$. For each event $\omega \in \Omega$, let

$$\tau(\omega) = \min\{t : \ t_0 \leqslant t < T \ \text{ and } \ Y_t(\omega) \leqslant \tilde{Y}_t(\omega), \ \text{ or } \ t = T\}.$$

The function τ is then a stopping time.

In what follows we use $a \vee b$ and $a \wedge b$, for two reals a and b, to denote the larger and the smaller of a and b, respectively.

We construct the following portfolios. Let

$$U_t(\omega) = (L_{(t_0 \vee t) \wedge \tau(\omega)}(\omega), 0, \ldots, 0) \tag{9}$$

where L_t is determined by $L_{t_0} \equiv 0$ and

$$\mathrm{d}L_t = [Y_t - \tilde{Y}_t] \exp(-rt)/\beta_0 \, \mathrm{d}t. \tag{10}$$

Let

$$W_t(\omega) = \begin{cases} U_t(\omega) & \text{for } t < t_0, \\ V_t(\omega) - \tilde{V}_t(\omega) + U_t(\omega) & \text{for } t \in [t_0, \tau(\omega)], \\ U_t(\omega) & \text{for } t > \tau(\omega). \end{cases} \tag{11}$$

We want to show that W_t is a risk-free bond arbitrage. Observe, that for $t \in [0, t_0]$, $W_t = U_t = (0, 0, \ldots, 0)$, and for $t \in [\tau(\omega), T]$, $W_t(\omega) = (L_{\tau(\omega)}(\omega), 0, \ldots, 0)$. Note also that the assumption $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ implies that the price process

$$W_t^{\mathrm{p}} = U_t^{\mathrm{p}} = L_{(t_0 \vee t) \wedge \tau} \beta_t$$

is continuous.

We first show that W_t is self-financing. To this end, fix an arbitrary ω event, and then take an interval $[t_1, t_2] \subset [0, T]$ with $t_1 \leqslant \tau(\omega)$ and $t_2 \geqslant t_0$. The definition of W_t shows that, on this ω

$$\mathscr{C}_{t_1}^{t_2}(W_\cdot)(\omega) = \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V_\cdot - \tilde{V}_\cdot)(\omega) + \mathscr{C}_{t_1}^{t_2}(U_\cdot)(\omega). \tag{12}$$

Since $U_t(\omega)$ is constant on $[t_1, t_0]$ and on $[\tau(\omega), t_2]$, there is no influx over these intervals

$$\mathscr{C}_{t_1}^{t_2}(U_\cdot)(\omega) = \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(U_\cdot)(\omega).$$

Then (4) and (10) show that

$$\mathscr{C}_{t_1}^{t_2}(U.)(\omega) = \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(U.)(\omega) \tag{13}$$

$$= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} \beta_t \, \mathrm{d}L_t(\omega) \tag{14}$$

$$= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} \beta_0 \exp(rt)[Y_t(\omega) - \tilde{Y}_t(\omega)] \exp(-rt)/\beta_0 \, \mathrm{d}t$$

$$= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} [Y_t(\omega) - \tilde{Y}_t(\omega)] \, \mathrm{d}t. \tag{15}$$

On the other hand,

$$\mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) = \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V.)(\omega) - \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(\tilde{V}.)(\omega) \tag{16}$$

$$= V^{\mathrm{p}}_{t_2 \wedge \tau(\omega)} - V^{\mathrm{p}}_{t_1 \wedge t_0} - \tilde{V}^{\mathrm{p}}_{t_2 \wedge \tau(\omega)} + \tilde{V}^{\mathrm{p}}_{t_1 \wedge t_0}$$

$$- \mathscr{G}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V.) + \mathscr{G}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(\tilde{V}.)(\omega) \tag{17}$$

$$= - \mathscr{G}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) \tag{18}$$

$$= - \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} [Y_t(\omega) - \tilde{Y}_t(\omega)] \, \mathrm{d}t, \tag{19}$$

where we used (2), (6), (7) and $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$. But then

$$\mathscr{C}_{t_1}^{t_2}(W.)(\omega) = \mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) + \mathscr{C}_{t_1}^{t_2}(U.)(\omega) = 0.$$

So far we assumed that $t_1 \leqslant \tau(\omega)$ and $t_2 \geqslant t_0$. If $t_2 \leqslant t_0$ or $t_1 \geqslant \tau(\omega)$ then $\mathscr{C}_{t_1}^{t_2}(W.)(\omega) = 0$ trivially by (11). Hence we get $\mathscr{C}_{t_1}^{t_2}(W.)(\omega) = 0$ for any interval $[t_1, t_2] \subset [0, T]$. But ω was arbitrary, therefore W_t is self-financing.

Now, if ω is such an event that $Y_{t_0}(\omega) > \tilde{Y}_{t_0}(\omega)$, then, for any $t \in (t_0, \tau(\omega)) \neq \emptyset$, $Y_t(\omega) > \tilde{Y}_t(\omega)$, which implies $L_{\tau(\omega)}(\omega) > 0$ since remember, (10) holds. Thus, along the initial hypotheses $P(Y_{t_0} > \tilde{Y}_{t_0}) > 0$, we arrive at

$$P(L_\tau > 0) > 0$$

and therefore

$$P(W_T^{\mathrm{p}}/\beta_T > 0) = P(L_\tau > 0) > 0.$$

But $W_0^{\mathrm{p}}/\beta_0 = 0/\beta_0 = 0$ and $W_T^{\mathrm{p}}/\beta_T = L_\tau \geqslant 0$, that is, we have a risk-free arbitrage because the three requirements of Definition 1 are satisfied with $\lambda = 0$. Risk free arbitrage is not allowed, so our initial hypotheses was wrong: $Y_t \leqslant \tilde{Y}_t$ must hold almost everywhere. Proving $Y_t \geqslant \tilde{Y}_t$ a.e. is similar, hence $Y_t = \tilde{Y}_t$ almost everywhere. □

Theorem 3. *Suppose we have a portfolio V_t and $\mathcal{G}_{t_1}^{t_2}(V.)$ has the representation*

$$\mathcal{G}_{t_1}^{t_2}(V.) = \int_{t_1}^{t_2} Y_t \, \mathrm{d}t, \tag{20}$$

where Y_t a continuous adapted process. If the market model contains no risk-free bond arbitrage, then for all $t \in [0, T]$

$$Y_t = V_t^{\mathrm{p}} \cdot r \quad \text{a.e. on } \Omega. \tag{21}$$

Proof. We choose a specific $\tilde{V}_t$ for the other portfolio in Lemma 2: one that consists of bonds only, and exactly V_t^{p} worth bonds: $\tilde{V}_t = (V_t^{\mathrm{p}}/\beta_t, 0, \ldots, 0)$. Then

$$\tilde{V}_t^{\mathrm{p}} = \frac{V_t^{\mathrm{p}}}{\beta_t}\beta_t = V_t^{\mathrm{p}}. \tag{22}$$

For this new portfolio

$$\mathcal{G}_{t_1}^{t_2}(\tilde{V}.) = \int_{t_1}^{t_2} \frac{V_t^{\mathrm{p}}}{\beta_t} \, \mathrm{d}\beta_t = \int_{t_1}^{t_2} V_t^{\mathrm{p}} \cdot r \, \mathrm{d}t. \tag{23}$$

From Lemma 2, it follows that the integrands in the two market-based gains, (20) and (23) must agree

$$Y_t = V_t^{\mathrm{p}} \cdot r$$

almost everywhere in Ω, for all $t \in [0, T]$. □

Corollary 4. *Suppose we have two portfolios V_t and $\tilde{V}_t$, such that for each $[t_1, t_2] \subset [0, T]$,*

$$\mathcal{G}_{t_1}^{t_2}(V.) = \int_{t_1}^{t_2} Y_t \, \mathrm{d}t \tag{24}$$

and

$$\mathcal{G}_{t_1}^{t_2}(\tilde{V}.) = \int_{t_1}^{t_2} \tilde{Y}_t \, \mathrm{d}t \tag{25}$$

hold, where Y_t and $\tilde{Y}_t$ are continuous adapted processes. If the market model contains no risk-free bond arbitrage, then for all $t \in \Omega$

$$V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}} \quad \text{a.e. on } \Omega \tag{26}$$

if and only if for all $t \in [0, T]$

$$Y_t = \tilde{Y}_t \quad \text{a.e. on } \Omega. \tag{27}$$

Proof. According to Theorem 3, $Y_t = V_t^{\mathrm{p}} \cdot r$ a.e. and $\tilde{Y}_t = \tilde{V}_t^{\mathrm{p}} \cdot r$ a.e., so $Y_t = \tilde{Y}_t$ holds almost everywhere if and only if $V_t^{\mathrm{p}} \cdot r = \tilde{V}_t^{\mathrm{p}} \cdot r$ almost everywhere, which holds if and only if $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ a.e. (for all $t \in [0, T]$). □

It is now natural to ask what happens if not the internal gain, but the influx of external funds were represented by an ordinary integral. In this case we have the following theorem, which is analogous to Lemma 2; no proposition analogous to Theorem 3 or the corollary can be proven for this case.

Theorem 5. *Suppose we have two portfolios* V_t *and* $\tilde{V}_t$, *such that for each* $[t_1, t_2] \subset [0, T]$

$$\mathscr{C}_{t_1}^{t_2}(V.) = \int_{t_1}^{t_2} Y_t \, \mathrm{d}t \tag{28}$$

and

$$\mathscr{C}_{t_1}^{t_2}(\tilde{V}.) = \int_{t_1}^{t_2} \tilde{Y}_t \, \mathrm{d}t \tag{29}$$

hold, where Y_t *and* $\tilde{Y}_t$ *are continuous adapted processes. If* $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ *for all* $t \in [0, T]$ *and if the market model contains no risk-free bond arbitrage, then for all* $t \in \Omega$

$$Y_t = \tilde{Y}_t \quad \text{a.e. on } \Omega. \tag{30}$$

Proof. The proof of this theorem follows line by line the proof of Lemma 2, except that W_t now should be defined as

$$W_t(\omega) = \begin{cases} U_t(\omega) & \text{for } t < t_0, \\ -[V_t(\omega) - \tilde{V}_t(\omega)] + U_t(\omega) & \text{for } t \in [t_0, \tau(\omega)], \\ U_t(\omega) & \text{for } t > \tau(\omega) \end{cases} \tag{31}$$

instead of Eq. (12), we have

$$\mathscr{C}_{t_1}^{t_2}(W.)(\omega) = -\mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) + \mathscr{C}_{t_1}^{t_2}(U.)(\omega) \tag{32}$$

here, and finally, Eqs. (16)–(19) have to be replaced by

$$\mathscr{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) = \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} [Y_t(\omega) - \tilde{Y}_t(\omega)] \, \mathrm{d}t. \tag{33}$$

The rest is the same. □

3. The Black–Scholes PDE as an application

It has been noted in the literature that the original derivation by Black and Scholes [3] of the price of an option, $w(x, t)$ on a stock whose price is x at time t is not completely rigorous. Indeed, they made use of a portfolio that is not self-financing, but treat it as though it were self financing: see [4] or [5], p. 129–130. A brief analysis of this can also be found in [2]. But, using the results of the previous section, we shall see that *the structure of the Black–Scholes derivation is correct*, although the comments and explanations that they provide to support their derivation are incorrect, as it was also pointed out in [6].

3.1. The Black–Scholes derivation as an application of Theorem 3

Let the random variable S_t denote the price of the stock in question at time $t \in [0, T]$; we assume that the price-process $(t, \omega) \mapsto S_t(\omega)$ satisfies the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t, \tag{34}$$

where μ and σ are constants and where B_t is the standard Brownian motion.

We then ask what happens, if there is a twice continuously differentiable function $f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ such that $(t, \omega) \mapsto f(t, S_t(\omega))$ gives the price-process of the European call option on the stock? As we shall see (as Black and Scholes saw), the δ-hedge portfolio argument below determines a PDE that such a function f must satisfy. Black and Scholes solved the PDE, and the set of solutions provided a unique twice-differentiable f for each possible call option on the given stock. This made the initial idea of searching for functions among the twice differentiable ones quite plausible, since it gave a unique solution for each situation.

Let $\partial_1 f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ denote the partial derivative with respect to the first (time) variable, whereas $\partial_2 f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ denotes the partial derivative with respect to the second variable. For simplicity, let $f(t, S_t)$ denote the random variable $\omega \mapsto f(t, S_t(\omega))$.

Let β_t mean the price of the bond again, which we accept to be governed by

$$\mathrm{d}\beta_t = r\beta_t \,\mathrm{d}t. \tag{35}$$

The original derivation of Black and Scholes requires a portfolio that contains linear combinations only of the bond, the stock and the European option on the stock. A portfolio like this is represented by a triple

$$W_t = (L_t, M_{1,t}, M_{2,t}),$$

where $t \in [0, T]$ represents time, and $L_t, M_{1,t}, M_{2,t}$ are adapted processes; the first denotes the number of bonds in the portfolio, the second the number of stocks, the third stands for the number of options.

More specifically, consider the portfolio X_t that has exactly one stock and $-1/\partial_2 f(t, S_t)$ amount of options, i.e. $X_t = (0, 1, -1/\partial_2 f(t, S_t))$ and

$$X_t^{\mathrm{p}} = S_t - f(t, S_t)/\partial_2 f(t, S_t). \tag{36}$$

Then, by (3), the market-based internal gain in our portfolio is

$$\mathscr{G}_{t_1}^{t_2}(X_.) = \int_{t_1}^{t_2} \left[\mathrm{d}S_t - \frac{1}{\partial_2 f(t, S_t)} \mathrm{d}[f(t, S_t)] \right]. \tag{37}$$

By Itô's formula, we see that

$$\begin{aligned} \mathrm{d}[f(t, S_t)] &= \partial_1 f(t, S_t)\,\mathrm{d}t + \partial_2 f(t, S_t)\,\mathrm{d}S_t + \tfrac{1}{2}\partial_2^2 f(t, S_t)\,\sigma^2 S_t^2\,\mathrm{d}t \\ &= \partial_2 f(t, S_t)\,\mathrm{d}S_t + [\partial_1 f(t, S_t) + \tfrac{1}{2}\sigma^2 S_t^2 \partial_2^2 f(t, S_t)]\,\mathrm{d}t. \end{aligned} \tag{38}$$

Therefore,

$$\mathrm{d}S_t - \frac{1}{\partial_2 f(t, S_t)}\,\mathrm{d}[f(t, S_t)] = -\frac{1}{\partial_2 f(t, S_t)}\left[\partial_1 f(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2^2 f(t, S_t)\right]\mathrm{d}t. \tag{39}$$

Hence

$$\mathscr{G}_{t_1}^{t_2}(X.) = \int_{t_1}^{t_2} \frac{-1}{\partial_2 f(t, S_t)}\left[\partial_1 f(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2^2 f(t, S_t)\right]\mathrm{d}t. \tag{40}$$

The integral turns out to be just an ordinary one. Using Theorem 3 (in particular, Eq. (21)) and (36), we get that the integrand is given by

$$-\frac{1}{\partial_2 f(t, S_t)}\left[\partial_1 f(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2^2 f(t, S_t)\right] = [S_t - f(t, S_t)/\partial_2 f(t, S_t)]r \tag{41}$$

almost everywhere for all t. That is, if f is twice differentiable and gives the price of the option, it must certainly satisfy the equation above. Since S_t is a geometric Brownian motion, for any $t \in [0, T]$, and any $x \in \mathbf{R}^+$, $P[|S_t - x| \leqslant 1/n] > 0$ whenever $n \in \mathbf{N}$, so there is an $\omega_n \in \Omega$ on which Eq. (41) is satisfied and $|S_t(\omega_n) - x| \leqslant 1/n$. Since in Eq. (41), f and its derivatives are all continuous, we conclude that

$$-\frac{1}{\partial_2 f(t, x)}\left[\partial_1 f(t, x) + \frac{1}{2}\sigma^2 x^2 \partial_2^2 f(t, x)\right] = [x - f(t, x)/\partial_2 f(t, x)]r \tag{42}$$

for all $(t, x) \in [0, T] \times \mathbf{R}^+$. After a bit of rearranging we receive the famous PDE

$$\partial_1 f(t, x) = rf(t, x) - rx\partial_2 f(t, x) - \tfrac{1}{2}\sigma^2 x^2 \partial_2^2 f(t, x). \tag{43}$$

3.2. Connection with the original

Here, we briefly indicate that our derivation really follows the original line by line. To avoid confusion, we use double parentheses (( )) for referring to formulas of the paper of Black and Scholes, and parentheses ( ) for referring to ours.

Our Eq. (36) clearly corresponds to their formula ((2)). Their ((3)), which is not the increase in the value of the asset, but the increase that is due to the market only, appears as a stochastic integral on the right-hand side of (37). On their formulas ((4)) and ((5)), we have (38) and (40) to reflect. We arrive at the analogue of ((6)) by receiving Eqs. (41) and (42) via a slightly different arbitrage argument (Theorem 3) than that of Black and Scholes. Finally, ((7)) and (43) are identical.

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