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Stability for the functional equation of cubic type

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Abstract

In this article, we establish the stability of the orthogonally cubic type functional equation (1.2) for all x_1, x_2, x_3 with $x_i \perp x_j$ $(i, j = 1, 2, 3)$, where $\perp$ is the orthogonality in the sense of Rätz, and investigate the stability of the n-dimensional cubic type functional equation (1.3), where $n \geqslant 3$ is an integer.

Keywords: Stability; Cubic functional equation; Orthogonally cubic functional equation

1. Introduction

In 1940, S.M. Ulam [32] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D.H. Hyers [12] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [24]. Since then, the stability problems of several

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functional equation have been extensively investigated by a number of authors (for instance, [1–6,9–11,13,15,17–19,22,23,25–28,30,31]).

In particular, one of the important functional equations studied is the following functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x)=ax^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1,8,16,20].

The Hyers–Ulam stability problem of the quadratic functional equation was first proved by F. Skof [30] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [6] and S. Czerwik [8].

The cubic function $f(x)=ax^3$ satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

Hence, throughout this paper, we promise that Eq. (1.1) is called a cubic functional equation and every solution of Eq. (1.1) is said to be a cubic function.

The functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [14]. In fact, they proved that a function $f: X \to Y$ between real vector spaces is a solution of the functional equation (1.1) if and only if there exists a function $G: X \times X \times X \to Y$ such that $f(x)=G(x,x,x)$ for all $x \in X$, and G is symmetric for each fixed one variable and additive for fixed two variables. The function G is given by

$$G(x,y,z)=\frac{1}{24}\big[f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)\big]$$

for all $x,y,z \in X$. Moreover, they investigated the Hyers–Ulam–Rassias stability for the functional equation (1.1).

Recently, Chang, Jun and Jung [4] introduced the cubic type functional equation as follows:

$$\begin{aligned} &f(x_1+x_2+2x_3)+f(x_1+x_2-2x_3)+f(2x_1)+f(2x_2)+7\big[f(x_1)+f(-x_1)\big] \\ &\quad =2f(x_1+x_2)+4\big[f(x_1+x_3)+f(x_1-x_3)+f(x_2+x_3)+f(x_2-x_3)\big]. \end{aligned} \tag{1.2}$$

It is easy to see that the function $f(x)=ax^3+b$ is a solution of the functional equation (1.2).

In this paper, we establish the stability of the orthogonally cubic type functional equation (1.2) for all x_1,x_2,x_3 with $x_i \perp x_j$ $(i,j=1,2,3)$, where $\perp$ is the orthogonality in the sense of Rätz. Furthermore, we will extend Eq. (1.2) to the n-dimensional cubic type functional equation

$$\begin{aligned} &2f\left(\sum_{j=1}^{n-1} x_j+2x_n\right)+2f\left(\sum_{j=1}^{n-1} x_j-2x_n\right)+2\sum_{j=1}^{n-1} f(2x_j)+7(n-1)\big[f(x_1)+f(-x_1)\big] \\ &\quad =4f\left(\sum_{j=1}^{n-1} x_j\right)+8\sum_{j=1}^{n-1}\big[f(x_j+x_n)+f(x_j-x_n)\big], \end{aligned} \tag{1.3}$$

where $n \geqslant 3$ is an integer, and offer the stability results for this equation.

2. Stability of Eq. (1.2)

Let us recall the orthogonality in the sense of J. Rätz [29].

Suppose that X is a real vector space with $\dim X \geqslant 2$ and $\perp$ is a binary relation on X with the following properties:

(1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(2) independence: if $x \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
(3) homogeneity: if $x \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Definition 2.1. Let X and Y be an orthogonality space and a real vector space. A mapping $f : X \to Y$ is said to orthogonally cubic if it satisfies the so-called orthogonally cubic functional equation (1.1) for all $x, y \in X$ with $x \perp y$.

Lemma 2.2. *Let X and Y be an orthogonality space and a real vector space, respectively. If a function $f : X \to Y$ satisfies the functional equation* (1.2) *for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$* $(i, j = 1, 2, 3)$, *then C is orthogonally cubic, where $C : X \to Y$ is a function defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Proof. From the assumption, it follows that

$$C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2) + 7\big[C(x_1) + C(-x_1)\big] = 2C(x_1 + x_2) + 4\big[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)\big] \tag{2.1}$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$. Particularly, it is obvious that $C(0) = 0$. Observe that $x \perp 0$ for all $x \in X$. Putting $x_1 = x_2 = 0$ in (2.1), we arrive at

$$C(2x_3) + C(-2x_3) = 8\big[C(x_3) + C(-x_3)\big]. \tag{2.2}$$

Letting $x_3 = 0$ in (2.1) gives the equation

$$C(2x_1) + C(2x_2) + 7\big[C(x_1) + C(-x_1)\big] = 8\big[C(x_1) + C(x_2)\big]. \tag{2.3}$$

If we put $x_2 = 0$ in (2.3), then we conclude that

$$C(2x_1) = C(x_1) - 7C(-x_1). \tag{2.4}$$

Let us replace x_1 by $-x_1$ in (2.4), then we get

$$C(-2x_1) = C(-x_1) - 7C(x_1). \tag{2.5}$$

By adding (2.4) and (2.5), we find that

$$C(2x_1) + C(-2x_1) = -6\big[C(x_1) + C(-x_1)\big]$$

and by comparing with (2.2), it follows that

$$C(x_1) + C(-x_1) = 0. \tag{2.6}$$

Therefore (2.1) now becomes

$$C(x_1 + x_2 + 2x_3) + C(x_1 + x_2 - 2x_3) + C(2x_1) + C(2x_2) = 2C(x_1 + x_2) + 4\big[C(x_1 + x_3) + C(x_1 - x_3) + C(x_2 + x_3) + C(x_2 - x_3)\big] \tag{2.7}$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$.

We take $x_1 = 0$ in (2.7) and then use (2.6) to obtain

$$C(x_2 + 2x_3) + C(x_2 - 2x_3) + C(2x_2) = 2C(x_2) + 4\big[C(x_2 + x_3) + C(x_2 - x_3)\big]. \tag{2.8}$$

Setting $x_3 = 0$ in (2.8) leads to the identity $C(2x_2) = 8C(x_2)$. If $x_3 \perp x_2$, then $x_3 \perp 2x_2$, $2x_3 \perp x_2$ and $2x_3 \perp 2x_2$. By replacing x_2 by $2x_2$ in (2.8), we see that

$$8C(x_2 + x_3) + 8C(x_2 - x_3) + 64C(x_2) = 16C(x_2) + 4\big[C(2x_2 + x_3) + C(2x_2 - x_3)\big]$$

for all $x_2, x_3 \in X$ with $x_2 \perp x_3$, which means that C is orthogonally cubic. The proof of lemma is complete. □

From now forward, let X be an orthogonality normed space and Y be a Banach space. Given a mapping $f : X \to Y$, we set

$$\begin{aligned} D_1 f(x_1, x_2, x_3) := {} & f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2) \\ & + 7\big[f(x_1) + f(-x_1)\big] - 2f(x_1 + x_2) \\ & - 4\big[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\big] \end{aligned}$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$.

Theorem 2.3. *Suppose that $f : X \to Y$ is a mapping for which there exists a function $\phi : X^3 \to [0, \infty)$ such that*

$$\sum_{i=0}^{\infty} \frac{1}{2^{3i}} \phi\big(0, 2^i x_2, 0\big) < \infty, \tag{2.9}$$

$$\lim_{n\to\infty} \frac{1}{2^{3i}} \phi\big(2^i x_1, 2^i x_2, 2^i x_3\big) = 0 \tag{2.10}$$

and

$$\big\| D_1 f(x_1, x_2, x_3)\big\| \leqslant \delta + \phi(x_1, x_2, x_3) \tag{2.11}$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$, where $\delta \geqslant 0$. Then there exists a unique orthogonally cubic function $C : X \to Y$ satisfying the inequality

$$\big\| f(x) - C(x)\big\| \leqslant \frac{1}{8}\Bigg[\sum_{i=0}^{\infty} \frac{1}{2^{3i}}\big(\delta + \phi\big(0, 2^i x, 0\big)\big)\Bigg] + \big\|f(0)\big\| \tag{2.12}$$

for all $x \in X$.

Proof. Let F be a function on X defined by

$$F(x) = f(x) - f(0)$$

for all $x \in X$. Then we have $F(0) = 0$. Note that $x \perp 0$ for all $x \in X$. Putting $x_1 = x_3 = 0$, $x_2 = x$ in (2.11) and dividing by 8, we have

$$\bigg\| F(x) - \frac{1}{8}F(2x)\bigg\| \leqslant \frac{1}{8}\big[\delta + \phi(0, x, 0)\big]. \tag{2.13}$$

By replacing x by $2x$ in (2.13) and dividing by 8 and summing the resulting inequality with (2.13), we get

$$\left\| F(x) - \left(\frac{1}{8}\right)^2 F\left(2^2 x\right) \right\| \leqslant \frac{1}{8}\left[\delta + \phi(0, x, 0)\right] + \left(\frac{1}{8}\right)^2 \left[\delta + \phi(0, 2x, 0)\right]. \tag{2.14}$$

An induction implies that

$$\left\| F(x) - \left(\frac{1}{8}\right)^n F\left(2^n x\right) \right\| \leqslant \frac{1}{8} \sum_{i=0}^{n-1} \left(\frac{1}{8}\right)^i \left[\delta + \phi\left(0, 2^i x, 0\right)\right]. \tag{2.15}$$

In order to prove convergence of the sequence $\{\frac{F(2^n x)}{2^{3n}}\}$, we divide inequality (2.15) by 8^m and also replace x by $2^m x$ to find that for $n > m > 0$,

$$\begin{aligned} &\left\| \left(\frac{1}{8}\right)^m F\left(2^m x\right) - \left(\frac{1}{8}\right)^{n+m} F\left(2^n 2^m x\right) \right\| \\ &\quad = \left(\frac{1}{8}\right)^m \left\| F\left(2^m x\right) - \left(\frac{1}{8}\right)^n F\left(2^n 2^m x\right) \right\| \\ &\quad \leqslant \left(\frac{1}{8}\right)^{m+1} \sum_{i=0}^{n-1} \left(\frac{1}{8}\right)^i \left[\delta + \phi\left(0, 2^{m+i} x, 0\right)\right]. \end{aligned} \tag{2.16}$$

Sine the right-hand side of the inequality tends to 0 as $m \to \infty$, $\{\frac{F(2^n x)}{2^{3n}}\}$ is Cauchy sequence. Therefore, we may define a function $C : X \to Y$ by

$$C(x) := \lim_{n\to\infty} \frac{F(2^n x)}{2^{3n}}$$

for all $x \in X$. By letting $n \to \infty$ in (2.15), we arrive at the formula (2.12).

Now we show that C satisfies the functional equation (1.2) for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$: If $x_i \perp x_j$, then $2^n x_i \perp 2^n x_j$ for $i, j = 1, 2, 3$. Let us replace x_1, x_2 and x_3 by $2^n x_1, 2^n x_2$ and $2^n x_3$ in (2.11) and divide by 8^n. Then it follows that

$$\begin{aligned} D_1 C(x_1, x_2, x_3) &= \lim_{n\to\infty} \frac{1}{2^{3n}} \left\| D_1 F\left(2^n x_1, 2^n x_2, 2^n x_3\right) \right\| \\ &\leqslant \lim_{n\to\infty} \frac{1}{2^{3n}} \left[\delta + \phi\left(2^n x_1, 2^n x_2, 2^n x_3\right)\right] = 0. \end{aligned}$$

Hence we obtain the desired result. Since $C(0) = 0$, Lemma 2.2 implies that C is an orthogonally cubic.

It only remains to claim that C is unique: Let us assume that there exists an orthogonally cubic function C' which satisfies (1.2) and the inequality (2.12). It is clear that $C(2^n x) = 8^n C(x)$ and $C'(2^n x) = 8^n C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (2.12) that

$$\begin{aligned} \left\| C(x) - C'(x) \right\| &= \left(\frac{1}{8}\right)^n \left\| C\left(2^n x\right) - C'\left(2^n x\right) \right\| \\ &\leqslant \left(\frac{1}{8}\right)^n \left[\left\| C\left(2^n x\right) - f\left(2^n x\right) \right\| + \left\| f\left(2^n x\right) - C'\left(2^n x\right) \right\|\right] \\ &\leqslant \left(\frac{1}{8}\right)^n \left\{ \frac{1}{4}\left[\sum_{i=0}^{\infty} \frac{1}{2^{3i}} \left(\delta + \phi\left(0, 2^i x, 0\right)\right) \right] + 2\left\| f(0) \right\| \right\}. \end{aligned}$$

By letting $n \to \infty$, we have $C(x) = C'(x)$ for all $x \in X$, which completes the proof of the theorem. □

Corollary 2.4. *Let p, q (< 3), r, δ, ε_1, ε_2 and ε_3 be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\| D_1 f(x_1, x_2, x_3) \right\| \leqslant \delta + \varepsilon_1 \|x_1\|^p + \varepsilon_2 \|x_2\|^q + \varepsilon_3 \|x_3\|^r$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ $(i, j = 1, 2, 3)$. Then there exists a unique orthogonally cubic function $C : X \to Y$ satisfying the inequality

$$\left\| f(x) - C(x) \right\| \leqslant \frac{1}{7}\delta + \frac{1}{8 - 2^q}\varepsilon_2 \|x\|^q + \left\| f(0) \right\|$$

for all $x \in X$.

Proof. In Theorem 2.3, if we consider that

$$\phi(x_1, x_2, x_3) = \varepsilon_1 \|x_1\|^p + \varepsilon_2 \|x_2\|^q + \varepsilon_3 \|x_3\|^r,$$

then we arrive at the conclusion of the corollary. □

3. Stability of Eq. (1.3)

For explicitly later use, we demonstrate the following theorem:

Theorem 3.1 *(The alternative of fixed point).* [21] *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d\big(T^n x, T^{n+1} x\big) = \infty \quad \text{for all } n \geqslant 0,$$

or

there exists a natural number n_0 such that
- *$d(T^n x, T^{n+1} x) < \infty$ for all $n \geqslant n_0$;*
- *the sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
- *y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega\colon\ d(T^{n_0} x, y) < \infty\}$;*
- *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

For completeness, we will first present solution of the functional equation (1.3).

Lemma 3.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.3) *for all $x_1, x_2, \ldots, x_n \in X$ if and only if C is cubic, where $C : X \to Y$ is a function defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Proof. (*Necessity.*) Note that, by the assumption, we arrive at

$$C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n-1)}{2}\big[C(x_1) + C(-x_1)\big]$$

$$= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1}\big[C(x_j + x_n) + C(x_j - x_n)\big] \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n \in X$. In particular, it is clear that $C(0) = 0$. Substituting $x_j = 0$ ($j = 1, 2, \ldots, n-1$) and $x_n = x$ in (3.1) yields

$$C(2x) + C(-2x) = 4(n-1)\big[C(x) + C(-x)\big]. \tag{3.2}$$

Letting $x_1 = x$, $x_2 = -x$, and $x_j = 0$ ($j = 3, \ldots, n$) in (3.1) gives the equation

$$C(2x) + C(-2x) = \frac{23 - 7n}{2}\big[C(x) + C(-x)\big]. \tag{3.3}$$

Now, by combining (3.2) and (3.3), we lead to

$$C(x) + C(-x) = 0$$

for all $x \in X$, i.e., C is an odd function.

Hence (3.1) now becomes

$$C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j)$$
$$= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1}\big[C(x_j + x_n) + C(x_j - x_n)\big].$$

Thus [7, Lemma 2.2] implies that C is cubic.

(*Sufficiency.*) Suppose that C is cubic, i.e.,

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{3.4}$$

for all $x, y \in X$. Then it is easy to check that

$$C(0) = 0, \quad C(x) + C(-x) = 0 \quad \text{and} \quad C(2x) = 8C(x).$$

On the other hand, by [7, Lemma 2.2], we obtain

$$C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j)$$
$$= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1}\big[C(x_j + x_n) + C(x_j - x_n)\big].$$

Since C is an odd function, we note that

$$C\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + C\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} C(2x_j) + \frac{7(n-1)}{2}\big[C(x_1) + C(-x_1)\big]$$
$$= 2C\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1}\big[C(x_j + x_n) + C(x_j - x_n)\big],$$

which gives the functional equation (1.3) for all $x_1, x_2, \ldots, x_n \in X$. This completes the proof of the lemma. □

Remark 3.3. Lemma 3.2 states that the functional equation (1.3) has a solution of the form $C(x) + B$, where C is cubic and B is a constant.

From now on, let X be a real vector space and Y be a real Banach space. As a matter of convenience, for a given mapping $f : X \to Y$, we use the following abbreviation:

$$\begin{aligned} D_2 f(x_1, x_2, \dots, x_n) := {} & 2f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + 2f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + 2\sum_{j=1}^{n-1} f(2x_j) \\ & + 7(n-1)\big[f(x_1) + f(-x_1)\big] - 4f\left(\sum_{j=1}^{n-1} x_j\right) \\ & - 8\sum_{j=1}^{n-1}\big[f(x_j + x_n) + f(x_j - x_n)\big] \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$.

Let $\varphi : X^n \to [0, \infty)$ be a function satisfying

$$\lim_{k\to\infty} \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \dots, \lambda_i^k x_n)}{\lambda_i^{3k}} = 0 \tag{3.5}$$

for all $x_1, x_2, \dots, x_n \in X$, where

$$\begin{cases} \lambda_i = 2, & \text{if } i = 0, \\ \lambda_i = \dfrac{1}{2}, & \text{if } i = 1. \end{cases}$$

Now, by the use of fixed point alternative, we obtain the main result as follow.

Theorem 3.4. *Let $n \geqslant 3$ be an integer. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\left\| D_2 f(x_1, x_2, \dots, x_n) \right\| \leqslant \varphi(x_1, x_2, \dots, x_n) \tag{3.6}$$

for all $x_1, x_2, \dots, x_n \in X$. If there exists $L < 1$ such that the function

$$x \mapsto \psi(x) = \varphi\Bigg(0, \underbrace{\frac{x}{2}, \frac{x}{2}, \dots, \frac{x}{2}}_{n-2}, 0\Bigg)$$

has the property

$$\psi(x) \leqslant L \cdot \lambda_i^3 \cdot \psi\left(\frac{x}{\lambda_i}\right) \tag{3.7}$$

for all $x \in X$, then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\left\| f(x) - C(x) \right\| \leqslant \frac{1}{2(n-2)} \frac{L^{1-i}}{1-L} \psi(x) + \left\| f(0) \right\| \tag{3.8}$$

for all $x \in X$.

Proof. Consider the set

$$\Omega := \{g\colon\ g : X \to Y,\ g(0) = 0\}$$

and introduce the generalized metric on Ω:

$$d(g,h) = d_\psi(g,h) = \inf\{K \in (0,\infty)\colon\ \|g(x) - h(x)\| \leqslant K\psi(x),\ x \in X\}.$$

It is easy to see that (Ω, d) is complete.

Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^3} g(\lambda_i x)$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned}
d(g,h) < K \quad &\Longrightarrow \quad \|g(x) - h(x)\| \leqslant K\psi(x), \quad x \in X \\
&\Longrightarrow \quad \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leqslant \frac{1}{\lambda_i^3} K\psi(\lambda_i x), \quad x \in X \\
&\Longrightarrow \quad \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leqslant LK\psi(x), \quad x \in X \\
&\Longrightarrow \quad d(Tg, Th) \leqslant LK.
\end{aligned}$$

Hence we see that

$$d(Tg, Th) \leqslant Ld(g,h)$$

for all $g, h \in \Omega$, i.e., T is a strictly contractive mapping of Ω with the Lipschitz constant L.

Here we define a function $F : X \to Y$ by

$$F(x) = f(x) - f(0)$$

for all $x \in X$. Then we have $F(0) = 0$.

If we put $x_1 = 0$, $x_2 = \cdots = x_{n-1} = y$, $x_n = 0$ in (3.6) and use (3.7), then

$$\begin{aligned}
&\|(n-2)F(2y) - 8(n-2)F(y)\| \\
&\quad = \|(n-2)[f(2y) - f(0)] - 8(n-2)[f(y) - f(0)]\| \\
&\quad \leqslant \frac{1}{2}\varphi(0, \underbrace{y, y, \ldots, y}_{n-2}, 0),
\end{aligned} \tag{3.9}$$

which is reduced to

$$\left\| F(y) - \frac{1}{2^3} F(2y) \right\| \leqslant \frac{1}{2^3} \frac{1}{2(n-2)} \psi(2y) \leqslant \frac{L}{2(n-2)} \psi(y)$$

for all $y \in X$, i.e., $d(F, TF) \leqslant \frac{L}{2(n-2)} \leqslant \infty$.

If we substitute $y := \frac{y}{2}$ in (3.9) and use (3.7), then

$$\left\| F(y) - 2^3 F\left(\frac{y}{2}\right) \right\| \leqslant \frac{1}{2(n-2)} \psi(y)$$

for all $y \in X$, i.e., $d(F, TF) \leqslant \frac{1}{2(n-2)} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x) = \lim_{k\to\infty} \frac{F(\lambda_i^k x)}{\lambda_i^{3k}} \tag{3.10}$$

for all $x \in X$, since $\lim_{k\to\infty} d(T^k F, C) = 0$.

To show that the function $C : X \to Y$ is cubic, let $x_j := \lambda_i^k x_j$ for $j = 1, 2, \ldots, n$ in (3.6) and divide by λ_i^{3k}. Then it follows from (3.5) and (3.10) that

$$\begin{aligned}\left\| D_2C(x_1, x_2, \ldots, x_n)\right\| &= \lim_{k\to\infty} \frac{\| D_2F(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)\|}{\lambda_i^{3k}} \\ &\leqslant \lim_{k\to\infty} \frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^{3k}} = 0\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in X$, i.e., C satisfies the functional equation (1.3). Therefore Lemma 3.2 guarantees that C is cubic, since $C(0) = 0$.

According to the fixed point alternative, since C is the *unique* fixed point of T in the set $\Delta = \{g \in \Omega \colon d(F, g) < \infty\}$, C is the unique function such that

$$\left\| F(x) - C(x)\right\| \leqslant K\psi(x)$$

for all $x \in X$ and some $K > 0$. Again, using the fixed point alternative, we have

$$d(F, C) \leqslant \frac{1}{1-L} d(F, TF),$$

and so we obtain the inequality

$$d(F, C) \leqslant \frac{1}{2(n-2)} \frac{L^{1-i}}{1-L},$$

which yields the inequality (3.8). This completes the proof of the theorem. □

From Theorem 3.4, we obtain the following corollary concerning the Hyers–Ulam–Rassias stability [24] of the functional equation (1.3).

Corollary 3.5. *Let X and Y be a normed space and a Banach space, respectively. Let $p \geqslant 0$ be given with $p \neq 3$ and $n \geqslant 3$ an integer. Assume that $\delta \geqslant 0$ and $\varepsilon \geqslant 0$ are fixed. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\left\| D_2 f(x_1, x_2, \ldots, x_n)\right\| \leqslant \delta + \varepsilon\big(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\big) \tag{3.11}$$

for all $x_1, x_2, \ldots, x_n \in X$. Moreover, assume that $\delta = 0$ in (3.11) for the case $p > 3$. Then there exists a unique cubic function $C : X \to Y$ satisfying

the inequality

$$\left\| f(x) - C(x)\right\| \leqslant \frac{1}{2(n-2)} \frac{\delta}{2^{3-p} - 1} + \frac{1}{2} \frac{\varepsilon}{8 - 2^p} \|x\|^p + \left\| f(0)\right\| \tag{3.12}$$

which holds for all $x \in X$, where $p < 3$, *or*

the inequality

$$\left\| f(x) - C(x) \right\| \leqslant \frac{1}{2} \frac{\varepsilon}{2^p - 8} \|x\|^p + \left\| f(0) \right\| \tag{3.13}$$

which holds for all $x \in X$, where $p > 3$.

Proof. Let

$$\varphi(x_1, x_2, \ldots, x_n) := \delta + \varepsilon\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right)$$

for all $x_1, x_2, \ldots, x_n \in X$. Then it follows that

$$\frac{\varphi(\lambda_i^k x_1, \lambda_i^k x_2, \ldots, \lambda_i^k x_n)}{\lambda_i^{3k}} = \frac{\delta}{\lambda_i^{3k}} + \left(\lambda_i^k\right)^{p-3} \varepsilon\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right) \to 0$$

as $k \to \infty$, where

$$\begin{cases} p < 3, & \text{if } i = 0, \\ p > 3, & \text{if } i = 1, \end{cases}$$

i.e., (3.5) is true.

Since the inequality

$$\frac{1}{\lambda_i^3} \psi(\lambda_i x) = \frac{\delta}{\lambda_i^3} + \frac{\lambda_i^{p-3}}{2^p} (n-2)\varepsilon \|x\|^p \leqslant \lambda_i^{p-3} \psi(x)$$

holds for all $x \in X$, where

$$\begin{cases} p < 3, & \text{if } i = 0, \\ p > 3, & \text{if } i = 1, \end{cases}$$

we see that the inequality (3.7) holds with either $L = 2^{p-3}$ or $L = \frac{1}{2^{p-3}}$. Now the inequality (3.8) yields the inequalities (3.12) and (3.13), which complete the proof of the corollary. □

The following corollary is the Hyers–Ulam stability [12] of the functional equation (1.3).

Corollary 3.6. *Let X and Y be a normed space and a Banach space, respectively. Assume that $\theta \geqslant 0$ is fixed and $n \geqslant 3$ an integer. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\left\| D_2 f(x_1, x_2, \ldots, x_n) \right\| \leqslant \theta \tag{3.14}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\left\| f(x) - C(x) \right\| \leqslant \frac{1}{14n} \theta + \left\| f(0) \right\| \tag{3.15}$$

for all $x \in X$.

Proof. Considering $\delta := 0$, $p := 0$ and $\varepsilon := \frac{\theta}{n}$ in Corollary 3.5, we arrive at the conclusion of the corollary. □

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