Stability for the functional equation of cubic type

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Abstract

In this article, we establish the stability of the orthogonally cubic type functional equation (1.2) for all \( x_1, x_2, x_3 \) with \( x_i \perp x_j \) \((i, j = 1, 2, 3)\), where \( \perp \) is the orthogonality in the sense of Rätz, and investigate the stability of the \( n \)-dimensional cubic type functional equation (1.3), where \( n \geq 3 \) is an integer.

Keywords: Stability; Cubic functional equation; Orthogonally cubic functional equation

1. Introduction

In 1940, S.M. Ulam [32] proposed the following question concerning the stability of group homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h: G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H: G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)?

In next year, D.H. Hyers [12] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [24]. Since then, the stability problems of several
functional equation have been extensively investigated by a number of authors (for instance, [1–6,9–11,13,15,17–19,22,23,25–28,30,31]).

In particular, one of the important functional equations studied is the following functional equation:

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y). \]

The quadratic function \( f(x) = ax^2 \) is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1,8,16,20].

The Hyers–Ulam stability problem of the quadratic functional equation was first proved by F. Skof [30] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [6] and S. Czerwik [8].

The cubic function \( f(x) = ax^3 \) satisfies the functional equation

\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \]

Hence, throughout this paper, we promise that Eq. (1.1) is called a cubic functional equation and every solution of Eq. (1.1) is said to be a cubic function.

The functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [14]. In fact, they proved that a function \( f : X \to Y \) between real vector spaces is a solution of the functional equation (1.1) if and only if there exists a function \( G : X \times X \times X \to Y \) such that \( f(x) = G(x, x, x) \) for all \( x \in X \), and \( G \) is symmetric for each fixed one variable and additive for fixed two variables. The function \( G \) is given by

\[ G(x, y, z) = \frac{1}{24} \left[ f(x + y + z) + f(x - y - z) - f(x + y - z) - f(x - y + z) \right] \]

for all \( x, y, z \in X \). Moreover, they investigated the Hyers–Ulam–Rassias stability for the functional equation (1.1).

Recently, Chang, Jun and Jung [4] introduced the cubic type functional equation as follows:

\[
\begin{align*}
&f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2) + 7\left[f(x_1) + f(-x_1)\right] \\
&= 2f(x_1 + x_2) + 4\left[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\right].
\end{align*}
\]

It is easy to see that the function \( f(x) = ax^3 + b \) is a solution of the functional equation (1.2).

In this paper, we establish the stability of the orthogonally cubic type functional equation (1.2) for all \( x_1, x_2, x_3 \) with \( x_i \perp x_j \) (\( i, j = 1, 2, 3 \)), where \( \perp \) is the orthogonality in the sense of Rätz. Furthermore, we will extend Eq. (1.2) to the \( n \)-dimensional cubic type functional equation

\[
2f \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + 2f \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + 2 \sum_{j=1}^{n-1} f(2x_j) + 7(n-1) \left[f(x_1) + f(-x_1)\right] \\
= 4f \left( \sum_{j=1}^{n-1} x_j \right) + 8 \sum_{j=1}^{n-1} f(x_j + x_n) + f(x_j - x_n),
\]

where \( n \geq 3 \) is an integer, and offer the stability results for this equation.

2. Stability of Eq. (1.2)

Let us recall the orthogonality in the sense of J. Rätz [29].

Suppose that \( X \) is a real vector space with \( \text{dim} X \geq 2 \) and \( \perp \) is a binary relation on \( X \) with the following properties:
Lemma 2.2. Let \((i, j \in \mathbb{N})\) from the assumption, it follows that
and by comparing with \((2.2)\), it follows that
for all \(x \in X\):

(1) totality of \(\perp\) for zero: \(x \perp 0, 0 \perp x\) for all \(x \in X\);
(2) independence: if \(x \in X - \{0\}, x \perp y\), then \(x, y\) are linearly independent;
(3) homogeneity: if \(x \in X, x \perp y\), then \(\alpha x \perp \beta y\) for all \(\alpha, \beta \in \mathbb{R}\);
(4) the Thalesian property: if \(P\) is a 2-dimensional subspace of \(X, x \in P\) and \(\lambda \in \mathbb{R}_+\), then there exists \(y_0 \in P\) such that \(x \perp y_0\) and \(x + y_0 \perp \lambda x - y_0\).

The pair \((X, \perp)\) is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Definition 2.1. Let \(X\) and \(Y\) be an orthogonality space and a real vector space. A mapping \(f : X \to Y\) is said to orthogonally cubic if it satisfies the so-called orthogonally cubic functional equation \((1.1)\) for all \(x, y \in X\) with \(x \perp y\).

Lemma 2.2. Let \(X\) and \(Y\) be an orthogonality space and a real vector space, respectively. If a function \(f : X \to Y\) satisfies the functional equation \((1.2)\) for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\), then \(C\) is orthogonally cubic, where \(C : X \to Y\) is a function defined by \(C(x) = f(x) - f(0)\) for all \(x \in X\).

Proof. From the assumption, it follows that

\[
C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2) + 7[C(x_1) + C(-x_1)]
\]

\[
= 2C(x_1 + x_2) + 4[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)] \quad (2.1)
\]

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) \((i, j = 1, 2, 3)\). Particularly, it is obvious that \(C(0) = 0\). Observe that \(x \perp 0\) for all \(x \in X\). Putting \(x_1 = x_2 = 0\) in \((2.1)\), we arrive at

\[
C(2x_3) + C(-2x_3) = 8[C(x_3) + C(-x_3)]. \quad (2.2)
\]

Letting \(x_3 = 0\) in \((2.1)\) gives the equation

\[
C(2x_1) + C(2x_2) + 7[C(x_1) + C(-x_1)] = 8[C(x_1) + C(x_2)]. \quad (2.3)
\]

If we put \(x_2 = 0\) in \((2.3)\), then we conclude that

\[
C(2x_1) = C(x_1) - 7C(-x_1). \quad (2.4)
\]

Let us replace \(x_1\) by \(-x_1\) in \((2.4)\), then we get

\[
C(-2x_1) = C(-x_1) - 7C(x_1). \quad (2.5)
\]

By adding \((2.4)\) and \((2.5)\), we find that

\[
C(2x_1) + C(-2x_1) = -6[C(x_1) + C(-x_1)]
\]

and by comparing with \((2.2)\), it follows that

\[
C(x_1) + C(-x_1) = 0. \quad (2.6)
\]

Therefore \((2.1)\) now becomes

\[
C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2)
\]

\[
= 2C(x_1 + x_2) + 4[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)] \quad (2.7)
\]
for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) (\(i, j = 1, 2, 3\)).

We take \(x_1 = 0\) in (2.7) and then use (2.6) to obtain

\[
C(x_2 + 2x_3) + C(x_2 - 2x_3) + C(2x_2) = 2C(x_2) + 4\left[C(x_2 + x_3) + C(x_2 - x_3)\right].
\]

Setting \(x_3 = 0\) in (2.8) leads to the identity \(C(2x_2) = 8C(x_2)\). If \(x_3 \perp x_2\), then \(x_3 \perp 2x_2, 2x_3 \perp x_2\) and \(2x_3 \perp 2x_2\). By replacing \(x_2\) by \(2x_2\) in (2.8), we see that

\[
8C(x_2 + x_3) + 8C(x_2 - x_3) + 64C(x_2) = 16C(x_2) + 4\left[C(2x_2 + x_3) + C(2x_2 - x_3)\right]
\]

for all \(x_2, x_3 \in X\) with \(x_2 \perp x_3\), which means that \(C\) is orthogonally cubic. The proof of lemma is complete. \(\square\)

From now forward, let \(X\) be an orthogonality normed space and \(Y\) be a Banach space. Given a mapping \(f : X \to Y\), we set

\[
D_1 f(x_1, x_2, x_3) := f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2)
+ 7\left[f(x_1) + f(-x_1)\right] - 2f(x_1 + x_2)
- 4\left[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\right]
\]

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) (\(i, j = 1, 2, 3\)).

**Theorem 2.3.** Suppose that \(f : X \to Y\) is a mapping for which there exists a function \(\phi : X^3 \to [0, \infty)\) such that

\[
\sum_{i=0}^{\infty} \frac{1}{23^i} \phi(0, 2^i x_2, 0) < \infty,
\]

\[
\lim_{n \to \infty} \frac{1}{23^n} \phi(2^i x_1, 2^i x_2, 2^i x_3) = 0
\]

and

\[
\|D_1 f(x_1, x_2, x_3)\| \leq \delta + \phi(x_1, x_2, x_3)
\]

for all \(x_1, x_2, x_3 \in X\) with \(x_i \perp x_j\) (\(i, j = 1, 2, 3\)), where \(\delta \geq 0\). Then there exists a unique orthogonally cubic function \(C : X \to Y\) satisfying the inequality

\[
\|f(x) - C(x)\| \leq \frac{1}{8} \left[\sum_{i=0}^{\infty} \frac{1}{23^i} (\delta + \phi(0, 2^i x, 0)) + \|f(0)\|\right]
\]

for all \(x \in X\).

**Proof.** Let \(F\) be a function on \(X\) defined by

\[
F(x) = f(x) - f(0)
\]

for all \(x \in X\). Then we have \(F(0) = 0\). Note that \(x \perp 0\) for all \(x \in X\). Putting \(x_1 = x_3 = 0, x_2 = x\) in (2.11) and dividing by 8, we have

\[
\|F(x) - \frac{1}{8}F(2x)\| \leq \frac{1}{8} (\delta + \phi(0, x, 0)).
\]
By replacing $x$ by $2x$ in (2.13) and dividing by 8 and summing the resulting inequality with (2.13), we get
\[ \left\| F(x) - \left( \frac{1}{8} \right)^2 F(2^2 x) \right\| \leq \frac{1}{8} \left[ \delta + \phi(0, x, 0) \right] + \left( \frac{1}{8} \right)^2 \left[ \delta + \phi(0, 2x, 0) \right]. \] (2.14)

An induction implies that
\[ \left\| F(x) - \left( \frac{1}{8} \right)^n F(2^n x) \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} \left( \frac{1}{8} \right)^i \left[ \delta + \phi(0, 2^i x, 0) \right]. \] (2.15)

In order to prove convergence of the sequence \( \{ \frac{F(2^n x)}{2^{3n}} \} \), we divide inequality (2.15) by \( 8^m \) and also replace \( x \) by \( 2^m x \) to find that for \( n > m > 0 \),
\[
\left\| \left( \frac{1}{8} \right)^m F(2^m x) - \left( \frac{1}{8} \right)^{n+m} F(2^n 2^m x) \right\| = \left( \frac{1}{8} \right)^m \left\| F(2^m x) - \left( \frac{1}{8} \right)^n F(2^n 2^m x) \right\| \leq \left( \frac{1}{8} \right)^m \sum_{i=0}^{n-1} \left( \frac{1}{8} \right)^i \left[ \delta + \phi(0, 2^m+i x, 0) \right]. \] (2.16)

Since the right-hand side of the inequality tends to 0 as \( m \to \infty \), \( \{ \frac{F(2^n x)}{2^{3n}} \} \) is Cauchy sequence. Therefore, we may define a function \( C : X \to Y \) by
\[ C(x) := \lim_{n \to \infty} \frac{F(2^n x)}{2^{3n}} \]
for all \( x \in X \). By letting \( n \to \infty \) in (2.15), we arrive at the formula (2.12).

Now we show that \( C \) satisfies the functional equation (1.2) for all \( x_1, x_2, x_3 \in X \) with \( x_i \perp x_j \) \((i, j = 1, 2, 3)\): If \( x_i \perp x_j \), then \( 2^n x_i \perp 2^n x_j \) for \( i, j = 1, 2, 3 \). Let us replace \( x_1, x_2 \) and \( x_3 \) by \( 2^n x_1, 2^n x_2 \) and \( 2^n x_3 \) in (2.11) and divide by \( 8^n \). Then it follows that
\[
D_1 C(x_1, x_2, x_3) = \lim_{n \to \infty} \frac{1}{2^{3n}} \left\| D_1 F(2^n x_1, 2^n x_2, 2^n x_3) \right\| \leq \lim_{n \to \infty} \frac{1}{2^{3n}} \left[ \delta + \phi(2^n x_1, 2^n x_2, 2^n x_3) \right] = 0.
\]

Hence we obtain the desired result. Since \( C(0) = 0 \), Lemma 2.2 implies that \( C \) is an orthogonally cubic.

It only remains to claim that \( C \) is unique: Let us assume that there exists an orthogonally cubic function \( C' \) which satisfies (1.2) and the inequality (2.12). It is clear that \( C(2^n x) = 8^n C(x) \) and \( C'(2^n x) = 8^n C'(x) \) for all \( x \in X \) and \( n \in \mathbb{N} \). Hence it follows from (2.12) that
\[
\left\| C(x) - C'(x) \right\| = \left( \frac{1}{8} \right)^n \left\| C(2^n x) - C'(2^n x) \right\| \leq \left( \frac{1}{8} \right)^n \left\{ \left\| C(2^n x) - f(2^n x) \right\| + \left\| f(2^n x) - C'(2^n x) \right\| \right\} \leq \left( \frac{1}{8} \right)^n \left\{ \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{2^{3i}} \left( \delta + \phi(0, 2^i x, 0) \right) \right\} + 2 \left\| f(0) \right\|.
\]
By letting \( n \to \infty \), we have \( C(x) = C'(x) \) for all \( x \in X \), which completes the proof of the theorem. \( \square \)

**Corollary 2.4.** Let \( p, q (< 3), r, \delta, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be nonnegative real numbers. Suppose that \( f : X \to Y \) is a mapping such that
\[
\| D_1 f(x_1, x_2, x_3) \| \leq \delta + \varepsilon_1 \| x_1 \|^p + \varepsilon_2 \| x_2 \|^q + \varepsilon_3 \| x_3 \|^r
\]
for all \( x_1, x_2, x_3 \in X \) with \( x_i \perp x_j \) (i, j = 1, 2, 3). Then there exists a unique orthogonally cubic function \( C : X \to Y \) satisfying the inequality
\[
\| f(x) - C(x) \| \leq \frac{1}{7} \delta + \frac{1}{8 - 2q} \varepsilon_2 \| x \|^q + \| f(0) \|
\]
for all \( x \in X \).

**Proof.** In Theorem 2.3, if we consider that
\[
\phi(x_1, x_2, x_3) = \varepsilon_1 \| x_1 \|^p + \varepsilon_2 \| x_2 \|^q + \varepsilon_3 \| x_3 \|^r,
\]
then we arrive at the conclusion of the corollary. \( \square \)

3. Stability of Eq. (1.3)

For explicitly later use, we demonstrate the following theorem:

**Theorem 3.1** (The alternative of fixed point). [21] Suppose that we are given a complete generalized metric space \((\Omega, d)\) and a strictly contractive mapping \( T : \Omega \to \Omega \) with Lipschitz constant \( L \). Then, for each given \( x \in \Omega \), either
\[
d(T^n x, T^{n+1} x) = \infty \quad \text{for all } n \geq 0,
\]
or
\[
\text{there exists a natural number } n_0 \text{ such that}
\]
- \( d(T^n x, T^{n+1} x) < \infty \) for all \( n \geq n_0 \);
- the sequence \( (T^n x) \) is convergent to a fixed point \( y^* \) of \( T \);
- \( y^* \) is the unique fixed point of \( T \) in the set \( \Delta = \{ y \in \Omega : d(T^{n_0} x, y) < \infty \} \);
- \( d(y, y^*) \leq \frac{1}{1 - L} d(y, Ty) \) for all \( y \in \Delta \).

For completeness, we will first present solution of the functional equation (1.3).

**Lemma 3.2.** Let \( X \) and \( Y \) be real vector spaces. A function \( f : X \to Y \) satisfies the functional equation (1.3) for all \( x_1, x_2, \ldots, x_n \in X \) if and only if \( C \) is cubic, where \( C : X \to Y \) is a function defined by \( C(x) = f(x) - f(0) \) for all \( x \in X \).

**Proof.** (Necessity.) Note that, by the assumption, we arrive at
\[
C \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + C \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n-1)}{2} \left[ C(x_1) + C(-x_1) \right]
\]
\[= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)] \tag{3.1}\]

for all \(x_1, x_2, \ldots, x_n \in X\). In particular, it is clear that \(C(0) = 0\). Substituting \(x_j = 0\) (\(j = 1, 2, \ldots, n - 1\)) and \(x_n = x\) in (3.1) yields

\[C(2x) + C(-2x) = 4(n - 1)[C(x) + C(-x)]. \tag{3.2}\]

Letting \(x_1 = x\), \(x_2 = -x\), and \(x_j = 0\) (\(j = 3, \ldots, n\)) in (3.1) gives the equation

\[C(2x) + C(-2x) = \frac{23 - 7n}{2}[C(x) + C(-x)]. \tag{3.3}\]

Now, by combining (3.2) and (3.3), we lead to

\[C(x) + C(-x) = 0\]

for all \(x \in X\), i.e., \(C\) is an odd function.

Hence (3.1) now becomes

\[C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j)\]

\[= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)]. \]

Thus [7, Lemma 2.2] implies that \(C\) is cubic.

(Sufficiency.) Suppose that \(C\) is cubic, i.e.,

\[C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{3.4}\]

for all \(x, y \in X\). Then it is easy to check that

\[C(0) = 0, \quad C(x) + C(-x) = 0 \quad \text{and} \quad C(2x) = 8C(x).\]

On the other hand, by [7, Lemma 2.2], we obtain

\[C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j)\]

\[= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)]. \]

Since \(C\) is an odd function, we note that

\[C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n - 1)}{2}[C(x_1) + C(-x_1)]\]

\[= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)], \]

which gives the functional equation (1.3) for all \(x_1, x_2, \ldots, x_n \in X\). This completes the proof of the lemma. \(\Box\)
Remark 3.3. Lemma 3.2 states that the functional equation (1.3) has a solution of the form $C(x) + B$, where $C$ is cubic and $B$ is a constant.

From now on, let $X$ be a real vector space and $Y$ be a real Banach space. As a matter of convenience, for a given mapping $f : X \to Y$, we use the following abbreviation:

$$D_2 f(x_1, x_2, \ldots, x_n) := 2 f \left( \sum_{j=1}^{n-1} x_j + 2x_n \right) + 2 f \left( \sum_{j=1}^{n-1} x_j - 2x_n \right) + 2 \sum_{j=1}^{n-1} f(2x_j)$$

$$+ 7(n-1) \left[ f(x_1) + f(-x_1) \right] - 4 f \left( \sum_{j=1}^{n-1} x_j \right)$$

$$- 8 \sum_{j=1}^{n-1} \left[ f(x_j + x_n) + f(x_j - x_n) \right]$$

for all $x_1, x_2, \ldots, x_n \in X$.

Let $\varphi : X^n \to [0, \infty)$ be a function satisfying

$$\lim_{k \to \infty} \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^{3k}} = 0$$

(3.5)

for all $x_1, x_2, \ldots, x_n \in X$, where

$$\begin{cases} \lambda_i = 2, & \text{if } i = 0, \\ \lambda_i = \frac{1}{2}, & \text{if } i = 1. \end{cases}$$

Now, by the use of fixed point alternative, we obtain the main result as follow.

Theorem 3.4. Let $n \geq 3$ be an integer. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|D_2 f(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n)$$

(3.6)

for all $x_1, x_2, \ldots, x_n \in X$. If there exists $L < 1$ such that the function

$$x \mapsto \psi(x) = \varphi \left( 0, \frac{x}{2}, \frac{x}{2}, \ldots, \frac{x}{2}, 0 \right)$$

has the property

$$\psi(x) \leq L \cdot \lambda_i^3 \cdot \psi \left( \frac{x}{\lambda_i} \right)$$

(3.7)

for all $x \in X$, then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\| f(x) - C(x) \| \leq \frac{1}{2(n-2)} \frac{L^{1-i}}{1 - L} \psi(x) + \| f(0) \|$$

(3.8)

for all $x \in X$. 
Proof. Consider the set
\[ \Omega := \{ g : g : X \to Y, \, g(0) = 0 \} \]
and introduce the generalized metric on \( \Omega \):
\[ d(g, h) = d_\psi(g, h) = \inf\{ K \in (0, \infty) : \| g(x) - h(x) \| \leq K \psi(x), \, x \in X \} \]
It is easy to see that \((\Omega, d)\) is complete.

Now we define a function \( T : \Omega \to \Omega \) by
\[ Tg(x) = \frac{1}{\lambda^3} g(\lambda x) \]
for all \( x \in X \). Note that for all \( g, h \in \Omega \),
\[ d(g, h) < K \Rightarrow \| g(x) - h(x) \| \leq K \psi(x), \, x \in X \]
\[ \Rightarrow \frac{1}{\lambda^3} \| g(\lambda x) - h(\lambda x) \| \leq \frac{1}{\lambda^3} K \psi(\lambda x), \, x \in X \]
\[ \Rightarrow \| \frac{1}{\lambda^3} g(\lambda x) - \frac{1}{\lambda^3} h(\lambda x) \| \leq L K \psi(x), \, x \in X \]
\[ \Rightarrow d(Tg, Th) \leq L K. \]

Hence we see that
\[ d(Tg, Th) \leq L d(g, h) \]
for all \( g, h \in \Omega \), i.e., \( T \) is a strictly contractive mapping of \( \Omega \) with the Lipschitz constant \( L \).

Here we define a function \( F : X \to Y \) by
\[ F(x) = f(x) - f(0) \]
for all \( x \in X \). Then we have \( F(0) = 0 \).

If we put \( x_1 = 0, x_2 = \ldots = x_{n-1} = y, x_n = 0 \) in (3.6) and use (3.7), then
\[ \| (n - 2) F(2y) - 8(n - 2) F(y) \|
= \| (n - 2) [f(2y) - f(0)] - 8(n - 2) [f(y) - f(0)] \|
\leq \frac{1}{2} \varphi(0, y, y, \ldots, y, 0), \]
(3.9)
which is reduced to
\[ \| F(y) - \frac{1}{2^3} F(2y) \| \leq \frac{1}{2^3} \frac{1}{2(n - 2)} \psi(2y) \leq \frac{L}{2(n - 2)} \psi(y) \]
for all \( y \in X \), i.e., \( d(F, TF) \leq \frac{L}{2(n - 2)} \leq \infty \).

If we substitute \( y := \frac{y}{2} \) in (3.9) and use (3.7), then
\[ \| F(y) - 2^3 F\left(\frac{y}{2}\right) \| \leq \frac{1}{2^3(n - 2)} \psi(y) \]
for all \( y \in X \), i.e., \( d(F, TF) \leq \frac{1}{2(n - 2)} < \infty \).
Now, from the fixed point alternative in both cases, it follows that there exists a fixed point \( C \) of \( T \) in \( \Omega \) such that
\[
C(x) = \lim_{k \to \infty} \frac{F(\lambda_i^k x)}{\lambda_i^{3k}}
\]
for all \( x \in X \), since \( \lim_{k \to \infty} d(T^k F, C) = 0 \).

To show that the function \( C : X \to Y \) is cubic, let \( x_j := \lambda_i^k x_j \) for \( j = 1, 2, \ldots, n \) in (3.6) and divide by \( \lambda_i^{3k} \). Then it follows from (3.5) and (3.10) that
\[
\| D_2 C(x_1, x_2, \ldots, x_n) \| = \lim_{k \to \infty} \frac{\| D_2 F(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n) \|}{\lambda_i^{3k}}
\leq \lim_{k \to \infty} \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^{3k}} = 0
\]
for all \( x_1, x_2, \ldots, x_n \in X \), i.e., \( C \) satisfies the functional equation (1.3). Therefore Lemma 3.2 guarantees that \( C \) is cubic, since \( C(0) = 0 \).

According to the fixed point alternative, since \( C \) is the unique fixed point of \( T \) in the set \( \Delta = \{ g \in \Omega : d(F, g) < \infty \} \), \( C \) is the unique function such that
\[
\| F(x) - C(x) \| \leq K \psi(x)
\]
for all \( x \in X \) and some \( K > 0 \). Again, using the fixed point alternative, we have
\[
d(F, C) \leq \frac{1}{1 - L} d(F, TF),
\]
and so we obtain the inequality
\[
d(F, C) \leq \frac{1}{2(n - 2)} L^{1-i},
\]
which yields the inequality (3.8). This completes the proof of the theorem. \( \square \)

From Theorem 3.4, we obtain the following corollary concerning the Hyers–Ulam–Rassias stability [24] of the functional equation (1.3).

**Corollary 3.5.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Let \( p \geq 0 \) be given with \( p \neq 3 \) and \( n \geq 3 \) an integer. Assume that \( \delta \geq 0 \) and \( \epsilon \geq 0 \) are fixed. Suppose that a function \( f : X \to Y \) satisfies the inequality
\[
\| D_2 f(x_1, x_2, \ldots, x_n) \| \leq \delta + \epsilon \left( \| x_1 \|^p + \| x_2 \|^p + \cdots + \| x_n \|^p \right)
\]
for all \( x_1, x_2, \ldots, x_n \in X \). Moreover, assume that \( \delta = 0 \) in (3.11) for the case \( p > 3 \). Then there exists a unique cubic function \( C : X \to Y \) satisfying

the inequality
\[
\| f(x) - C(x) \| \leq \frac{1}{2(n - 2)} \frac{\delta}{2^{3-p} - 1} + \frac{1}{2} \frac{\epsilon}{8 - 2p} \| x \|^p + \| f(0) \|
\]
which holds for all \( x \in X \), where \( p < 3 \),
or

the inequality

$$\| f(x) - C(x) \| \leq \frac{\varepsilon}{2} \| x \|^p + \| f(0) \|$$  \hspace{1cm} (3.13)

which holds for all \( x \in X \), where \( p > 3 \).

Proof. Let

$$\varphi(x_1, x_2, \ldots, x_n) := \delta + \varepsilon \left( \| x_1 \|^p + \| x_2 \|^p + \cdots + \| x_n \|^p \right)$$

for all \( x_1, x_2, \ldots, x_n \in X \). Then it follows that

$$\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n) = \frac{\delta}{\lambda_i^{3k}} + (\lambda_i^k)^{p-3} \varepsilon \left( \| x_1 \|^p + \| x_2 \|^p + \cdots + \| x_n \|^p \right) \to 0$$

as \( k \to \infty \), where

$$\begin{cases} p < 3, & \text{if } i = 0, \\ p > 3, & \text{if } i = 1, \end{cases}$$

i.e., (3.5) is true.

Since the inequality

$$\frac{1}{\lambda_i^3} \psi(\lambda_i x) = \frac{\delta}{\lambda_i^3} + \frac{\lambda_i^{p-3}}{2^p} (n - 2) \varepsilon \| x \|^p \leq \lambda_i^{p-3} \psi(x)$$

holds for all \( x \in X \), where

$$\begin{cases} p < 3, & \text{if } i = 0, \\ p > 3, & \text{if } i = 1, \end{cases}$$

we see that the inequality (3.7) holds with either \( L = 2^p \) or \( L = \frac{1}{2^p} \). Now the inequality (3.8) yields the inequalities (3.12) and (3.13), which complete the proof of the corollary. \( \Box \)

The following corollary is the Hyers–Ulam stability [12] of the functional equation (1.3).

Corollary 3.6. Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Assume that \( \theta \geq 0 \) is fixed and \( n \geq 3 \) an integer. Suppose that a function \( f : X \to Y \) satisfies the inequality

$$\| D_2 f(x_1, x_2, \ldots, x_n) \| \leq \theta$$  \hspace{1cm} (3.14)

for all \( x_1, x_2, \ldots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) satisfying the inequality

$$\| f(x) - C(x) \| \leq \frac{1}{14n} \theta + \| f(0) \|$$  \hspace{1cm} (3.15)

for all \( x \in X \).

Proof. Considering \( \delta := 0, p := 0 \) and \( \varepsilon := \frac{\theta}{n} \) in Corollary 3.5, we arrive at the conclusion of the corollary. \( \Box \)
Acknowledgments

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References