Stability for the functional equation of cubic type

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Abstract

In this article, we establish the stability of the orthogonally cubic type functional equation (1.2) for all $x_1, x_2, x_3$ with $x_i \perp x_j$ ($i, j = 1, 2, 3$), where $\perp$ is the orthogonality in the sense of Rätz, and investigate the stability of the $n$-dimensional cubic type functional equation (1.3), where $n \geq 3$ is an integer.

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1. Introduction

In 1940, S.M. Ulam [32] proposed the following question concerning the stability of group homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D.H. Hyers [12] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [24]. Since then, the stability problems of several
The functional equation have been extensively investigated by a number of authors (for instance, [1–6, 9–11, 13, 15, 17–19, 22, 23, 25–28, 30, 31]).

In particular, one of the important functional equations studied is the following functional equation:

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y). \]

The quadratic function \( f(x) = ax^2 \) is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 8, 16, 20].

The Hyers–Ulam stability problem of the quadratic functional equation was first proved by P.W. Cholewa [6] and S. Czerwik [8].

The cubic function \( f(x) = ax^3 \) satisfies the functional equation

\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \]

Hence, throughout this paper, we promise that Eq. (1.1) is called a cubic functional equation and every solution of Eq. (1.1) is said to be a cubic function.

The functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [14]. In fact, they proved that a function \( f : X \to Y \) between real vector spaces is a solution of the functional equation (1.1) if and only if there exists a function \( G : X \times X \times X \to Y \) such that \( f(x) = G(x, x, x) \) for all \( x \in X \), and \( G \) is symmetric for each fixed one variable and additive for fixed two variables. The function \( G \) is given by

\[ G(x, y, z) = \frac{1}{24}[(f(x + y + z) + f(x - y - z) - f(x + y - z) - f(x - y + z)] \]

for all \( x, y, z \in X \). Moreover, they investigated the Hyers–Ulam–Rassias stability for the functional equation (1.1).

Recently, Chang, Jun and Jung [4] introduced the cubic type functional equation as follows:

\[
\begin{align*}
  f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2) + 7[f(x_1) + f(-x_1)] \\
  = 2f(x_1 + x_2) + 4[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)].
\end{align*}
\]

It is easy to see that the function \( f(x) = ax^3 + b \) is a solution of the functional equation (1.2).

In this paper, we establish the stability of the orthogonally cubic type functional equation (1.2) for all \( x_1, x_2, x_3 \) with \( x_i \bot x_j \) \( (i, j = 1, 2, 3) \), where \( \bot \) is the orthogonality in the sense of Rätz. Furthermore, we will extend Eq. (1.2) to the \( n \)-dimensional cubic type functional equation

\[
\begin{align*}
  2f\left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + 2f\left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + 2\sum_{j=1}^{n-1} f(2x_j) + 7(n - 1)[f(x_1) + f(-x_1)] \\
  = 4f\left( \sum_{j=1}^{n-1} x_j \right) + 8\sum_{j=1}^{n-1} [f(x_j + x_n) + f(x_j - x_n)].
\end{align*}
\]

where \( n \geq 3 \) is an integer, and offer the stability results for this equation.

2. Stability of Eq. (1.2)

Let us recall the orthogonality in the sense of J. Rätz [29].

Suppose that \( X \) is a real vector space with \( \text{dim} \ X \geq 2 \) and \( \bot \) is a binary relation on \( X \) with the following properties:
Lemma 2.2. From the assumption, it follows that

Proof. Therefore (2.1) now becomes

and by comparing with (2.2), it follows that

The pair \((X, \perp)\) is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Definition 2.1. Let \(X\) and \(Y\) be an orthogonality space and a real vector space. A mapping \(f : X \to Y\) is said to orthogonally cubic if it satisfies the so-called orthogonally cubic functional equation (1.1) for all \(x, y \in X\) with \(x \perp y\).

Lemma 2.2. Let \(X\) and \(Y\) be an orthogonality space and a real vector space, respectively. If a function \(f : X \to Y\) satisfies the functional equation (1.2) for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\), then \(C\) is orthogonally cubic, where \(C : X \to Y\) is a function defined by

\[
C(x) = f(x) - f(0) \quad \text{for all} \quad x \in X.
\]

Proof. From the assumption, it follows that

\[
C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2) + 7[C(x_1) + C(-x_1)]
\]

\[
= 2C(x_1 + x_2) + 4[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)]
\]

(2.1)

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\). Particularly, it is obvious that \(C(0) = 0\). Observe that \(x \perp 0\) for all \(x \in X\). Putting \(x_1 = x_2 = 0\) in (2.1), we arrive at

\[
C(x_3) + C(-x_3) = 8[C(x_3) + C(-x_3)].
\]

(2.2)

Letting \(x_3 = 0\) in (2.1) gives the equation

\[
C(2x_1) + C(2x_2) + 7[C(x_1) + C(-x_1)] = 8[C(x_1) + C(x_2)].
\]

(2.3)

If we put \(x_2 = 0\) in (2.3), then we conclude that

\[
C(2x_1) = C(x_1) - 7C(-x_1).
\]

(2.4)

Let us replace \(x_1\) by \(-x_1\) in (2.4), then we get

\[
C(-2x_1) = C(-x_1) - 7C(x_1).
\]

(2.5)

By adding (2.4) and (2.5), we find that

\[
C(2x_1) + C(-2x_1) = -6[C(x_1) + C(-x_1)]
\]

and by comparing with (2.2), it follows that

\[
C(x_1) + C(-x_1) = 0.
\]

(2.6)

Therefore (2.1) now becomes

\[
C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2)
\]

\[
= 2C(x_1 + x_2) + 4[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)]
\]

(2.7)
for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\).

We take \(x_1 = 0\) in (2.7) and then use (2.6) to obtain

\[
C(x_2 + 2x_3) + C(x_2 - 2x_3) + C(2x_2) = 2C(x_2) + 4\left[C(x_2 + x_3) + C(x_2 - x_3)\right].
\] (2.8)

Setting \(x_3 = 0\) in (2.8) leads to the identity \(C(2x_2) = 8C(x_2)\). If \(x_3 \perp x_2\), then \(x_3 \perp 2x_2, 2x_3 \perp x_2\) and \(2x_3 \perp 2x_2\). By replacing \(x_2\) by \(2x_2\) in (2.8), we see that

\[
8C(x_2 + x_3) + 8C(x_2 - x_3) + 64C(x_2) = 16C(x_2) + 4\left[C(2x_2 + x_3) + C(2x_2 - x_3)\right]
\]

for all \(x_2, x_3 \in X\) with \(x_2 \perp x_3\), which means that \(C\) is orthogonally cubic. The proof of lemma is complete. \(\square\)

From now forward, let \(X\) be an orthogonality normed space and \(Y\) be a Banach space. Given a mapping \(f : X \to Y\), we set

\[
D_1 f(x_1, x_2, x_3) := f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2)
\]

\[
+ 7\left[f(x_1) + f(-x_1)\right] - 2f(x_1 + x_2)
\]

\[
- 4\left[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\right]
\]

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\).

**Theorem 2.3.** Suppose that \(f : X \to Y\) is a mapping for which there exists a function \(\phi : X^3 \to [0, \infty)\) such that

\[
\sum_{i=0}^{\infty} \frac{1}{2^{3i}} \phi(0, 2^i x_2, 0) < \infty,
\] (2.9)

\[
\lim_{n \to \infty} \frac{1}{2^{3i}} \phi(2^i x_1, 2^i x_2, 2^i x_3) = 0
\] (2.10)

and

\[
\|D_1 f(x_1, x_2, x_3)\| \leq \delta + \phi(x_1, x_2, x_3)
\] (2.11)

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\), where \(\delta \geq 0\). Then there exists a unique orthogonally cubic function \(C : X \to Y\) satisfying the inequality

\[
\|f(x) - C(x)\| \leq \frac{1}{8} \left[\sum_{i=0}^{\infty} \frac{1}{2^{3i}} (\delta + \phi(0, 2^i x, 0)) \right] + \|f(0)\|
\] (2.12)

for all \(x \in X\).

**Proof.** Let \(F\) be a function on \(X\) defined by

\[
F(x) = f(x) - f(0)
\]

for all \(x \in X\). Then we have \(F(0) = 0\). Note that \(x \perp 0\) for all \(x \in X\). Putting \(x_1 = x_3 = 0, x_2 = x\) in (2.11) and dividing by 8, we have

\[
\|F(x) - \frac{1}{8} F(2x)\| \leq \frac{1}{8} [\delta + \phi(0, x, 0)].
\] (2.13)
By replacing \( x \) by \( 2x \) in (2.13) and dividing by 8 and summing the resulting inequality with (2.13), we get
\[
\left\| F(x) - \left( \frac{1}{8} \right)^2 F(2^2 x) \right\| \leq \frac{1}{8} \left[ \delta + \phi(0, x, 0) \right] + \left( \frac{1}{8} \right)^2 \left[ \delta + \phi(0, 2x, 0) \right].
\] (2.14)

An induction implies that
\[
\left\| F(x) - \left( \frac{1}{8} \right)^n F(2^n x) \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} \left( \frac{1}{8} \right)^i \left[ \delta + \phi(0, 2^i x, 0) \right].
\] (2.15)

In order to prove convergence of the sequence \( \{ F(2^n x) \} \), we divide inequality (2.15) by \( 8^m \) and also replace \( x \) by \( 2^m x \) to find that for \( n > m > 0 \),
\[
\left\| \left( \frac{1}{8} \right)^m F(2^m x) - \left( \frac{1}{8} \right)^{n+m} F(2^n 2^m x) \right\|
= \left( \frac{1}{8} \right)^m \left\| F(2^m x) - \left( \frac{1}{8} \right)^n F(2^n 2^m x) \right\|
\leq \left( \frac{1}{8} \right)^{m+1} \sum_{i=0}^{n-1} \left( \frac{1}{8} \right)^i \left[ \delta + \phi(0, 2^m+i x, 0) \right].
\] (2.16)

Sine the right-hand side of the inequality tends to 0 as \( m \to \infty \), \( \{ F(2^n x) \} \) is Cauchy sequence. Therefore, we may define a function \( C : X \to Y \) by
\[
C(x) := \lim_{n \to \infty} \frac{F(2^n x)}{2^m}
\]
for all \( x \in X \). By letting \( n \to \infty \) in (2.15), we arrive at the formula (2.12).

Now we show that \( C \) satisfies the functional equation (1.2) for all \( x_1, x_2, x_3 \in X \) with \( x_i \perp x_j \) \((i, j = 1, 2, 3)\): If \( x_i \perp x_j \), then \( 2^i x_i \perp 2^j x_j \) for \( i, j = 1, 2, 3 \). Let us replace \( x_1, x_2 \) and \( x_3 \) by \( 2^n x_1, 2^n x_2 \) and \( 2^n x_3 \) in (2.11) and divide by \( 8^n \). Then it follows that
\[
D_1 C(x_1, x_2, x_3) = \lim_{n \to \infty} \frac{1}{2^{3n}} \left\| D_1 F(2^n x_1, 2^n x_2, 2^n x_3) \right\|
\leq \lim_{n \to \infty} \frac{1}{2^{3n}} \left[ \delta + \phi(2^n x_1, 2^n x_2, 2^n x_3) \right] = 0.
\]

Hence we obtain the desired result. Since \( C(0) = 0 \), Lemma 2.2 implies that \( C \) is an orthogonally cubic.

It only remains to claim that \( C \) is unique: Let us assume that there exists an orthogonally cubic function \( C' \) which satisfies (1.2) and the inequality (2.12). It is clear that \( C(2^n x) = 8^n C(x) \) and \( C'(2^n x) = 8^n C'(x) \) for all \( x \in X \) and \( n \in \mathbb{N} \). Hence it follows from (2.12) that
\[
\left\| C(x) - C'(x) \right\| = \left\| \left( \frac{1}{8} \right)^n \right\| C(2^n x) - C'(2^n x) \right\|
\leq \left\| \left( \frac{1}{8} \right)^n \right\| \left[ \left\| C(2^n x) - f(2^n x) \right\| + \left\| f(2^n x) - C'(2^n x) \right\| \right]
\leq \left\| \left( \frac{1}{8} \right)^n \right\| \left\{ \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{2^{3i}} \left[ \delta + \phi(0, 2^i x, 0) \right] \right\} + 2 \left\| f(0) \right\|.
\]
By letting $n \to \infty$, we have $C(x) = C'(x)$ for all $x \in X$, which completes the proof of the theorem. \hfill \square

**Corollary 2.4.** Let $p, q (< 3), r, \delta, \varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping such that

$$
\|D_1 f(x_1, x_2, x_3)\| \leq \delta + \varepsilon_1 \|x_1\|^p + \varepsilon_2 \|x_2\|^q + \varepsilon_3 \|x_3\|^r
$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$. Then there exists a unique orthogonally cubic function $C : X \to Y$ satisfying the inequality

$$
\|f(x) - C(x)\| \leq \frac{1}{7} \delta + \frac{1}{8 - 2^q} \varepsilon_2 \|x\|^q + \|f(0)\|
$$

for all $x \in X$.

**Proof.** In Theorem 2.3, if we consider that

$$
\phi(x_1, x_2, x_3) = \varepsilon_1 \|x_1\|^p + \varepsilon_2 \|x_2\|^q + \varepsilon_3 \|x_3\|^r,
$$

then we arrive at the conclusion of the corollary. \hfill \square

3. Stability of Eq. (1.3)

For explicitly later use, we demonstrate the following theorem:

**Theorem 3.1 (The alternative of fixed point).** [21] Suppose that we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $L$. Then, for each given $x \in \Omega$, either

$$
d(T^n x, T^{n+1} x) = \infty \quad \text{for all } n \geq 0,
$$

or

there exists a natural number $n_0$ such that

- $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- the sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;
- $y^*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

For completeness, we will first present solution of the functional equation (1.3).

**Lemma 3.2.** Let $X$ and $Y$ be real vector spaces. A function $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, x_2, \ldots, x_n \in X$ if and only if $C$ is cubic, where $C : X \to Y$ is a function defined by $C(x) = f(x) - f(0)$ for all $x \in X$.

**Proof.** (Necessity.) Note that, by the assumption, we arrive at

$$
C \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + C \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n-1)}{2} \left[ C(x_1) + C(-x_1) \right]
$$

\[ = 2C \left( \sum_{j=1}^{n-1} x_j \right) + 4 \sum_{j=1}^{n-1} \left[ C(x_j + x_n) + C(x_j - x_n) \right] \]  

(3.1)

for all \( x_1, x_2, \ldots, x_n \in X \). In particular, it is clear that \( C(0) = 0 \). Substituting \( x_j = 0 \) \( (j = 1, 2, \ldots, n - 1) \) and \( x_n = x \) in (3.1) yields

\[ C(2x) + C(-2x) = 4(n - 1) \left[ C(x) + C(-x) \right]. \]  

(3.2)

Letting \( x_1 = x, x_2 = -x, \) and \( x_j = 0 \) \( (j = 3, \ldots, n) \) in (3.1) gives the equation

\[ C(2x) + C(-2x) = \frac{23 - 7n}{2} \left[ C(x) + C(-x) \right]. \]  

(3.3)

Now, by combining (3.2) and (3.3), we lead to

\[ C(x) + C(-x) = 0 \]

for all \( x \in X \), i.e., \( C \) is an odd function.

Hence (3.1) now becomes

\[ C \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + C \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + \sum_{j=1}^{n-1} C(2x_j) \]

\[ = 2C \left( \sum_{j=1}^{n-1} x_j \right) + 4 \sum_{j=1}^{n-1} \left[ C(x_j + x_n) + C(x_j - x_n) \right]. \]

Thus [7, Lemma 2.2] implies that \( C \) is cubic.

(Sufficiency.) Suppose that \( C \) is cubic, i.e.,

\[ C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \]  

(3.4)

for all \( x, y \in X \). Then it is easy to check that

\[ C(0) = 0, \quad C(x) + C(-x) = 0 \quad \text{and} \quad C(2x) = 8C(x). \]

On the other hand, by [7, Lemma 2.2], we obtain

\[ C \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + C \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + \sum_{j=1}^{n-1} C(2x_j) \]

\[ = 2C \left( \sum_{j=1}^{n-1} x_j \right) + 4 \sum_{j=1}^{n-1} \left[ C(x_j + x_n) + C(x_j - x_n) \right]. \]

Since \( C \) is an odd function, we note that

\[ C \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + C \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n - 1)}{2} \left[ C(x_1) + C(-x_1) \right] \]

\[ = 2C \left( \sum_{j=1}^{n-1} x_j \right) + 4 \sum_{j=1}^{n-1} \left[ C(x_j + x_n) + C(x_j - x_n) \right], \]

which gives the functional equation (1.3) for all \( x_1, x_2, \ldots, x_n \in X \). This completes the proof of the lemma. \( \Box \)
Remark 3.3. Lemma 3.2 states that the functional equation (1.3) has a solution of the form \( C(x) + B \), where \( C \) is cubic and \( B \) is a constant.

From now on, let \( X \) be a real vector space and \( Y \) be a real Banach space. As a matter of convenience, for a given mapping \( f : X \to Y \), we use the following abbreviation:

\[
D_2 f(x_1, x_2, \ldots, x_n) := 2f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + 2f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + 2\sum_{j=1}^{n-1} f(2x_j) \\
+ 7(n - 1)\left[ f(x_1) + f(-x_1) \right] - 4f\left(\sum_{j=1}^{n-1} x_j\right) \\
- 8\sum_{j=1}^{n-1}\left[ f(x_j + x_n) + f(x_j - x_n) \right]
\]

for all \( x_1, x_2, \ldots, x_n \in X \).

Let \( \varphi : \mathbb{R}^n \to [0, \infty) \) be a function satisfying

\[
\lim_{k \to \infty} \varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n) = 0 \tag{3.5}
\]

for all \( x_1, x_2, \ldots, x_n \in X \), where

\[
\begin{aligned}
\lambda_i = 2, & \quad \text{if } i = 0, \\
\lambda_i = \frac{1}{2}, & \quad \text{if } i = 1.
\end{aligned}
\]

Now, by the use of fixed point alternative, we obtain the main result as follow.

**Theorem 3.4.** Let \( n \geq 3 \) be an integer. Suppose that a function \( f : X \to Y \) satisfies the inequality

\[
\|D_2 f(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n) \tag{3.6}
\]

for all \( x_1, x_2, \ldots, x_n \in X \). If there exists \( L < 1 \) such that the function

\[
x \mapsto \psi(x) = \varphi\left(0, \frac{x}{2}, \frac{x}{2}, \ldots, \frac{x}{2}, 0\right)
\]

has the property

\[
\psi(x) \leq L \cdot \lambda_i^3 \cdot \psi\left(\frac{x}{\lambda_i}\right) \tag{3.7}
\]

for all \( x \in X \), then there exists a unique cubic function \( C : X \to Y \) satisfying the inequality

\[
\|f(x) - C(x)\| \leq \frac{1}{2(n - 2)} \frac{L^{1-i}}{1-L} \psi(x) + \|f(0)\| \tag{3.8}
\]

for all \( x \in X \).
Proof. Consider the set
\[ \Omega := \{ g : g : X \to Y, \ g(0) = 0 \} \]
and introduce the generalized metric on \( \Omega \):
\[ d(g, h) = d_\psi(g, h) = \inf \{ K \in (0, \infty) : \| g(x) - h(x) \| \leq K \psi(x), \ x \in X \} \]
It is easy to see that \( (\Omega, d) \) is complete.
Now we define a function \( T : \Omega \to \Omega \) by
\[ Tg(x) = \frac{1}{\lambda_i^3} g(\lambda_i x) \]
for all \( x \in X \). Note that for all \( g, h \in \Omega \),
\[ d(g, h) < K \quad \implies \quad \| g(x) - h(x) \| \leq K \psi(x), \ x \in X \]
\[ \implies \quad \| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \| \leq \frac{1}{\lambda_i^3} K \psi(\lambda_i x), \ x \in X \]
\[ \implies \quad \| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \| \leq LK \psi(x), \ x \in X \]
\[ \implies \quad d(Tg, Th) \leq LK. \]
Hence we see that
\[ d(Tg, Th) \leq Ld(g, h) \]
for all \( g, h \in \Omega \), i.e., \( T \) is a strictly contractive mapping of \( \Omega \) with the Lipschitz constant \( L \).
Here we define a function \( F : X \to Y \) by
\[ F(x) = f(x) - f(0) \]
for all \( x \in X \). Then we have \( F(0) = 0 \).
If we put \( x_1 = 0, x_2 = \cdots = x_{n-1} = y, x_n = 0 \) in (3.6) and use (3.7), then
\[ \| (n-2)F(2y) - 8(n-2)F(y) \|
= \| (n-2)[f(2y) - f(0)] - 8(n-2)[f(y) - f(0)] \|
\leq \frac{1}{2} \varphi(0, y, y, \ldots, y, 0), \tag{3.9} \]
which is reduced to
\[ \| F(y) - \frac{1}{2^3} F(2y) \| \leq \frac{1}{2^3 \frac{1}{2(n-2)}} \psi(2y) \leq \frac{L}{2(n-2)} \psi(y) \]
for all \( y \in X \), i.e., \( d(F, TF) \leq \frac{L}{2(n-2)} \leq \infty \).
If we substitute \( y := \frac{y}{2} \) in (3.9) and use (3.7), then
\[ \| F(y) - 2^3 F\left(\frac{y}{2}\right) \| \leq \frac{1}{2(n-2)} \psi(y) \]
for all \( y \in X \), i.e., \( d(F, TF) \leq \frac{1}{2(n-2)} < \infty \).
Now, from the fixed point alternative in both cases, it follows that there exists a fixed point \( C \) of \( T \) in \( \Omega \) such that
\[
C(x) = \lim_{k \to \infty} \frac{F(\lambda_i^k x)}{\lambda_i^3 k}
\tag{3.10}
\]
for all \( x \in X \), since \( \lim_{k \to \infty} d(T^k F, C) = 0 \).

To show that the function \( C : X \to Y \) is cubic, let \( x_j := \lambda_i^k x_j \) for \( j = 1, 2, \ldots, n \) in (3.6) and divide by \( \lambda_i^3 k \). Then it follows from (3.5) and (3.10) that
\[
\|D_2 C(x_1, x_2, \ldots, x_n)\| = \lim_{k \to \infty} \|D_2 F(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)\| \leq \lim_{k \to \infty} \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^3 k} = 0
\]
for all \( x_1, x_2, \ldots, x_n \in X \), i.e., \( C \) satisfies the functional equation (1.3). Therefore Lemma 3.2 guarantees that \( C \) is cubic, since \( C(0) = 0 \).

According to the fixed point alternative, since \( C \) is the unique fixed point of \( T \) in the set \( \Delta = \{ g \in \Omega : d(F, g) < \infty \} \), \( C \) is the unique function such that
\[
\| F(x) - C(x)\| \leq K \psi(x)
\]
for all \( x \in X \) and some \( K > 0 \). Again, using the fixed point alternative, we have
\[
d(F, C) \leq \frac{1}{1 - L} d(F, TF),
\]
and so we obtain the inequality
\[
d(F, C) \leq \frac{1}{2(n - 2)} \frac{L^{1-i}}{1 - L},
\]
which yields the inequality (3.8). This completes the proof of the theorem. \( \square \)

From Theorem 3.4, we obtain the following corollary concerning the Hyers–Ulam–Rassias stability [24] of the functional equation (1.3).

**Corollary 3.5.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Let \( p \geq 0 \) be given with \( p \neq 3 \) and \( n \geq 3 \) an integer. Assume that \( \delta \geq 0 \) and \( \varepsilon \geq 0 \) are fixed. Suppose that a function \( f : X \to Y \) satisfies the inequality
\[
\|D_2 f(x_1, x_2, \ldots, x_n)\| \leq \delta + \varepsilon \left( \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \right)
\tag{3.11}
\]
for all \( x_1, x_2, \ldots, x_n \in X \). Moreover, assume that \( \delta = 0 \) in (3.11) for the case \( p > 3 \). Then there exists a unique cubic function \( C : X \to Y \) satisfying
the inequality
\[
\| f(x) - C(x)\| \leq \frac{1}{2(n - 2)} \frac{\delta}{2^{3-p} - 1} + \frac{1}{2} \frac{\varepsilon}{8 - 2p} \|x\|^p + \| f(0)\|
\tag{3.12}
\]
which holds for all \( x \in X \), where \( p < 3 \),
or

the inequality

\[ \| f(x) - C(x) \| \leq \frac{1}{2} \frac{\varepsilon}{2p - 8} \| x \|^p + \| f(0) \| \] (3.13)

which holds for all \( x \in X \), where \( p > 3 \).

**Proof.** Let

\[ \varphi(x_1, x_2, \ldots, x_n) := \delta + \varepsilon \left( \| x_1 \|^p + \| x_2 \|^p + \cdots + \| x_n \|^p \right) \]

for all \( x_1, x_2, \ldots, x_n \in X \). Then it follows that

\[ \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^{3k}} = \frac{\delta}{\lambda_i^k} + (\lambda_i^k)^{p-3} \varepsilon (\| x_1 \|^p + \| x_2 \|^p + \cdots + \| x_n \|^p) \to 0 \]

as \( k \to \infty \), where

\[
\begin{cases}
  p < 3, & \text{if } i = 0, \\
  p > 3, & \text{if } i = 1,
\end{cases}
\]

i.e., (3.5) is true.

Since the inequality

\[ \frac{1}{\lambda_i^3} \psi(\lambda_i x) = \frac{\delta}{\lambda_i^3} + \lambda_i^{p-3} \frac{(n - 2)\varepsilon \| x \|^p}{2p} \leq \lambda_i^{p-3} \psi(x) \]

holds for all \( x \in X \), where

\[
\begin{cases}
  p < 3, & \text{if } i = 0, \\
  p > 3, & \text{if } i = 1,
\end{cases}
\]

we see that the inequality (3.7) holds with either \( L = 2p^{-3} \) or \( L = \frac{1}{2p^2} \). Now the inequality (3.8) yields the inequalities (3.12) and (3.13), which complete the proof of the corollary. \(\square\)

The following corollary is the Hyers–Ulam stability [12] of the functional equation (1.3).

**Corollary 3.6.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Assume that \( \theta \geq 0 \) is fixed and \( n \geq 3 \) an integer. Suppose that a function \( f : X \to Y \) satisfies the inequality

\[ \| D_2 f(x_1, x_2, \ldots, x_n) \| \leq \theta \] (3.14)

for all \( x_1, x_2, \ldots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) satisfying the inequality

\[ \| f(x) - C(x) \| \leq \frac{1}{14n} \theta + \| f(0) \| \] (3.15)

for all \( x \in X \).

**Proof.** Considering \( \delta := 0 \), \( p := 0 \) and \( \varepsilon := \frac{\theta}{n} \) in Corollary 3.5, we arrive at the conclusion of the corollary. \(\square\)
Acknowledgments

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References