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On Σ_1 and Π_1 sentences and degrees of interpretability

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1. Let D_T be the lattice of degrees of interpretability defined in [4] (cf. also [7]). Let Γ be a set of sentences. A degree $a \in D_T$ is Γ if there is a Γ sentence φ such that $a = d(T + \varphi)$. All degrees are Π_2 and Σ_2 (cf. [3], [4]). This was improved by Franco Montagna (private communication) who observed that all degrees are Δ_2 (Theorem 1). But then it is natural to ask if all degrees are B_1, where B_1 is the set of Boolean combinations of Σ_1 sentences. Not unexpectedly the answer is negative (Theorem 2(i)); in fact, every nontrivial interval $[a, b]$ $(=\{c: a \leqslant c \leqslant b\}$, where $a < b)$ has a nontrivial subinterval containing no B_1 degree (Theorem 2(ii)). We also show that there are Σ_1 degrees a_0, a_1 such that $a_0 \cup a_1$ is not B_1 (Theorem 2(iii)). This would follow trivially from Theorem 2(i) if every degree is the least upper bound (l.u.b) of two (finitely many) Σ_1 degrees. Thus it is relevant to show that that is not true (Corollary 1); in fact, there is a Π_1 degree which cannot be obtained from 0 by taking finite l.u.b.s and g.l.b.s (greatest lower bounds) and Σ_1-extensions (defined below) (Theorem 3). We then go on to prove (a result implying) that there is a degree which is not the l.u.b. of a finite set of Σ_1 and Π_1 degrees (Theorem 4). A degree a is said to cup to b if there is a $c < b$ such that $a \cup c = b$. One way to improve Corollary 1 would be to show that there is a Π_1 degree $a > 0$ such that no Σ_1 degree cups to a. We prove (a result implying) that there is no such degree (Theorem 5). In [4] it is shown that there is a degree $a < 1$ which cups to every degree b such that $a \leqslant b < 1$. We improve this by showing that a can be taken to be Σ_1 (Theorem 6). The above mentioned consequence of Theorem 4 leads to the question if there is a degree $a > 0$ such that no Π_1 degree cups to a. We show that the answer is affirmative (Theorem 7; this result was not obtained until after the PIA conference at Utrecht). Finally we consider the existence of pseudocomplements. In [4] it is shown that there is a

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degree which has no pseudocomplement (p.c.). We now improve this by showing that there is a Σ_1 degree with no p.c. (Theorem 8). Trivially every Π_1 degree has a p.c. This leads to the question if there is a Σ_1 and non-Π_1 degree which has a p.c. We show that the answer is affirmative (Theorem 9). We mention several open problems. * (See Note added in proof.)

2. As in [4] T is a consistent primitive recursive essentially reflexive extension of Peano arithmetic PA. (A theory is identified with the set of its axioms.) Let $\tau(x)$ be a PR binumeration of T. A, B, etc. are primitive recursive extensions of T in the language of T. Thus A, B, etc. are essentially reflexive. We write $A \vdash X$ or $X \dashv A$, where X is a set of sentences, to mean that all members of X are provable in A. Thus $A \dashv B$ means that A is a subtheory of B. We write $A \dashv_\Gamma B$, A is a *Γ-subtheory* of B, to mean that every sentence in Γ provable in A is provable in B. A sentence φ is *Γ-conservative* over A if $A + \varphi \dashv_\Gamma A$ (cf. [2]). A is a *Γ-conservative extension* of B if $B \dashv A \dashv_\Gamma B$. $A \leqslant B$ means that A is interpretable in B, $A < B$ that $A \leqslant B \nleqslant A$, and $A \equiv B$ that $A \leqslant B \leqslant A$. *The basic lemma of the subject is the result that $A \leqslant B$ iff $A \dashv_{\Pi_1} B$.* This lemma will be used in what follows without further comment. (If we drop the assumption that T is (essentially) reflexive our results remain true provided $\leqslant$ is replaced by $\dashv_{\Pi_1}$.) $\equiv$ is an equivalence relation; its equivalence classes are called *degrees* (*of interpretability*) and are written a, b, c, etc. $a \leqslant b$ iff $A \leqslant B$ where $A \in a$ and $B \in b$. $d(A)$ is the degree of A. D_T is the partially ordered set of degrees thus defined. As is easily verified D_T is a distributive lattice (cf. [4, Theorem 1]). $a \cap b$ and $a \cup b$ are the g.l.b. and the l.u.b. of a and b, respectively. D_T has a smallest element $0 = d(T)$ and a largest element 1, the common degree of all inconsistent theories. $S_0 \downarrow S_1 = \{\varphi \vee \psi\colon\ \varphi \in S_0 \,\&\, \psi \in S_1\}$ and so $d(A \downarrow B) = d(A) \cap d(B)$; $A \uparrow B$ is a theory such that $d(A \uparrow B) = d(A) \cup d(B)$ (cf. [4, Lemma 8]). When there is no risk of confusion we use φ and X in place of $T + \varphi$ and $T + X$. Thus $d(\varphi)$ is $d(T + \varphi)$, $X \leqslant \varphi$ means that $T + X \leqslant T + \varphi$, etc. We use $a \ll b$ to mean that $a < b$ and for every c, if $b \cap c = a$, then $c = a$. $A \ll B$ means that $d(A) \ll d(B)$. σ, σ_0, etc. denote Σ_1 sentences and π, π_0, etc. denote Π_1 sentences. Notation and terminology not explained here are standard (cf. [1]).

In what follows Γ is either Π_n or Σ_n, $n > 0$, and Γ^d is the dual of Γ. Γ-true(x) is a Γ partial truth-definition for Γ sentences. $\alpha(x)$ is a PR binumeration of A. Let $[\Gamma]_\alpha(x, y)$ be the formula

$$\forall uv \leqslant y\ (u \text{ is } \Gamma \wedge \mathrm{Prf}_{\alpha(z) \vee z = x}(u, v) \rightarrow \Gamma\text{-true}(u)).$$

The following lemma is then easily verified (cf. [3, Lemma 1]).

Lemma 1. *$[\Gamma]_\alpha(x, y)$ is a Γ formula such that*

(i) $\mathrm{PA} \vdash [\Gamma]_\alpha(x, y) \wedge z \leqslant y \rightarrow [\Gamma]_\alpha(x, z)$,

(ii) $A + \varphi \vdash [\Gamma]_\alpha(\bar{\varphi}, \bar{m})$ *for all* φ *and* m,

(iii) *if* ψ *is* Γ *and* $A + \varphi \vdash \psi$, *then there is a* q *such that* $\mathrm{PA} + [\Gamma]_\alpha(\bar{\varphi}, \bar{q}) \vdash \psi$.

If $\Gamma = \Pi_n$ let $\xi(x)$ be such that

$$\mathrm{PA} \vdash \xi(\bar{k}) \leftrightarrow \forall y\, ([\Sigma_n]_\alpha(\overline{\xi(\bar{k})}, y) \rightarrow \chi(\bar{k}, y)).$$

If $\Gamma = \Sigma_n$ let $\xi(x)$ be such that

$$\mathrm{PA} \vdash \xi(\bar{k}) \leftrightarrow \exists y\, (\neg[\Pi_n]_\alpha(\overline{\xi(\bar{k})}, y) \wedge \forall z \leq y\, \chi(\bar{k}, z)).$$

From Lemma 1 we get (cf. [3], Lemma 2):

Lemma 2. *If $\chi(x, y)$ is Γ, then $\xi(x)$ is Γ and*
(i) $A + \xi(\bar{k}) \vdash \chi(\bar{k}, \bar{m})$,
(ii) $A + \xi(\bar{k}) \dashv_{\Gamma^d} A + \{\chi(\bar{k}, \bar{q}) : q \in \mathbb{N}\}$.

Clearly, if for some m, $A \vdash \neg\chi(\bar{k}, \bar{m})$, then $A \vdash \neg\xi(\bar{k})$. Also note that if $A \vdash \chi(\bar{k}, \bar{m})$, for all m, then (ii) implies that $\xi(\bar{k})$ is Γ^d-conservative over A.

A set χ of sentences is *monoconsistent with A* if $A + \varphi$ is consistent for every $\varphi \in X$. Suppose X is recursively enumerable (r.e.) and let $R(k, m)$ be a primitive recursive relation such that $X = \{k : \exists m\, R(k, m)\}$. Let $\rho(x, y)$ be a PR binumeration of $R(k, m)$. Let $\xi(x)$ be as in Lemma 2 with $\chi(x, y) := \neg\rho(x, y)$ and let φ be such that $\mathrm{PA} \vdash \varphi \leftrightarrow \xi(\bar{\varphi})$. Then (cf. [3, Corollary 1]):

Lemma 3. *If X is r.e. and monoconsistent with A, then there is, and we can effectively find, a Γ sentence φ such that $\varphi \notin X$ and φ is Γ^d-conservative over A.*

Let Y be any primitive recursive set and let $\eta(x)$ be a PR binumeration of Y. Then, by Lemma 2 with $\chi(x, y) := \eta(y) \rightarrow \Gamma\text{-true}(y)$ we get (i) of our next lemma (cf. [3, Theorem 4]); (ii) and (iii) are obtained by a straightforward extension of this construction.

Lemma 4. (i) *To any r.e. set Y of Γ sentences, there is a Γ sentence φ such that $T + \varphi$ is a Γ^d-conservative extension of $T + Y$.*

Let $R(k, m)$ be an r.e. relation such that $X_k = \{m : R(k, m)\}$ is a set of Γ sentences.

(ii) *There is a Γ formula $\gamma(x)$ such that for every k, $A + \gamma(\bar{k})$ is a Γ^d-conservative extension of $A + X_k$.*

(iii) *There is a Γ formula $\gamma(x, y)$ such that for every k and every sentence φ, $A + \varphi + \gamma(\bar{\varphi}, \bar{k})$ is a Γ^d-conservative extension of $A + X_k + \varphi$.*

The following lemma is an immediate consequence of Lemma 4(i) (cf. [3, Theorem 11]).

Lemma 5. *To every r.e. set X of Σ_1 sentences, there is a Σ_1 sentence σ such that $A + \sigma$ is a Π_1-conservative extension of $A + X$ and consequently $A + \sigma \equiv A + X$.*

From a slight generalization of Lemma 2 for $\Gamma = \Sigma_1$ we also get the following (cf. [4, Lemma 6]):

Lemma 6. *Let X be an r.e. set. There is then a PR formula $\eta(y, x, z)$ such that for all k and θ,*

(i) *if $k \in X$, then $T + \theta \vdash \neg\exists z\, \eta(\bar{\theta}, \bar{k}, z)$,*

(ii) *if $k \notin X$, then $\exists z\, \eta(\bar{\theta}, \bar{k}, z)$ is Π_1-conservative over $T + \theta$.*

3. Let us begin with Montagna's observation.

Theorem 1. *Every degree is Δ_2.*

Proof. Let a be any degree. Using Craig's trick, there is a primitive recursive set X of Π_1 sentences such that $a = d(X)$. Let $\xi(x)$ be a PR binumeration of X and let φ be such that

$$\mathrm{PA} \vdash \varphi \leftrightarrow \forall z\, ([\Pi_1]_\tau(\bar{\varphi}, z) \rightarrow (\xi(z) \rightarrow \Pi_1\text{-true}(z))).$$

Then φ is Π_2 and $T + \varphi$ is a Π_1-conservative extension of $T + X$ (cf. the above proof of Lemma 4(i)). It follows that $a = d(\varphi)$. By Lemma 1(i),

$$[\Pi_1]_\tau(x, z) \wedge u \leqslant z \rightarrow [\Pi_1]_\tau(x, u).$$

Using this it is easily verified that φ is also Σ_2:

$$\mathrm{PA} \vdash \varphi \leftrightarrow \forall z\, (\xi(z) \rightarrow \Pi_1\text{-true}(z)) \vee \exists z\, (\neg[\Pi_1]_\tau(\bar{\varphi}, z) \wedge \forall u < z\, (\xi(u) \rightarrow \Pi_1\text{-true}(u))).$$

Thus φ is Δ_2. $\square$

In much the same way and using Corollary 3 of [5] it can be shown that for every A, there is a Δ_2 sentence φ such that A and $T + \varphi$ are mutually faithfully interpretable (defined in [5]) thus somewhat improving Corollary 4 of [5].

We now show that Theorem 1 is optimal in the sense that Δ_2 cannot be replaced by B_1.

Theorem 2. (i) *Not all degrees are B_1.*

(ii) *Every nontrivial interval $[a, b]$ has a nontrivial subinterval containing no B_1 degree.*

(iii) *If T is Σ_1-sound, then there are Σ_1 degrees a_0, a_1 such that $a_0 \cup a_1$ is not B_1.* *

To prove Theorem 2 we need Lemmas 8 and 9 below. The following easy lemma will be used repeatedly (cf. [4, Lemma 14]).

Lemma 7. *$A \ll B$ iff $A < B$ and for every σ, if $B \leqslant A + \sigma$, then $A \vdash \neg\sigma$. Thus, in particular, if A is consistent and $\neg\pi$ is Π_1-conservative over A, then $A \ll A + \pi$.*

Lemma 8. *Suppose φ is B_1 and X is r.e. and for every k, $X \restriction k \ll X$.*
(i) *If $X \leqslant \varphi$, then $X \ll \varphi$.*
(ii) *If $\varphi \leqslant X$, then $\varphi \ll X$.*

Proof. (i) φ can be written in the form $(\pi_0 \wedge \sigma_0) \vee \cdots \vee (\pi_n \wedge \sigma_n)$. Now for any degrees a, b, c, if $a \ll b$ and $a \ll c$, then $a \ll b \cap c$. Thus it suffices to show that if $X \leqslant \pi \wedge \sigma$, then $X \ll \pi \wedge \sigma$. Let χ be a Σ_1 sentence such that $\pi \wedge \sigma \leqslant X + \chi$. Then, by Lemma 7, it suffices to show that $T + X \vdash \neg\chi$. Now, by assumption, there is a k such that $T + X \restriction k + \chi \vdash \pi$. Hence $T + \pi \wedge \sigma \dashv T + X \restriction k + (\chi \wedge \sigma)$ and so $X \leqslant X \restriction k + (\chi \wedge \sigma)$. But then since $X \restriction k \ll X$, by Lemma 7, $T + X \vdash \neg (\chi \wedge \sigma)$. But $X \leqslant \pi \wedge \sigma$. It follows that $T + \pi \wedge \sigma \vdash \neg\chi$, whence $T + X \vdash \neg\chi$, as was to be shown.

(ii) Let σ be such that $X \leqslant \varphi \wedge \sigma$. It suffices to show that $T + \varphi \vdash \neg\sigma$. Now $\varphi \wedge \sigma$ is B_1. Hence, by (i), $X \ll \varphi \wedge \sigma$. It follows that $\varphi \ll \varphi \wedge \sigma$. But this is possible only if $T + \varphi \vdash \neg\sigma$. □

To prove part (iii) of Theorem 2 we need the following lemma from [4] (Lemma 11).

Lemma 9. *Given a true Π_1 sentence θ and an r.e. set X monoconsistent with* PA *we can effectively find Π_1 sentences θ_i such that*
(i) $\mathrm{PA} \vdash \theta_0 \vee \theta_1$,
(ii) $\mathrm{PA} \vdash \theta_0 \wedge \theta_1 \rightarrow \theta$,
(iii) $\theta_i \notin X$, $i = 0, 1$.

Proof. The following proof is a bit more elegant than the one given in [4]. Let $\theta := \forall y\, \gamma(y)$ where $\gamma(y)$ is PR. Let $\rho(x, y)$ be a PR formula such that $X = \{k : \exists n\, \mathrm{PA} \vdash \rho(\bar{k}, \bar{n})\}$. Finally let θ_0 and θ_1 be such that

$$\mathrm{PA} \vdash \theta_0 \leftrightarrow \forall y\, ((\rho(\bar{\theta}_0, y) \vee \neg\gamma(y)) \rightarrow \exists z < y\, \rho(\bar{\theta}_1, z)),$$

$$\mathrm{PA} \vdash \theta_1 \leftrightarrow \forall z\, (\rho(\bar{\theta}_1, z) \rightarrow \exists y \leqslant z\, (\rho(\bar{\theta}_0, y) \vee \neg\gamma(y))).$$

Then θ_0 and θ_1 are as desired. □

Proof of Theorem 2. (i) By Lemma 3 with $X = \mathrm{Th}(T + \{\pi_k : k < n\})$, we can effectively construct sentences π_k such that $\neg\pi_k$ is Π_1-conservative over but not provable in $T + \{\pi_k : k < n\}$. Let $X = \{\pi_k : k \in \mathbb{N}\}$. Then, by Lemma 7, $X \restriction k \ll X$ for all k, so, by Lemma 8(i), $d(X)$ is not B_1.

(ii) Let e be such that $a < e < b$. Let $A \in a$, $B \in b$, $E \in e$. Let Y be the set of Π_1 sentences provable in A. Set $A^T = T + Y$. (A^T is the deductively weakest extension of T of degree a (cf. [4]).) By Orey's compactness theorem [4, Lemma

4], there is an m such that $B \restriction m \nleqslant E$ and $E \restriction m \nleqslant A$. We now effectively define Π_1 sentences ψ_n such that

(1) $B \restriction m \downarrow (Q + \psi_0) \nleqslant E$,

and for every k,

(2) $E \restriction m \nleqslant A^T + \psi_0 \wedge \cdots \wedge \psi_k$,

(3) $\neg\psi_{k+1}$ is Π_1 conservative over $A^T + \psi_0 \wedge \cdots \wedge \psi_k$.

The set $\{\varphi: B \restriction m \downarrow (Q + \varphi) \leqslant E\} \cup \{\varphi: E \restriction m \leqslant A^T + \neg\varphi\}$ is r.e. and monoconsistent with Q. Hence, by Lemma 10 of [4], there is a Π_1 sentence ψ_0 such that (1) holds and (2) holds for $k = 0$. Now, suppose (2) holds for $k = n$ and (3) holds for $k = n - 1$, if $n > 0$. Let $A_n = A^T + \psi_0 \wedge \cdots \wedge \psi_n$. The set $Z = \{\varphi: E \restriction m \leqslant A_n + \neg\varphi\}$ is then r.e. and monoconsistent with A_n. Hence, by Lemma 3, we can effectively find a Σ_1 sentence $\sigma \notin Z$ which is Π_1-conservative over A_n. Let $\psi_{n+1} = \neg\sigma$. Then (2) holds for $k = n + 1$ and (3) holds for $k = n$.

Now let $X = Y \cup \{\psi_k: k \in \mathbb{N}\}$. Then $a \leqslant d(X)$. Also, by (2), $e \nleqslant d(X)$, by (1), $d(X) \cap b > d(X) \cap e$, and, by (3) and Lemma 7, $X \restriction k \ll X$ for every k. Let $c = d(X) \cap e$ and $d = d(X) \cap b$. Then $a \leqslant c < d \leqslant b$. Also if $c \leqslant f \leqslant d$, then $f \leqslant d(X)$ and not $f \ll d(X)$, since $d(X) \cap e \leqslant f$ and $e \nleqslant f$. Hence, by Lemma 8(ii), f is not B_1.

(iii) We effectively construct sentences π_k, $\sigma_{i,k}$ such that for all k,

(1) $T \vdash \sigma_{i,k+1} \to \sigma_{i,k}$, $i = 0, 1$,

(2) $T + \sigma_{i,k}$ is consistent, $i = 0, 1$,

(3) $T + \pi_k$ is consistent,

(4) $T \vdash \pi_{k+1} \to \pi_k$,

(5) $\neg\pi_{k+1}$ is Π_1-conservative over $T + \pi_k$,

(6) $T + \pi_k \leqslant (T + \sigma_{0,k}) \uparrow (T + \sigma_{1,k})$,

(7) $(T + \sigma_{0,k}) \uparrow (T + \sigma_{1,k}) \leqslant T + \pi_{k+1}$.

Let $\sigma_{0,0} := \sigma_{1,0} := \pi_0 := 0 = 0$. Then (2), (3), and (6) hold for $k = 0$. Suppose $\sigma_{0,n}$ and $\sigma_{1,n}$ have been defined and that (2), (3), and (6) hold for $k = n$. Since T is Σ_1-sound, (2) implies that $(T + \sigma_{0,n}) \uparrow (T + \sigma_{1,n})$ is consistent. But then we can find a Π_1 sentence θ such that

(8) $T + \theta$ is consistent,

(9) $(T + \sigma_{0,n}) \uparrow (T + \sigma_{1,n}) \leqslant T + \theta$.

(For instance let $\theta := \mathrm{Con}_\gamma$ where $\gamma(x)$ is a PR binumeration of $(T + \sigma_{0,n}) \uparrow (T + \sigma_{1,n})$.) From (9) and (6) for $k = n$ it follows that

(10) $T + \theta \vdash \pi_n$.

Hence, by (8) and Lemma 3, we can find a Π_1 sentence ψ such that

(11) $T + \theta \nvdash \neg\psi$

(12) $\neg\psi$ is Π_1-conservative over $T + \pi_n$.

Let $\pi_{n+1} = \theta \wedge \psi$. Then, by (10), (4) holds for $k = n$ and, by (9), (7) holds for $k = n$. Also, by (11), (3) holds for $k = n + 1$ and, by (12), (5) holds for $k = n$. π_{n+1} is true, since otherwise (3) would not hold for $n = k + 1$. Hence, by (2) for $k = n$ and Lemma 9, we can find Π_1 sentences θ_i such that

(13) $\mathrm{PA} \vdash \theta_0 \vee \theta_1$,

(14) $\mathrm{PA} \vdash \theta_0 \wedge \theta_1 \rightarrow \pi_{n+1}$,

(15) $\theta_i \notin \mathrm{Th}(T + \sigma_{i,n})$, $i = 0, 1$.

Let $\sigma_{i,n+1} := \neg\theta_i \wedge \sigma_{i,n}$. Then (1) holds for $k = n$ and, by (15), (2) holds for $k = n + 1$. Finally, by (13), $\mathrm{PA} + \sigma_{i,n+1} \vdash \theta_{1-i}$, $i = 0, 1$. Hence, by (14), (6) holds for $k = n + 1$.

Now let $a_i = d(\{\sigma_{i,n}: n \in \mathbb{N}\})$ and $b = d(\{\pi_n: n \in \mathbb{N}\})$. Then, by (1) and Lemma 5, a_i is Σ_1. By (3), (4), (5), and Lemmas 7 and 8(i), b is not B_1. Finally, by (1), (6), and (7), $a_0 \cup a_1 = b$. □

I don't know if Theorem 2(iii) holds without the assumption that T is Σ_1-sound. A more interesting question is if (assuming that T is Σ_1-sound) there are degrees a and b such that a is Σ_1, b is Π_1, and $a \cup b$ is not B_1. *

4. Part (iii) of Theorem 2 would follow trivially from part (i) if we could show that every degree is the l.u.b. of two (finitely many) Σ_1 degrees. (In the proof of Theorem 6 below we define Σ_1 degrees a_0 and a_1 such that $a_0 \cup a_1 = d(\mathrm{Con}_\tau)$.) We now prove that this is not the case (and more). (Note that, by Theorem 8 of [4], for every degree $a > 0$, there is a Σ_1 degree b such that $0 < b \leqslant a$.) If $A \leqslant B$, then for any σ, $A + \sigma \leqslant B + \sigma$. Thus $d(A + \sigma)$ is uniquely determined by $d(A)$; it will be said to be a *Σ_1-extension* of $d(A)$. Let E_T be the least set of degrees containing 0 and closed under $\cup$, $\cap$, and Σ_1-extensions.

Theorem 3. *There is a Π_1 degree not in E_T.*

This is an immediate consequence of the following two lemmas.

Lemma 10. *To every $a \in E_T$ there is a smallest Σ_1 degree $\geqslant a$.*

Proof. It is easily shown by induction that if $a \in E_T$, then there are $\sigma_0, \ldots, \sigma_n$ such that

(1) $d(\sigma_0) \cup \cdots \cup d(\sigma_n) \leqslant a \leqslant d(\sigma_0 \wedge \cdots \wedge \sigma_n)$.

Now

(2) $d(\sigma_0 \wedge \cdots \wedge \sigma_n)$ is the smallest Σ_1 degree $\geqslant d(\sigma_0) \cup \cdots \cup d(\sigma_n)$.

This can be seen as follows. Suppose $d(\sigma_0) \cup \cdots \cup d(\sigma_n) \leqslant d(\sigma)$. Let π be such that $T + \sigma_0 \wedge \cdots \wedge \sigma_n \vdash \pi$. Then $T + \sigma_0 \vdash \sigma_1 \wedge \cdots \wedge \sigma_n \to \pi$. Now $\sigma_1 \wedge \cdots \wedge \sigma_n \to \pi$ is a Π_1 sentence. It follows that $T + \sigma \vdash \sigma_1 \wedge \cdots \wedge \sigma_n \to \pi$. But then $T + \sigma_1 \vdash \sigma \wedge \sigma_2 \wedge \cdots \wedge \sigma_n \to \pi$ and so $T + \sigma \vdash \sigma_2 \wedge \cdots \wedge \sigma_n \to \pi$. Continuing in this way we eventually get $T + \sigma \vdash \pi$, as desired.

Finally, by (1) and (2), $d(\sigma_0 \wedge \cdots \wedge \sigma_n)$ is the smallest Σ_1 degree $\geqslant a$. □

Lemma 11. *There is a Π_1 degree a for which there is no smallest Σ_1 degree $\geqslant a$.*

Proof. We can effectively construct sentences σ_k such that for every k,

(1) $T + \sigma_{k+1} \vdash \sigma_k$,

(2) $\sigma_k < \sigma_{k+1}$.

Let $\sigma_0 := 0 = 0$. Given σ_k such that $T + \sigma_k$ is consistent let $X = \mathrm{Th}(T + \sigma_k)$. By Lemma 9, we can find Π_1 sentences θ_i such that $P \vdash \theta_0 \vee \theta_1$ and $\theta_i \notin X$, $i = 0, 1$. (Let θ be any true Π_1 sentence.) Let $\sigma_{k+1} := \sigma_k \wedge \neg\theta_0$. Then $T + \sigma_{k+1}$ is consistent and (1) holds trivially. Finally $T + \sigma_{k+1} \vdash \theta_1$ and $T + \sigma_k \nvdash \theta_1$ and so (2) is true.

By Lemma 5, there is a sentence σ such that

(3) $T + \sigma$ is a Π_1-conservative extension of $T + \{\sigma_k : k \in \mathbb{N}\}$.

Let $a = d(\neg\sigma)$. Then a is Π_1. Now let χ be any Σ_1 sentence such that $a \leqslant d(\chi)$. Then $T + \chi \vdash \neg\sigma$ and so $T + \sigma \vdash \neg\chi$. But then, by (1) and (3), there is a k such that $T + \sigma_k \vdash \neg\chi$ and so

(4) $T + \chi \vdash \neg\sigma_k$.

Also, by (2) and (3), there is a sentence π such that $T + \sigma \vdash \pi$ and $T + \sigma_k \nvdash \pi$. It follows that

(5) $T + \neg\pi \vdash \neg\sigma$,

(6) $T + \neg\pi \nvdash \neg\sigma_k$.

But then, by (5), $a \leqslant d(\neg\pi)$ and, by (4) and (6), $d(\chi) \nleqslant d(\neg\pi)$. Thus $d(\chi)$ is not the smallest Σ_1 degree $\geqslant a$. □

Corollary 1. *There is a Π_1 degree which is not the l.u.b. of finitely many Σ_1 degrees.*

Suppose $a \notin E_T$. Then $0 < a < 1$. But then, by Theorem 3 of [4], there are b_0, $b_1 < a$ such that $b_0 \cup b_1 = a$ and $c_0, c_1 > a$ such that $c_0 \cap c_1 = a$. Now E_T is closed under $\cap$ and $\cup$. It follows that either $[b_0, a]$ or $[b_1, a]$ is disjoint from E_T. Let b

be the b_i for which this holds. Then either $[b, c_0]$ or $[b, c_1]$ is disjoint from E_T. Let c be the c_i for which this holds. Then $[b, c]$ is disjoint from E_T. Thus to every $a \notin E_T$, there are b, c such that $b < a < c$ and $[b, c] \cap E_T = \emptyset$.

The degree a defined in the proof of Lemma 11 can be made arbitrarily small: let π be such that $T \nvdash \pi$ and let $\sigma_0 := \neg\pi$. Then $a \leqslant d(\pi)$. However a cannot be made arbitrarily large since to every degree $b \gg 0$, there is, by Lemma 7, a smallest Σ_1 degree $\geqslant b$, namely 1. It is an open problem if there are arbitrarily large (Π_1) degrees not in E_T. *

At this point it is natural to ask if there are degrees which cannot be written as the l.u.b. of a finite set of Σ_1 and Π_1 degrees. We now prove that the answer is affirmative (and more?). Let I_T be the set of degrees a for which there are $\pi, \sigma_0, \ldots, \sigma_n$ such that $d(\pi) \cup d(\sigma_0) \cup \cdots \cup d(\sigma_n) \leqslant a \leqslant d(\pi + \sigma_0 \wedge \cdots \wedge \sigma_n)$. We shall also need the following definition: $A \lll B$ iff $A < B$ and for every set X of Σ_1 sentences, if $B \dashv_{\Pi_1} A + X$, then $A + X$ is inconsistent. (Here X need not be r.e.) We write $a \lll b$ to mean that $A \lll B$ where $A \in a$ and $B \in b$. By Lemma 7, $A \lll B$ implies $A \ll B$. As will become clear, the converse of this is not true.

Lemma 12. *Suppose $a \in I_T$ and for all π, if $d(\pi) \leqslant a$, then $d(\pi) \ll a$. Then $0 \lll a$.*

Proof. By assumption there are $\pi, \sigma_0, \ldots, \sigma_n$ such that $d(\pi) \cup d(\sigma_0) \cup \cdots \cup d(\sigma_n) \leqslant a \leqslant d(\pi + \sigma_0 \wedge \cdots \wedge \sigma_n)$. Also $d(\pi) \ll a$. Let $A \in a$. Then

(1) $\quad T + \sigma_i \leqslant A \quad$ for $i \leqslant n$.

Moreover, $d(\pi) \ll d(\pi + \sigma_0 \wedge \cdots \wedge \sigma_n)$ and so, by Lemma 7, $T + \pi \vdash \neg\sigma_0 \vee \cdots \vee \neg\sigma_n$. But $A \vdash \pi$ and so

(2) $\quad A \vdash \neg\sigma_0 \vee \cdots \vee \neg\sigma_n$.

Let X be any set of Σ_1 sentences such that

(3) $\quad A \dashv_{\Pi_1} T + X$.

Then, by (2), $T + X \vdash \neg\sigma_0 \vee \cdots \vee \neg\sigma_n$, whence there is a k_0 such that $T + \sigma_0 \vdash \neg \bigwedge X \restriction k_0 \vee \neg\sigma_1 \vee \cdots \vee \neg\sigma_n$, and so, by (1) and (3), $T + X \vdash \neg\sigma_1 \vee \cdots \vee \neg\sigma_n$. Continuing in this way we eventually obtain the desired conclusion that $T + X$ is inconsistent. □

Lemma 13. $I_T \neq D_T$.

To prove this we need the following lemma from [3] (Lemma 5).

Lemma 14. *Suppose X is r.e. Then there are a Σ_1 formula $\xi_0(x)$ and a Π_1 formula $\xi_1(x)$ such that*

(i) *if $k \in X$, then* $\mathrm{PA} \vdash \xi_0(\bar{k})$,

(ii) $\mathrm{PA} \vdash \xi_0(\bar{k}) \rightarrow \xi_1(\bar{k})$,

(iii) $T + \{\neg\xi_1(\bar{k}): k \notin X\}$ *is consistent.*

Proof of Lemma 13. Our proof is an elaboration of the proof of Corollary 2 of [4]. We effectively construct sentences $\psi_0, \psi_1, \ldots$ such that if $A_n = T + \{\psi_k : k < n\}$ and $A = T + \{\psi_k : k \in \mathbb{N}\}$ then

(1) $A_n \ll A_{n+1}$,

(2) not $T \lll A$.

Let $a = d(A)$. Then for all π, if $d(\pi) \leq a$, then there is an n such that $d(\pi) \leq d(A_n)$. Also $d(A_n) \ll d(A_{n+1}) \leq a$ and so $d(\pi) \ll a$. It follows, by Lemma 12, $a \notin I_T$.

By Lemma 7, there is an r.e. relation $S(n, k, p, q)$ such that (not $T + \psi \ll T + \psi + \varphi$) iff $\exists p\, \forall q\, S(\psi, \varphi, p, q)$. By (a straightforward extension of) Lemma 14, there are a Σ_1 formula $\sigma_0(x, y, z, u)$ and a Π_1 formula $\sigma_1(x, y, z, u)$ such that

(3) if $S(n, k, p, q)$, then $T \vdash \sigma_0(\bar{n}, \bar{k}, \bar{p}, \bar{q})$,

(4) $T \vdash \sigma_0(\bar{n}, \bar{k}, \bar{p}, \bar{q}) \to \sigma_1(\bar{n}, \bar{k}, \bar{p}, \bar{q})$,

(5) $T + Y$ is consistent where $Y = \{\neg\sigma_1(\bar{n}, \bar{k}, \bar{p}, \bar{q})\colon \text{not } S(n, k, p, q)\}$.

Set $A_0 = T$. Suppose A_n has been defined and set $\theta_n := \bigwedge\{\psi_k : k < n\}$. Then

(6) not $A_n \ll A_n + \varphi$ iff $\exists p\, \forall q\, S(\theta_n, \varphi, p, q)$.

By (3) and Lemma 2, there is a Σ_1 formula $\rho_n(x, y)$ such that

(7) $A_n \vdash \rho_n(\bar{\varphi}, \bar{p}) \to \sigma_0(\overline{\theta_n}, \bar{\varphi}, \bar{p}, \bar{q})$,

(8) if $\forall q\, S(\theta_n, \varphi, p, q)$, then $\rho_n(\bar{\varphi}, \bar{p})$ is Π_1-conservative over A_n.

Moreover, by Lemma 4(ii), there is a formula $\eta_n(x)$ such that

(9) $A_n + \eta_n(\bar{\varphi})$ is a Π_1-conservative extension of $A_n + \{\neg\rho_n(\bar{\varphi}, \bar{p}) : p \in \mathbb{N}\}$.

Finally let ψ_n be such that

(10) $T \vdash \psi_n \leftrightarrow \eta_n(\overline{\psi_n})$.

The formulas $\rho_n(x, y)$, $\eta_n(x)$ and the sentences ψ_n can be found effectively in n.

To prove (1) assume it is false. Then, by (6), there is a p such that $\forall q\, S(\theta_n, \psi_n, p, q)$. But then, by (8), $\rho_n(\overline{\psi_n}, \bar{p})$ is Π_1-conservative over A_n. Moreover, by (9) and (10), $A_n + \psi_n \vdash \neg\rho_n(\overline{\psi_n}, \bar{p})$. But, by Lemma 7, this implies that $A_n \ll A_n + \psi_n$, a contradiction. This proves (1).

Next we prove (2). Let Y be as in (5). Then $T + Y$ is consistent. To prove that $A \dashv_{\Pi_1} T + Y$ we first show that

(11) $A_{n+1} + Y \dashv_{\Pi_1} A_n + Y$

Indeed suppose $A_{n+1} + Y \vdash \pi$. Then there is a k such that

(12) $A_{n+1} \vdash \neg\bigwedge Y \restriction k \vee \pi$.

By (1) and (6), to each p there is a q_p such that

(13) not $S(\theta_n, \psi_n, p, q_p)$.

Moreover, by (12), (9), and (10),

$$A_n + \{\neg\rho_n(\overline{\psi_n}, \bar{p}): p \in \mathbb{N}\} \vdash \neg\bigwedge Y \upharpoonright k \vee \pi.$$

But then, by (7), (4), (13), $A_n + Y \vdash \pi$. This proves (11).

From (11) it follows that $A \dashv_{\Pi_1} T + Y$. This proves (2) and so the proof of Lemma 13 is complete. □

Let F_T be the set of degrees obtained from E_T together with the set of Π_1 degrees by closing under ∪ and Σ_1-extensions. By Theorem 3, $F_T \nsubseteq E_T$. $E_T \subseteq I_T$ (cf. the proof of Lemma 10) and trivially every Π_1 degree is a member of I_T. Moreover, as is easily verified, I_T is closed under ∪ and Σ_1-extensions. Hence $F_T \subseteq I_T$ and so, by Lemma 13, we get the following:

Theorem 4. $F_T \neq D_T$.

Corollary 2. *There is a degree which is not the l.u.b. of a finite set of degrees of the form $d(\pi \wedge \sigma)$.*

Let a be the degree constructed in the proof of Lemma 13. Then $0 \ll a$. We can obtain a degree $b \notin I_T$, and so $b \notin F_T$, such that not $b \gg 0$ as follows. By Theorem 3 of [4], there are b_0, $b_1 < a$ such that $b_0 \cap b_1 = 0$ and $b_0 \cup b_1 = a$. Since I_T is closed under ∪, it follows that $b_0 \notin I_T$ or $b_1 \notin I_T$; in fact, one of the intervals $[b_i, a]$ is disjoint from I_T.

Let G_T be the set of degrees obtained from the set of Π_1 and Σ_1 degrees by closing under ∪ and ∩. The above results do not seem to imply that $G_T \neq D_T$ and the problem if this is true remains open. All degrees in G_T can be written in the form $d(\pi_0 \vee \sigma_0) \cup \cdots \cup d(\pi_n \vee \sigma_n)$. So if all degrees of the form $d(\pi \vee \sigma)$ are in I_T, then $G_T \subseteq I_T$ and so $G_T \neq D_T$. If, on the other hand, $d(\pi \vee \sigma) \notin I_T$, then $d(\pi \vee \sigma) \notin F_T$, a much better result than Theorem 4: by Lemma 8, the degree a in the proof of Lemma 13 is not even B_1.

5. Let us say that a *cups to* b if there is a $c < b$ such that $a \cup c = b$. (Thus no degree cups to 0 and every degree $a > 0$ cups to itself.) One way to strengthen Corollary 1 would be to show that there is a Π_1 degree $a > 0$ such that no Σ_1 degree cups to a. We now prove a result which implies that this is false. (On the other hand, to every degree $a > 0$, there is a degree b such that $0 < b < a$ and b does not cup to a (cf. [4, Theorem 4(i)]).)

Theorem 5. *For every a, if there is a degree in G_T which cups to a, then there is a Σ_1 degree which cups to a.*

Proof. If $d(\pi_0 \vee \sigma_0) \cup \cdots \cup d(\pi_n \vee \sigma_n)$ cups to a, then there is a $k \leq n$ such that $d(\pi_k \vee \sigma_k)$ cups to a. Thus we may assume that there is a degree $d(\pi \vee \sigma)$ which cups to a. Let $b < a$ be such that $d(\pi \vee \sigma) \cup b = a$ and let $B \in b$. Let $\pi := \forall u\, \delta(u)$, where $\delta(u)$ is PR. We may assume that

(1) $\quad T \vdash \neg\delta(u) \wedge \neg\delta(v) \rightarrow u = v;$

if necessary replace $\delta(u)$ by $\delta(u) \vee \exists v < u \neg\delta(v)$. Let π^* be such that $T + \pi \vee \sigma \vdash \pi^*$ and $B \nvdash \pi^*$. By Lemma 6, there is a PR formula $\eta(x, y, z)$ such that for all φ, θ,

(2) $\quad$ if $(T + \varphi) \upharpoonright B \vdash \pi^*$, then $T + \theta \vdash \neg\exists z\, \eta(\bar{\theta}, \bar{\varphi}, z)$,

(3) $\quad$ if $(T + \varphi) \upharpoonright B \nvdash \pi^*$, then $\exists z\, \eta(\bar{\theta}, \bar{\varphi}, z)$ is Π_1-conservative over $T + \theta$.

Next let ψ and θ be such that

(4) $\quad T \vdash \psi \leftrightarrow \forall u\, (\neg\delta(u) \rightarrow \neg\exists z \leq u\, \eta(\bar{\theta}, \bar{\psi}, z)),$

(5) $\quad T \vdash \theta \leftrightarrow \forall u\, (\neg\delta(u) \rightarrow \exists z \leq u\, \eta(\bar{\theta}, \bar{\psi}, z)).$

Then

(6) $\quad T \vdash (\psi \wedge \theta) \leftrightarrow \pi$

and, by (1)

(7) $\quad T \vdash \psi \vee \theta.$

We now show that

(8) $\quad (T + \psi) \upharpoonright B \nvdash \pi^*.$

Supoose not. Then, by (2) and (5), $T + \theta \vdash \pi$. But then $T + \theta \vdash \pi^*$. Also, by assumption, $(T + \psi) \upharpoonright B \vdash \pi^*$ and so, by (7), $B \vdash \pi^*$, contrary to assumption. This proves (8).

Now let

$$\chi := \exists z\, (\eta(\bar{\theta}, \bar{\psi}, z) \wedge \forall u < z\, \delta(u)).$$

Then χ is Σ_1 and

$$T \vdash \chi \leftrightarrow \exists z\, \eta(\bar{\theta}, \bar{\psi}, z) \wedge \theta.$$

But then, by (3), and (8), $d(\chi) = d(\theta)$ and so, by (6), $d(\chi) \cup d(\psi) = d(\pi)$. It follows that $d(\chi \vee \sigma) \cup d(\psi \vee \sigma) \cup b = d(\pi \vee \sigma) \cup b$. Finally $d(\chi \vee \sigma)$ is Σ_1 and, by (8), $d(\psi \vee \sigma) \cup b < d(\pi \vee \sigma) \cup b$ and so $d(\chi \vee \sigma)$ cups to a. □

It is an open problem if to every $a > 0$, there is a Σ_1 degree which cups to a. (If not, then, by Theorem 5, $G_T \neq D_T$.) However, our next result implies that this is true of all sufficiently large degrees.

Theorem 6. *There is a Σ_1 degree $a < 1$ which cups to every b for which $a \leqslant b < 1$; in fact, there are two such degrees a_0 and a_1 such that $a_0 \cap a_1 = 0$.*

Proof. In [4, Theorem 5] we prove that $d(\mathrm{Con}_\tau)$ cups to every b such that $d(\mathrm{Con}_\tau) \leqslant b < 1$; the following proof is an elaboration of that proof. Let θ_i be such that

$$\mathrm{PA} \vdash \theta_0 \leftrightarrow \forall z\, (\mathrm{Prf}_\tau(\bar{\theta}_0, z) \rightarrow \exists u \leqslant z\, \mathrm{Prf}_\tau(\bar{\theta}_1, u)),$$

$$\mathrm{PA} \vdash \theta_1 \leftrightarrow \forall z\, (\mathrm{Prf}_\tau(\bar{\theta}_1, z) \rightarrow \exists u < z\, \mathrm{Prf}_\tau(\bar{\theta}_0, u)).$$

Then, by standard arguments,

(1) $\quad \mathrm{T} \nvdash \theta_i, \quad i = 0, 1,$

(2) $\quad \mathrm{PA} \vdash \theta_0 \vee \theta_1,$

(3) $\quad \mathrm{PA} \vdash \theta_0 \wedge \theta_1 \rightarrow \neg \mathrm{Pr}_\tau(\bar{\theta}_i), \quad i = 1, 0.$

Let $a_i = d(\theta_i)$. Then $a_0 \cap a_1 = 0$. Also clearly

(4) $\quad \mathrm{PA} \vdash \neg\theta_i \leftrightarrow \mathrm{Pr}_\tau(\bar{\theta}_i) \wedge \theta_{1-i}, \quad i = 0, 1.$

By (2), $\mathrm{PA} \vdash \mathrm{Pr}_\tau(\overline{\neg\theta_{1-i}}) \rightarrow \mathrm{Pr}_\tau(\bar{\theta}_i)$. As is well known, $\mathrm{Pr}_\tau(\overline{\neg\theta_{1-i}})$ is Π_1-conservative over $T + \theta_{1-i}$. But then, so is $\mathrm{Pr}_\tau(\bar{\theta}_i)$ and so, by (4), $d(\neg\theta_i) = d(\theta_{1-i})$. Thus a_0 and a_1 are Σ_1. (Formalizing the proof of (1) we get $\mathrm{PA} \vdash \mathrm{Con}_\tau \rightarrow \theta_0 \wedge \theta_1$ and so, by (3), $a_0 \cup a_1 = d(\mathrm{Con}_\tau)$.)

Suppose now $a_i \leqslant b < 1$. Let $\beta(x)$ be a PR binumeration of a theory of degree b. Let φ be such that

$$\mathrm{PA} \vdash \varphi \leftrightarrow \forall z\, (\mathrm{Prf}_\tau(\overline{\varphi \vee \theta_i}, z) \rightarrow \exists u \leqslant z\, \mathrm{Prf}_\beta(\overline{0=1}, u))$$

and let

$$\hat{\varphi} := \forall u\, (\mathrm{Prf}_\beta(\overline{0=1}, u) \rightarrow \exists z < u\, \mathrm{Prf}_\tau(\overline{\varphi \vee \theta_i}, z)).$$

Then, by (1) and again using standard arguments,

(4) $\quad T \nvdash \varphi \vee \theta_i,$

(5) $\quad \mathrm{PA} \vdash \varphi \vee \hat{\varphi},$

(6) $\quad \mathrm{PA} \vdash \varphi \wedge \hat{\varphi} \rightarrow \mathrm{Con}_\beta.$

Clearly $\mathrm{PA} \vdash \neg\varphi \rightarrow \mathrm{Pr}_\tau(\overline{\varphi \vee \theta_i})$. Since $\neg\varphi$ is Σ_1, we also have $\mathrm{PA} \vdash \neg\varphi \rightarrow \mathrm{Pr}_\tau(\overline{\neg\varphi})$. It follows that $\mathrm{PA} \vdash \neg\varphi \rightarrow \mathrm{Pr}_\tau(\bar{\theta}_i)$ and so, by (3),

(7) $\quad \mathrm{PA} \vdash \theta_0 \wedge \theta_1 \rightarrow \varphi.$

Now let $d = d(\theta_{1-i} \wedge \hat{\varphi})$. Then, by (6) and (7), $T + \theta_0 \wedge \theta_1 \wedge \hat{\varphi} \vdash \mathrm{Con}_\beta$. Hence $a_i \cup d \geqslant d(\mathrm{Con}_\beta) \geqslant b$. Suppose $a_i \leqslant d$. Then $T + \theta_{1-i} \wedge \hat{\varphi} \vdash \theta_i$. But then, by (2) and (5), $T \vdash \varphi \vee \theta_i$, contradicting (4). Thus $a_i \nleqslant d$. Now let $c = b \cap d$. Then $c < b$ and $a_i \cup c = b$ as desired. □

One way to improve Corollary 2 would be to show that there is a degree $a > 0$ such that no degree of the form $d(\pi \wedge \sigma)$ cups to a. It is an open question if this is true. (If it is then, of course, there is a degree $a > 0$ such that no Σ_1 degree cups to a, solving a problem already mentioned.) But we do have the following weaker:

Theorem 7. *There is a degree $a > 0$ such that no Π_1 degree cups to a.*

Proof. The idea is to construct Π_1 sentences ψ_k such that for all k,

(1) $T \nvdash \psi_k$,

(2) $T \vdash \psi_{k+1} \rightarrow \psi_k$,

(3) ψ_k is Σ_1-conservative over $T + \neg\psi_{k+1}$.

Let $a = d(\{\psi_k : k \in \mathbb{N}\})$. By (1), $a > 0$. By (3), $d(\psi_k)$ does not cup to $d(\psi_{k+1})$ (cf. [4, Theorem 4(i)]). Suppose $d(\pi) \leqslant a$. Then, by (2), $d(\pi) \leqslant d(\psi_k)$ for some k, whence $d(\pi)$ does not cup to $d(\psi_{k+1})$. It follows that $d(\pi)$ does not cup to a. (Note that the theories $T + \psi_k$ are consistent: if $T \vdash \neg\psi_k$, then, by (2), $T \vdash \neg\psi_{k+1}$, whence, by (3), $T \nvdash \neg\psi_k$.) However, the sentences ψ_k cannot be constructed by first defining ψ_0, then ψ_1, then ψ_2 etc.; at least this cannot be done in any straightforward way. (First of all, there is no known way of constructing, given ψ_k, a ψ_{k+1} satisfying (2) and (3). Secondly, $d(\psi_{k+1})$ must not cup to every degree $\geqslant d(\psi_{k+1})$ and, by Theorem 6, that is a nontrivial condition.) Instead we shall use a construction inspired by that used in the solution, due to H. Friedman, of a problem of H. Gaifman (cf. [6, Exercise 4, p. 179]).

Let $\delta(u)$ be an arbitrary PR formula. Let $\kappa(z, u, x, y)$ be a Π_1 formula such that

(4) $\mathrm{PA} \vdash \neg\kappa(z, u, x, \bar{0})$,

(5) $$\mathrm{PA} \vdash \kappa(\bar{\delta}, u, \bar{k}, y+1) \leftrightarrow \kappa(\bar{\delta}, u, \overline{k+1}, y) \vee \forall v\, ([\Sigma_1]_\tau(\overline{\neg\eta_\delta(\bar{k}) \wedge \xi_\delta(\bar{k})}, v) \rightarrow \neg\mathrm{Prf}_\tau(\overline{\xi_\delta(\bar{k})}, v)),$$

where

$$\xi_\delta(x) := \forall u\, (\delta(u) \rightarrow \kappa(\bar{\delta}, u, x, (u \dot{-} x) + 1)),$$

$$\eta_\delta(x) := \forall u\, (\delta(u) \rightarrow \kappa(\bar{\delta}, u, x+1, u \dot{-} x)).$$

($\dot{-}$ is the function such that $k \dot{-} m = k - m$ if $k \geqslant m$ and $= 0$ otherwise.) In (5) set $y = u \dot{-} \bar{k}$. Then, since u is not free in the second disjunct (to the right of $\leftrightarrow$) of (5), we get

(6) $$\mathrm{PA} \vdash \xi_\delta(\bar{k}) \leftrightarrow \eta_\delta(\bar{k}) \vee \forall v\, ([\Sigma_1]_\tau(\overline{\neg\eta_\delta(\bar{k}) \wedge \xi_\delta(\bar{k})}, v) \rightarrow \neg\mathrm{Prf}_\tau(\overline{\xi_\delta(\bar{k})}, v)).$$

It follows that

(7) if $T \vdash \xi_\delta(\bar{k})$, then $T \vdash \eta_\delta(\bar{k})$.

For let p be a proof of $\xi_\delta(\bar{k})$ in T. Then, by Lemma 1(ii), $T+\neg\eta_\delta(\bar{k})\wedge \xi_\delta(\bar{k})\vdash\neg\mathrm{Prf}_\tau(\overline{\xi_\delta(\bar{k})}, \bar{p})$, whence $T+\xi_\delta(\bar{k})\vdash\eta_\delta(\bar{k})$ and so $T\vdash\eta_\delta(\bar{k})$. Clearly

(8) if $T\vdash\delta(u)\rightarrow u>\bar{k}$, then $T\vdash\eta_\delta(\bar{k})\leftrightarrow\xi_\delta(\overline{k+1})$.

Next we show that

(9) if $\exists u\,\delta(u)$ is true, then $T\nvdash\xi_\delta(\bar{0})$.

Let m be the least number such that $\delta(\bar{m})$ is true. Then, by (7) and (8), if $k<m$ and $T\vdash\xi_\delta(\bar{k})$, then $T\vdash\xi_\delta(\overline{k+1})$. Thus it suffices to show that $T\nvdash\xi_\delta(\bar{m})$. But, by (4), $T\vdash\neg\eta_\delta(\bar{m})$ and so, by (7), $T\nvdash\xi_\delta(\bar{m})$. This proves (9).

The set of PR formulas $\delta(u)$ such that $\exists u\,\delta(u)$ is true is an r.e. nonrecursive set. Hence, by (9), there is a PR formula $\delta^*(u)$ such that $\exists u\,\delta^*(u)$ is false and $T\nvdash\xi_{\delta^*}(\bar{0})$. Let $\varphi_k:=\eta_{\delta^*}(\bar{k})$ and $\psi_k:=\xi_{\delta^*}(\bar{k})$. Then $T\nvdash\psi_0$. Hence, by (6) and (8), we get (1) and (2).

(3) can be verified as follows. Suppose

(10) $T+\neg\psi_{k+1}+\psi_k\vdash\sigma$.

Then $T+\neg\varphi_k+\psi_k\vdash\sigma$. Hence, by Lemma 1(iii), there is a q such that $T+[\Sigma_1]_\tau(\overline{\neg\varphi_k\wedge\psi_k}, \bar{q})\vdash\sigma$. But then, by Lemma 1(i), (1), and (6), $T+\neg\sigma\vdash\psi_k$, whence $T+\neg\psi_k\vdash\sigma$. But then, by (10), $T+\neg\psi_{k+1}\vdash\sigma$, proving (3).

Finally, as we have already observed, it follows from (1), (2), (3) that $a>0$ and that no Π_1 degree cups to a. □

It would be interesting to know if there is a Σ_1 degree $a>0$ such that no Π_1, degree cups to a. *

The dual of the notion of cupping is that of capping: *a caps to b* if there is a $c>b$ such that $a\cap c=b$. Thus if $b<a$, then a caps to b iff not $b\ll a$. From Lemma 8 and the proof of Theorem 2 we get the following:

Corollary 3. (i) *There is a degree $a<1$ such that no B_1 degree caps to a and a caps to no B_1 degree.*

(ii) *If T is Σ_1-sound, then there are Σ_1 degrees a_0 and a_1 such that $a=a_0\cup a_1$ is as in* (i).

The most interesting open problem about capping seems to be if there is a Σ_1 degree $a<1$ such that no Π_1 degree caps to a. *

6. As is easily verified for every π, $d(\neg\pi)$ is the *pseudocomplement* (p.c.) of $d(\pi)$, i.e., $d(\neg\pi)=\max\{b: b\cap d(\pi)=0\}$. (Clearly $d(\pi)\cap d(\neg\pi)=0$. Suppose $d(\pi)\cap a=0$. Let $a=d(A)$. Then $(T+\pi)\downarrow A\leqslant T$. But then for every σ, if $A\vdash\neg\sigma$, then $T+\pi\leqslant T+\sigma$, whence $T+\sigma\vdash\pi$, whence $T+\neg\pi\vdash\neg\sigma$ (cf. [4, Lemma 12]). It follows that $a\leqslant d(\neg\pi)$.) Thus every Π_1 degree has a p.c. In [4] it is shown that there is a degree with no p.c. This can be improved as follows.

Theorem 8. *There is a Σ_1 degree which has no p.c.*

This is an almost immediate consequence of the following lemma which improves Theorem 10(ii) of [4] and Lemma 11 above.

Lemma 15. *There is a sentence σ such that $\{b \geq d(\neg\sigma): b$ is $\Sigma_1\}$ has no g.l.b.*

Proof. The following proof is the same as the proof of Theorem 10(ii) of [4] except for the introduction of the sentence σ. Let $\pi := \forall u\, \delta(u)$, where $\delta(u)$ is PR, be any Π_1 sentence not provable in T. In the proof of Theorem 8 of [4] we construct a Π_1 sentence θ and a Σ_1 sentence χ such that $0 < d(\theta) = d(\chi) \leq d(\pi)$ in the following way. By Lemma 6, there is a PR formula $\eta_1(x, z)$ such that

$$\text{if } T \vdash \varphi, \text{ then } T \vdash \neg\exists z\, \eta_1(\bar{\varphi}, z),$$

$$\text{if } T \nvdash \varphi, \text{ then } \exists z\, \eta_1(\bar{\varphi}, z) \text{ is } \Pi_1\text{-conservative over } T + \varphi.$$

Now let θ be such that

$$T \vdash \theta \leftrightarrow \forall u\, (\neg\delta(u) \rightarrow \exists z \leq u\, \eta_1(\bar{\theta}, z)).$$

Finally, set

$$\chi := \exists z\, (\eta_1(\bar{\theta}, z) \wedge \forall u < z\, \delta(u))$$

(compare the proof of Theorem 5). We have $T \nvdash \theta$ and $T \vdash \chi \leftrightarrow \exists z\, \eta_1(\bar{\theta}, z) \wedge \theta$. Thus there are (primitive) recursive functions $f(n)$ and $g(n)$ such that if π is any Π_1 sentence, then $f(\pi)$ is a Π_1 sentence, $g(\pi)$ is a Σ_1 sentence, and if $T \nvdash \pi$, then $T < T + f(\pi) \equiv T + g(\pi) \leq T + \pi$.

We now define π_k and σ_k as follows. Let π_0 be any Π_1 sentence not provable in T. Next suppose π_k has been defined and $T \nvdash \pi_k$. Let ψ be a Π_1 sentence undecidable in $T + \neg\pi_k$. Then $T < T + \pi_k \vee \psi < T + \pi_k$. Let $\sigma_k := g(\pi_k \vee \psi)$ and $\pi_{k+1} := f(\pi_k \vee \psi)$. Then for every k,

(1) $\quad \pi_{k+1} \leq \sigma_k < \pi_k.$

By Lemma 5, there is a sentence σ such that

(2) $\quad T + \sigma$ is a Π_1-conservative extension of $T + \{\neg\pi_k : k \in \mathbb{N}\}$.

Then

(3) $\quad d(\neg\sigma) \leq d(\sigma_k).$

Moreover

(4) $\quad$ if b is Σ_1 and $b \geq d(\neg\sigma)$, then there is a k such that $b \geq d(\pi_k)$.

For suppose $b = d(\chi)$ where χ is Σ_1. Then $T + \chi \vdash \neg\sigma$ whence $T + \sigma \vdash \neg\chi$. But then, by (2), there is a k such that $T + \neg\pi_k \vdash \neg\chi$ whence $T + \chi \vdash \pi_k$ and so $b \geq d(\pi_k)$.

Now if $\{b \geqslant d(\neg\sigma): b \text{ is } \Sigma_1\}$ has a g.l.b., then, by (1), (3), (4), so does $\{d(\pi_k): k \in \mathbb{N}\}$. But from (1) it follows that no $d(\pi_k)$ is g.l.b. of $\{\pi: T + \pi_k \vdash \pi$ for every $k\}$. Hence, by Lemma 17 of [4], $\{d(\pi_k): k \in \mathbb{N}\}$ has no g.l.b. Thus σ is as desired. □

Proof of Theorem 8. Let σ be as in Lemma 15. For all B,

$$(T + \sigma) \downarrow B \leqslant T \quad \text{iff} \quad B \leqslant T + \chi \text{ for all } \Sigma_1 \text{ sentences } \chi \text{ such that } T + \chi \vdash \neg\sigma$$

(cf. [4, Lemma 12]). But then the p.c. of $d(\sigma)$, if it had one, would also be the g.l.b. of $\{b \geqslant d(\neg\sigma): b \text{ is } \Sigma_1\}$. Thus, by Lemma 15, $d(\sigma)$ is as desired. □

If $0 \ll a < 1$, then, trivially, a is not the p.c. of any degree. A nontrivial example of a degree which is not a p.c. is given in the following:

Corollary 4. *There is a Π_1 degree a such that not $0 \ll a$ and a is not the p.c. of any degree.*

Proof. Let σ be such that $d(\sigma)$ has no p.c. and let $a = d(\neg\sigma)$. Then not $0 \ll a$. Suppose a is the p.c. of some degree b. Then $b \leqslant d(\sigma)$, since $d(\sigma)$ is the p.c. of a. It follows that a is the p.c. of $d(\sigma)$, a contradiction. □

Theorem 8 suggests the problem if there is a Σ_1 and non-Π_1 degree which has a p.c. We show that the answer is affirmative. Note that there are lots of non-Π_1, even non-B_1, degrees, which do have a p.c. Indeed if $a \gg 0$ and $a \not\geqslant d(\pi)$, then $d(\neg\pi)$ is the p.c. of every member of $[a \cap d(\pi), d(\pi)]$ and, by Theorem 2(ii), this interval contains non-B_1 degrees. However, no member of $[a \cap d(\pi), d(\pi)]$ is Σ_1, except possibly $d(\pi)$ (cf. [4, Corollary 9]).

Theorem 9. *There is a Σ_1 and non-Π_1 degree which has a p.c.*

Proof. Let $\forall u\, \delta(u)$, where $\delta(u)$ is PR, be a Π_1 sentence not provable in T. We have seen in the proof of Lemma 15 how to construct a Π_1 sentence θ and a Σ_1 sentence χ such that $0 < d(\chi) = d(\theta) \leqslant d(\forall u\, \delta(u))$. It follows that $d(\neg\chi) \leqslant d(\neg\theta)$. As we have already remarked, $d(\chi)$ is the p.c. of $d(\neg\chi)$. Also $d(\neg\theta) \cap d(\chi) = 0$. It follows that $d(\chi)$ is the p.c. of $d(\neg\theta)$. Thus it suffices to choose $\delta(u)$ in such a way that $d(\neg\theta)$ is not Π_1 (cf. the proof of Corollary 4 of [3]).

By Lemma 6, there is a PR formula $\eta_1(x, z)$ such that

(1) if $T \vdash \varphi$, then $T \vdash \neg\exists z\, \eta_1(\bar{\varphi}, z)$,

(2) if $T \nvdash \varphi$, then $\exists z\, \eta_1(\bar{\varphi}, z)$ is Π_1-conservative over $T + \varphi$.

For any formula $\gamma(x)$ let

$$\mu_\gamma(x) := \forall z\, (\eta_1(x, z) \rightarrow \exists u < z\, \gamma(u)).$$

We can then effectively in k and γ define Σ_1 formulas $\sigma_{\gamma,k}(x)$ such that

(3) $\quad T \vdash \sigma_{\gamma,k+1}(x) \to \sigma_{\gamma,k}(x)$,

(4) $\quad$ if $T + \mu_\gamma(\bar{\varphi})$ is consistent, then

$$T + \mu_\gamma(\bar{\varphi}) + \sigma_{\gamma,k}(\bar{\varphi}) < T + \mu_\gamma(\bar{\varphi}) + \sigma_{\gamma,k+1}(\bar{\varphi}).$$

By Lemma 4(iii), there is a PR formula $\rho(x, y, z)$ such that

(5) $\quad T + \mu_\gamma(\bar{\varphi}) + \exists u\, \rho(\bar{\gamma}, \bar{\varphi}, u)$ is a Π_1-conservative extension of

$$T + \mu_\gamma(\bar{\varphi}) + \{\sigma_{\gamma,k}(\bar{\varphi}) \colon k \in \mathbb{N}\}.$$

Now let $\kappa(x)$ be such that

(6) $\quad T \vdash \kappa(\bar{\gamma}) \leftrightarrow \forall u\, (\rho(\bar{\gamma}, \overline{\kappa(\bar{\gamma})}, u) \to \exists z \leqslant u\, \eta_1(\overline{\kappa(\bar{\gamma})}, z))$.

Then

(7) $\quad T \nvdash \kappa(\bar{\gamma})$.

For suppose not. Then, by (1),

(8) $\quad T \vdash \neg\exists z\, \eta_1(\overline{\kappa(\bar{\gamma})}, z)$.

It follows that $T \vdash \mu_\gamma(\overline{\kappa(\bar{\gamma})})$ and so, by (4) and (5), $T + \exists u\, \rho(\bar{\gamma}, \overline{\kappa(\bar{\gamma})}, u)$ is consistent. On the other hand, by (6) and (8), this theory is inconsistent, a contradiction. This proves (7).

Now let

$$\chi_\gamma := \exists z\, (\eta_1(\overline{\kappa(\bar{\gamma})}, z) \wedge \forall u < z\, \neg\rho(\bar{\gamma}, \overline{\kappa(\bar{\gamma})}, u)).$$

Then

$$T \vdash \chi_\gamma \leftrightarrow \exists z\, \eta_1(\overline{\kappa(\bar{\gamma})}, z) \wedge \kappa(\bar{\gamma}).$$

But then, by (2) and (7), for all γ,

(9) $\quad d(\chi_\gamma) = d(\kappa(\bar{\gamma}))$.

Moreover

(10) $\quad T \vdash \neg\kappa(\bar{\gamma}) \leftrightarrow \exists u\, \rho(\bar{\gamma}, \overline{\kappa(\bar{\gamma})}, u) \wedge \forall z\, (\eta_1(\overline{\kappa(\bar{\gamma})}, z) \to \exists u < z\, \rho(\bar{\gamma}, \overline{\kappa(\bar{\gamma})}, u))$.

Finally let $\nu(u)$ be such that

$$T \vdash \nu(u) \leftrightarrow \rho(\bar{\nu}, \overline{\kappa(\bar{\nu})}, u)$$

and set $\theta := \kappa(\bar{\nu})$. Then, by (10),

(11) $\quad T \vdash \neg\theta \leftrightarrow \exists u\, \rho(\bar{\nu}, \bar{\theta}, u) \wedge \mu_\nu(\bar{\theta})$.

Combining this with (7) we get

(12) $\quad T + \mu_\nu(\bar{\theta})$ is consistent.

That $d(\neg\theta)$ is not Π_1 can now be shown in the following way. Let π be such that $T + \neg\theta \vdash \pi$. Then, by (11), (3), (5), there is a k such that $T + \mu_\nu(\bar{\theta}) + \sigma_{\nu,k}(\bar{\theta}) \vdash \pi$. But then, by (12), (4), (5), (11), $d(\neg\theta) > d(\pi)$.

Finally, as we have already seen, by (9), $d(\theta)$ is the p.c. of $d(\neg\theta)$ and so the proof is complete.

*Note added in proof

I have now answered some of the questions left open in the paper by proving the following results.

Theorem A. *To every Σ_1 degree $a < 1$, there is a Π_1 degree $\geqslant a$ which caps to* 0.

Theorem B. (i) *Every sufficiently large degree is the l.u.b. of a Σ_1 and a Π_1 degree.*

(ii) *Every sufficiently large degree is the l.u.b. of two Σ_1 degrees.*

Theorem B(ii), in combination with the proof of Theorem 2(i), implies Theorem 2(iii).

Theorem C. *There is a Σ_1 degree a such that no Π_1 degree cups to a and* (*consequently*) *a cups to no Π_1 degree.*

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