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On Infinite-Dimensional Convex Programs

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ABSTRACT

One way to approach infinite-dimensional nonlinear programs is to append increasingly large cost or penalty terms to the objective function in such a way that the minima of the augmented but unconstrained functions converge to the constrained minimum in the limit. In this paper we establish the convergence of the penalty argument on reflexive B-spaces, and then apply it to obtain the Kuhn-Tucker necessary conditions for convex programs in Hilbert space. The proof is constructive and suggests a computationally feasible algorithm for solving such programs.

In this note we look at an iterative method for solving nonlinear programming problems based on the penalty function concept of Courant [*1*]. The idea is to append increasingly large cost or penalty terms to the objective function in such a way that the resulting minima of the augmented, but otherwise unconstrained, functions converge to the constrained solution in the limit. In this way a constrained minimization problem is reduced to solving a sequence of unconstrained ones. A similar technique was made the basis of a computational algorithm for convex programs on E^n by Fiacco-McCormick but under different hypothesis [*2*]. The method has the virtue that necessary conditions in mathematical programming can be obtained in a constructive (i.e., computationally feasible) manner by passing to the limit with the necessary conditions for each unconstrained problem as the penalty term increases. Several specific examples on E^n have been computed by this approach and the algorithm appears to be quite effective [*3*].

In this first section we establish existence and necessary conditions for convex programs on a Hilbert space by appealing to the behavior of convex functionals on closed convex sets in reflexive B-spaces. However, there is no pretense that our argument leads to the best statement of the necessary conditions (the paper by Halkin-Neustadt [*4*] gives a more general treatment, for example). Instead we simply wish to indicate a concrete approach to the problem of necessaey conditions within the larger framework of actually constructing optimal programs. In the last section, in fact, we show how the penalty argument, together with the classical Ritz method, allows us to reduce an infinite-dimensional program to one of solving a sequence of unconstrained finite-dimensional programs. Thus the penalty idea is at once an iterative method for proving existence, a constructuve device for establishing necessary conditions, and a specific technique for computing solutions.

A Multiplier Rule

The mathematical program to be discussed is that of minimizing a convex functional f subject to the m constraints $g_j = 0$, $j \leqslant r$, and $g_j \leqslant 0$, $r < j \leqslant m$, where $0 \leqslant r \leqslant m$ and f, g_j are mappings of a Hilbert space $\mathscr{H}$ to E^1.

As a matter of convenience all constraints will be written as equality constraints. To do this in the case $0 \leqslant r < j \leqslant m$ we define $H_j \in \mathscr{L}(\mathscr{H}, E^1)$ by $H_j = 1$ whenever $g_j(x) > 0$ and by $H_j = 0$ otherwise, and then note that $H_j g_j = 0$ if and only if $g_j \leqslant 0$; when $j \leqslant r$ let $H_j \equiv 1$. Thus we have m constraints $H_j g_j = 0$ or, equivalently, $(H_j g_j)^2 = 0$. Now let K an $m \times m$ diagonal matrix with positive entries k_j. Then $K_n \to \infty$ means that all entries $k_{n,j}$ in the sequence K_n increase without bound as $n \to \infty$. The quadratic form $\sum_{j \leqslant m} k_{n,j}(H_j g_j)^2$ will be denoted by $(Hg, K_n Hg)$ with Hg a mapping of $\mathscr{H}$ to E^m. In Theorem 1 below we assume that the $g_j{}^2$ are lower semicontinuous (l.s.c.) and convex. This implies the l.s.c. and convexity of $(H_j g_j)^2$ and hence of $(Hg, K_n Hg)$ as well.

We begin by quoting without proof two lemmas that will be needed in the sequel. For a proof see, for example, [*5*], pp. 125-6. Then two additional lemmas are proven by combining known results in functional analysis. They are followed by the first theorem, strengthening a result of Butler-Martin [*6*] which, in turn, is based on the argument given in the Courant notes [*7*]. This theorem establishes the validity of the penalty argument. In the proofs we deal with a reflexive Banach space B and by weak convengence to zero of $\{x_n\}$ in B is meant that $f(x_n) \to 0$ for all f in the dual B'.

Lemma 1. *Every bounded sequence in B has a weakly convengent subsequence.*

Lemma 2. *A closed convex set in B is weakly closed.*

Lemma 3. *If f is a convex and l.s.c. functional on B then f is weakly l.s.c.*

Proof. Let $S(\alpha) = \{x \mid f(x) \leqslant \alpha\}$. It is not hard to see that f is l.s.c. if and only if S is closed for all α (see, for example, [*15*], p. 40). Since f is convex, l.s.c. then S is convex, closed for all α and so, by Lemma 2, weakly closed. Hence f is weakly l.s.c.

Lemma 4. *Let f be convex and l.s.c. on a closed convex subset Ω of B. If $f \to +\infty$ as $\| x \| \to \infty$, then it attains a minimum on Ω.*

Proof. Let $d = \inf_{x \in \Omega} f$ and let $\{x_n\}$ be a minimizing sequence in Ω. Since $f \to +\infty$ as $\| x \| \to \infty$ the set $\{x_n\}$ is bounded. Hence, by Lemma 1, there exists a subsequence

$\{x_{n_k}\}$ which tends weakly to some $x_0 \in B$. By Lemma 2, $x_0 \in \Omega$ and finally, by Lemma 3, f is weakly l.s.c., so that $d = \underline{\lim} f(x_{n_k}) \geqslant f(x_0) \geqslant d$, which shows that $f(x_0) = d$.

THEOREM 1. *Let f, g_j^2, $j = 1,..., m$, be convex and l.s.c. on a closed convex set Ω_0 in B and suppose $f \to -\infty$ as $\|x\| \to \infty$. Also let us suppose that $\Omega_1 = \{x \mid H_j g_j = 0\}$ has a nonempty intersection with Ω_0.*

Then for every sequence of matrices $K_n \to \infty$ as $n \to \infty$ there exists a corresponding sequence $x_n \in \Omega_0$ with the property that

$$\min_{x \in \Omega_0} [f + \tfrac{1}{2}(Hg, K_n Hg)]$$

is attained by x_n and such that, for some subsequence x_{n_k}, x_{n_k} tends weakly to $x_0 \in \Omega_0 \cap \Omega_1$, where $f(x_0) = \inf_{x \in \Omega_0} f$ (i.e., x_0 is a minimum for f subject to the constraints).

Proof. First note that $f_n \to +\infty$ as $\|x\| \to \infty$ so that from Lemma 4 we have that f_n takes on a minimum d_n at some $x_n \in \Omega_0$. Since $f_n \to +\infty$ it also follows that $\{x_n\}$ is a bounded set in Ω_0 and so for some subsequence, also denoted by x_n, we have that $x_n \to x_0 \in \Omega_0$ weakly as $n \to \infty$ (Lemma 1). It remains to show that $f(x_0) = d$. Now $f = f_n \geqslant d_n$ for all $x\Omega_0 \cap \Omega_1$ and so $d_n \leqslant d$. Thus $d \geqslant \underline{\lim} f(x_n) \geqslant f(x_0)$ and, in fact, $f(x_n) \geqslant f(x_0) - 1$ for all n greater than some N since, by Lemma 3, f is weakly l.s.c. Hence

$$f(x_0) - 1 + \tfrac{1}{2}(Hg(x_n), K_n Hg(x_n)) \leqslant d \qquad \text{or} \qquad \tfrac{1}{2}(Hg(x_n), K_n Hg(x_n)) \leqslant \text{constant}$$

for all $n \geqslant N$. Since $K_n \to \infty$ as $n \to \infty$ it follows that $(H_j g_j(x_n))^2 \to 0$ for each j. But then

$$0 = \underline{\lim}\, (H_j g_j(x_n))^2 \geqslant (H_j g_j(x_0))^2 \geqslant 0$$

and so $x_0 \in \Omega_0 \cap \Omega_1 \subset \Omega_0$. Together with the fact that $f(x_0) \leqslant d$ it now follows that $f(x_0) = d$, which proves the theorem.

The content of the above Theorem is that the minima of the unconstrained functions f_n tend weakly to the minimum of the constrained f as $n \to \infty$ since $(Hg, K_n Hg)$ necessarily tends to zero as the cost of violating the constraints increases without bound. For this reason K is called a matrix of penalty constants. Note that the theorem also establishes the existence of a minimum for the constrained function since, as we saw, $f(x_0) = \inf f$ on $\Omega_0 \cap \Omega_1$. In the proof we assumed Ω_0 is closed, convex. Actually, $\Omega_0 \cap \Omega_1$ is also closed, convex as we now show; this allows us to infer the existence of a minimum of f on $\Omega_0 \cap \Omega_1$ directly from Lemma 4. Let

$$\Omega_j = \{x \mid (H_j g_j)^2 = 0\} = \{x \mid (H_j g_j)^2 \leqslant 0\};$$

for $r < j \leqslant m$ $\Omega_j = \{x \mid g_j \leqslant 0\}$. Since $(H_j g_j)^2$ is convex it follows that the sets Ω_j are also convex for all $j = 1,..., m$. But then so is $\Omega_1 = \cap \Omega_j$. Also if $z_k \in \Omega_j$ and if $z_k \to z$ then

$$0 = \underline{\lim}_{\inf} H_j^2 g_j^2(z_k) \geqslant H_j^2 g_j^2(z) \geqslant 0$$

since $(H_j g_j)^2$ is l.s.c. and so $z \in \Omega_j$. Hence Ω_1 is closed which shows that $\Omega_0 \cap \Omega_1$ is closed, convex. Finally, note that if Ω_0 is bounded one does not need the hypothesis that $f \to +\infty$ as $\| x \| \to \infty$.

The next theorem establishes a necessary condition for optimality in terms of a multiplier rule. To simplify notation and reduce wordiness in the proof we restrict ourselves to inequality constraints below (i.e., with $r = 0$), but essentially the same reasoning yields a theorem even when $r > 0$ except that the resulting multipliers will be unrestricted in sign for $j \leqslant r$. Also B is now specialized to a Hilbert space $\mathscr{H}$.

We recall that a continuous functional f is differentiable on an open set Ω' of $\mathscr{H}$ if for every $x \in \Omega'$ there exists a continuous linear mapping $f'(x) \in \mathscr{H}^*$, the conjugate space of $\mathscr{H}$, called the Frechet derivative of f such that

$$\frac{|f(x+h) - f(x) - (f'(x), h)|}{\| h \|} \to 0 \qquad \text{as} \qquad \| h \| \to 0, \qquad h \in \mathscr{H}$$

(see Dieudonne [8], p. 143). Thus $f'(x)$ or simply f' is a mapping of Ω' into $\mathscr{H}^*$ and if f' is weakly continuous on Ω' we say that f has a weakly continuous derivative on Ω'. It follows immediately that the existence of such a derivative for g_j implies that g_j^2 is l.s.c.

THEOREM 2. *Let x_0 be given as a local minimum of f on $\Omega' \cap \Omega_1 \neq \emptyset$ where Ω' is some open subset of $\mathscr{H}$ and suppose f, g_j have weakly continuous derivatives on Ω'. Assume also that f is strictly convex and the g_j^2 convex on Ω'. Moreover, suppose that $(g_j'(x_0), h) < 0$ for some $h \in \mathscr{H}$, $j \in J = \{j \mid g_j(x_0) = 0\}$. Then there exists $\lambda_j \geqslant 0$ such that*

$$f' = -\sum_{j=1}^{m} \lambda_j g_j' \tag{1}$$

at x_0 and such that $\lambda_j = 0$ whenever $j \notin J$.

Proof. Let $\Omega_0 \subset \Omega'$ be a closed ball about x_0. Then Ω_0 is closed, convex and bounded and we identify Ω_0, Ω_1 with the corresponding sets in Theorem 1. Moreover x_0 is a unique global minimum on $\Omega_0 \cap \Omega_1$ since f is strictly convex.

Let $f_n = f + \frac{1}{2}(Hg, K_n Hg)$ and note that if f_n attains its minimum at $x_n \in \Omega_0$ then, by Theorem 1, there exists a subsequence, also denoted by $\{x_n\}$, such that x_n tends weakly to some x in $\Omega_0 \cap \Omega_1$ for which $f(x) = d$. Since the minimum of f is unique, $x = x_0$. For n large enough, x_n is in the interior of Ω_0 and since f_n is unconstrained we have

$$f_n' = f' + \sum_{j \leqslant m} k_{n,j} H_j g_j g_j' = 0 \tag{2}$$

at x_n (see [8], p. 145). Also, x_n does not belong to the interior of $\Omega_J = \{x \mid g_j \leqslant 0, j \in J\}$ for all sufficiently large n. To see this, note that x_0 is on the boundary of Ω_J and since $f_n = f$ on Ω_J this implies $x_n = x_0$ whenever $x_n \in \Omega_J$. By the same reasoning, x_n belongs to the interior of $\Omega_1 - \Omega_J = \{x \mid g_j \leqslant 0, j \notin J\}$. Thus, for some N, we have $0 \leqslant k_{n,j} g_j(x_n)$ for all $n \geqslant N$ and $j \in J$ and $k_{n,j} H_j g_j(x_n) = 0$ for $n \geqslant N$ and $j \notin J$. Since $(g_j'(x_0), h) < 0$ for $j \in J$ and h some fixed element of $\mathscr{H}$, then $(g_j'(x_n), h) \leqslant -\delta$ when n is sufficiently large and δ is sufficiently small since g_j' is weakly continuous and $x_n \to x_0$ weakly. Moreover, using Lemma 1 one can show that $(f'(x), h)$ is bounded on Ω_0 for each h since f' is also weakly continuous. Hence from Eq. (2) we obtain

$$M \geqslant (f'(x_n), h) = -\sum_{j \in J} k_{n,j} g_j(x_n)\,(g_j'(x_n), h) \leqslant \delta \sum k_{n,j} g_j(x_n) \geqslant \delta k_{n,j} g_j(x_n)$$

for all sufficiently large n. But then $k_{n,j} g_j(x_n) \leqslant$ constant, $n \geqslant N$. Thus one can choose a suitable subsequence $\{x_{n_\nu}\}$ so that $k_{n_\nu,j} g_j(x_{n_\nu}) \to \lambda_j \geqslant 0$ as $\nu \to \infty$. Since f', g_j' are weakly continuous we obtain

$$f' = -\sum_{j \leqslant m} \lambda_j g_j = -\sum_{j \in J} \lambda_j g_j'$$

at x_0 by passing to the limit in (2) as $x_{n_\nu} \to s_0$. This establishes (1).

Theorem 2 establishes the validity of a multiplier rule for convex programs on a Hilbert space. The conditions can and have been weakened by others. However, as was already stated, our purpose here is not generality but simply to indicate how an algorithm for computing optimum programs can also be used in a constructive way to obtain necessary conditions for a minimum. Such an approach has an obvious conceptual advantage. The essential point of the proof is in showing that at least one representation of the form (1) holds with multipliers $\lambda_j \geqslant 0$.

In the case where the Hilbert space is E^n the theorem specializes to the usual Kuhn-Tucker necessary conditions [9] except that f', g_j' become the gradients ∇f, ∇g_j.

Remark. Suppose that l, the number of elements in J, satisfies $l \leqslant n$ for $\mathscr{H} = E^n$ and that rank $\nabla G = l$ at x_0. Here ∇G is the $l \times n$ Jacobian matrix with columns ∇g_j, $j \in J$; ∇G_f will denote the $(l+1) \times n$ matrix with $l+1$ columns ∇f, ∇g_j. Under these conditions and if $r = l = m$ (the classical constrained Lagrange problem) then $\nabla f = -\sum k_{n,j} g_j(x_n)\, \nabla g_j$ implies rank $\nabla G_f \leqslant m$ at x_n. Thus det $G_f = 0$ and it remains zero as $n \to \infty$ since f, $g_i \in C^1$. But then rank $G_f =$ rank $G = m$ at x_0 and so there exists a unique set of multipliers λ_j so that $\nabla f = -\sum_{j\leqslant m}\lambda_j\nabla g_j$. Our proof of Theorem 2 follows a similar argument given in the notes [7] for the case $n = 2$ and $r = m = 1$, in which case the existence of a multiplier is equivalent to asserting that the 2×2 Jacobian G_f has rank < 2.

Note that for $\mathscr{H} = E^n$ the closed ball Ω_0 used in the proof of Theorem 2 is compact and so it is not necessary to require that f, g_j be convex in this case, only that they be l.s.c. But then even if Ω_0 is chosen so that x_0 is a global minimum on $\Omega_0 \cap \Omega_1$ it need not be unique. To overcome this difficulty we replace f by the objective function $u = f + \| x - x_0 \|^2$ which does have a unique minimum at x_0. The multiplier rule holds for u but since $\nabla u = \nabla f$ at x_0 it holds for f also.

Finally one observes from Theorem 2 that the multipliers λ_j are obtained as the limits of $k_{n,j}g_j$ as the x_n tend weakly to x_0. This is actually seen in specific examples, as indicated un [3]. Indeed, that the theoretical arguments are vindicated in practice has already been established by computing on a number of examples on E^n.

A Feasible Algorithm

We now indicate on how to proceed to compute an optimal conver program on L_2. Actually the same procedure will be applicable to programs on other spaces in a formal way. First one replaces the constrained minimization problem by an unconstrained one determined by $f_n = f + \frac{1}{2}(Hg, K_nHg)$ as prescribed by Theorem 1. Then $f_n \in \mathscr{H} \to E^1$ is replaced by a problem in the setting $f_n : E^n \to E^1$. This is done as follows, using an idea of W. Ritz [10]. Let W denote the collection of finite linear combinations of constant functions with compact support (step functions). Then W is dense in L_2 and so for every $\epsilon > 0$ there exists $w \in W$ such that $f_n(w) - f_n(x) \leqslant \epsilon$ for any $x \in L_2$ (f_n is assumed continuous). Thus, if $\{x_i\}$ is a minimizing sequence for f_n (recall that f_n is bounded below) then $f_n(x_i) - d_n \leqslant \epsilon$ for i is large enough. Now let $d_{n,k} = \min f_n(w) = f_n(w_k)$ over all $w \in W$ consisting of exactly k steps (the minimum clearly exists since f_n is now restricted to E^k) and note that $d_{n,k}$ is monotone-decreasing to some d_n as $k \to \infty$. Thus for some k there exists a w in E^k for which

$$d_{n,k} = f_n(w_k) \leqslant f_n(w) \leqslant f_n(x_i) + \epsilon \leqslant d_n + 2\epsilon$$

and so $\{w_k\}$ is a minimizing sequence. At this point all that is needed is a procedure for finding $\min f_n : E^k \to E^1$ for such k. What follows is a brief outline of one such procedure. Let H be any positive definite k by k matrix; then H defines a metric on E^k via the norm $\| x \|_H$. For any starting vector $x \in E^k$ we move down the negative gradient of f_n in this metric, i.e., in the direction $-\alpha H\nabla f_n$ (if ∇f_n is not explicitly known then a suitable procedure approximation to it will do). We stop when the one-dimensional minimum of f_n versus α is attained. At the new x we evaluate ∇f_n and then update H according to the formula given by Fletcher-Powell in [11], which modifies an algorithm of Davidon [*12*]. It is proven in [*11*] that the updated H remains positive-definite and therefore the procedure is stable (i.e., strictly downhill).

In summary, then, the method is this. Replace the problem of minimizing f on $\Omega_0 \cap \Omega_1$ by the unconstrained problem of minimizing f_n on Ω_0. Then parametrize $\Omega_0 \subset L_2$ by some approximating set $\Omega_k \subset E^k$ and minimize f_n on Ω_k. Repeat the last step by updating k until no further improvement is obtained for that n. Now update n itself and repeat until the desired accuracy is achieved. A stopping rule in each case is, for example, to terminate when successive iterates agree by a prescribed amount. Further details on the method for nonlinear programs on E^n is given in [*3*]. Initial computer trials in this setting indicate that the proposed technique is quite effective and a program for computing on problems in infinite dimensional space is now in preparation.

We note here that the penalty idea has already been used successfully in conjunction with the maximum principle of Pontriagin in several nonlinear control problems with inequality constraints by McGill [*13*] and Beltrami-McGill [*14*]. Such control problems can, of course, be considered as infinite dimensional programs. The method used in [*13*] and [*14*] was indirect, however, in contrast to the one proposed above, and did not utilize a parametrization procedure.

It is worth mentioning, finally, that the essential ingredients for the computational approach outlined above were already available in a classic paper of Courant in 1943 [*1*], in which he discussed the combined use of the Ritz method, the penalty argument, as well as gradient techniques in function space to solve variational problems.

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