Generalized Extrapolation Principle and Convergence of Some Generalized Iterative Methods

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ABSTRACT

To solve the linear system $Ax = b$, this paper presents a generalized extrapolated method by replacing the extrapolation parameter ω with the diagonal matrix Ω, and systematically gives the basic results for its convergence. Based upon these results, the paper considers the convergence of the GJ and GAOR iterative methods and, using the set of the equimodularized diagonally similar matrices defined here, gives some new further convergence results for H-matrices and their subclasses, strictly or irreducibly diagonally dominant matrices, which unify, improve, and extend previously given various results. Finally, conditions equivalent to the statement that A is a nonsingular H-matrix or a strictly (or an irreducibly) diagonally dominant matrix are given in connection with the GJ and GAOR methods.

1. INTRODUCTION

In [1] and [2] A. Hadjidimos presented successively the accelerated overrelaxation (AOR) method and the generalized accelerated overrelaxation (GAOR) method to solve the linear system $Ax = b$. The extrapolation theorem and some further results for the AOR method were given in [3].

Let the coefficient matrix $A = D - E - F = D_A - E_A - F_A$, where D is a diagonal matrix and $D_A = \operatorname{diag} A$. In this paper the iteration matrices of the GAOR and AOR methods are uniformly expressed as

$$L_{\gamma\omega} = L_{\gamma\omega}(D, E) = (D - \gamma E)^{-1}[(1 - \omega)D + (\omega - \gamma)E + \omega F],$$

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where γ and ω are real numbers. In fact, when $D = D_A$ and when E and F are, respectively, the strictly lower and upper triangular parts, denoted by E_A and F_A respectively, of the matrix $-A$, it turns out that $L_{\gamma\omega}$ is the iteration matrix of the AOR method for A; when $D \neq D_A$ and the off-diagonal elements of E and F are, respectively, those of E_A and F_A, $L_{\gamma\omega}$ is the iteration matrix of the GAOR method for A. In our theoretical analysis, one only needs to assume that

$$|E + F| = |E| + |F|,$$

without the assumption that E and F are associated with E_A and F_A.

In Section 2 a generalized extrapolated method is presented by replacing the extrapolation parameter ω of the extrapolated method given in [3] with a diagonal matrix Ω, and the basic results for its convergence are given systematically. By applying the generalized extrapolation principle to the generalized Jacobi (GJ) iterative methods in Section 3, we give some new and important results on the convergence of the GJ method with H-matrices. Using the set of the equimodularized diagonally similar matrices defined here, we give new convergence results which extend and unify the previously given various results [2–5, 11, 12, 15] for H-matrices and their subclasses, strictly or irreducibly diagonally dominant matrices. Combining the results with the convergence theorem for regular splittings, we obtain the convergence domains of the GAOR iterative method, which greatly extend the previous results [1, 8–10, 13, 14]. Finally we give some conditions equivalent to the statement that A is a nonsingular H-matrix or a strictly (or an irreducibly) diagonally dominant matrix in connection with the GJ and GAOR methods.

NOTATION. For the matrices $A = (a_{ij})$ and $B = (b_{ij})$ we write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ $\forall_{i,j}$, and write $D = D_A$ for the diagonal matrix $D = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$. In general, we write $D = \mathrm{diag}\, D$ if the matrix D is a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$. For $D = \mathrm{diag}\, D$ and $\Omega = \mathrm{diag}\, \Omega$, we write $\Omega > D > 0$ if $\omega_i > d_i > 0$ $\forall i$. Let A and B be two complex matrices. B is called a diagonally similar matrix to A, denoted by $B \hat{\sim} A$, if there exists some nonsingular diagonal matrix $Q > 0$ such that $B = Q^{-1}AQ$. In addition, let $|A| = (|a_{ij}|)$, $\|A\| = \max_i \sum_j |a_{ij}|$, I be the identity matrix, A^T be the transpose of the matrix A, and $\rho(A)$ be the spectral radius of the matrix A.

BASIC LEMMA [6, Theorem 2]. *Let $A = (a_{ij})$ be an $n \times n$ complex matrix. Then for any given number $\varepsilon > 0$ there exists some $\tilde{A} \hat{\sim} A$ such that $\|\tilde{A}\| < \rho(|A|) + \varepsilon$.*

2. GENERALIZED EXTRAPOLATION

For the linear system $(I - T)x = C$ define two iterative schemes

$$x^{(m+1)} = Tx^{(m)} + C, \qquad m = 0, 1, 2, \dots,$$

and

$$x^{(m+1)} = (I - \Omega + \Omega T)x^{(m)} + \Omega C, \qquad m = 0, 1, 2, \dots .$$

The latter is called a generalized extrapolation of the former. Here $\Omega = \operatorname{diag} \Omega$ is called an extrapolation-parameter diagonal matrix. When $\Omega = \omega I$, where ω is a real number, the latter becomes an ordinary extrapolation [3] of the former.

Throughout this section we assume that T is an $n \times n$ complex matrix, Ω is a real diagonal matrix $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n) > 0$, and $T_\Omega = I - \Omega + \Omega T$.

THEOREM 2.1. *Suppose that the complex matrix T satisfies $\rho(|T|) < 1$ and $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n)$ satisfies*

$$0 < \Omega < \frac{2}{1 + \rho(|T|)} I. \tag{2.1}$$

Then it follows that

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) < 1. \tag{2.2}$$

Proof. Notice that for given Ω

$$\frac{2}{\max_i \omega_i} - 1 - \rho(|T|) > 0,$$

since $\max_i \omega_i < 2/[1 + \rho(|T|)]$. If ε satisfies $0 < \varepsilon < 2/\max_i \omega_i - 1 - \rho(|T|)$, then

$$\max_i \omega_i < \frac{2}{1 + \rho(|T|) + \varepsilon} \tag{2.3}$$

Now take ε satisfying

$$0 < \varepsilon < \min\left\{1 - \rho(|T|), \frac{2}{\max_i \omega_i} - 1 - \rho(|T|)\right\}. \qquad (2.3')$$

From the Basic Lemma, for T there exists some $\tilde{T} = Q^{-1}TQ$ where $Q = \operatorname{diag} Q > 0$ such that $\|\tilde{T}\| < \rho(|T|) + \varepsilon$. It is easy to show that all row sums of the matrix $|I - \Omega| + \Omega Q^{-1}|T|Q$ are less than one, since for $\omega_i \leqslant 1$ each one is less than

$$(1 - \omega_i) + \omega_i[\rho(|T|) + \varepsilon] = 1 - \omega_i[1 - \rho(|T|) - \varepsilon] < 1,$$

and for $\omega_i > 1$ that is less than

$$(\omega_i - 1) + \omega_i[\rho(|T|) + \varepsilon] = \omega_i[1 + \rho(|T|) + \varepsilon] - 1 < 1 \qquad [\text{by } (2.3)]$$

Hence $\||I - \Omega| + \Omega Q^{-1}|T|Q\| < 1$. Then

$$\rho(Q^{-1}|T_\Omega|Q) \leqslant \rho(|I - \Omega| + \Omega Q^{-1}|T|Q) \leqslant \||I - \Omega| + \Omega Q^{-1}|T|Q\| < 1.$$

Thus (2.2) holds and the proof is complete. ■

COROLLARY 2.1.1. *Suppose that T satisfies $\rho(|T|) < 1$ and $T_\omega = (1 - \omega)I + \omega T$, where $0 < \omega < 2/[1 + \rho(|T|)]$. Then it follows that $\rho(T_\omega) \leqslant \rho(|T_\omega|) \leqslant |1 - \omega| + \omega\rho(|T|) < 1$.*

REMARK. The result of Corollary 2.1.1 can be applied to extrapolation, but is weaker than the extrapolation theorem [3]. It has other simple proofs, but it is mentioned her as a corollary in order to emphasize that Theorem 2.1 is a generalization of it. However, as the following example shows, the condition $\rho(|T|) < 1$ given in Theorem 2.1 cannot be replaced by $\rho(T) < 1$, and so the extrapolation theorem [3] cannot have such a generalization.

EXAMPLE 2.1. Consider the matrix

$$T = \begin{pmatrix} 100 & -99.005 \\ 101 & -100 \end{pmatrix}.$$

It is easy to verify that $\rho(T) = \sqrt{0.495} < 1$. Let the diagonal matrix $\Omega = \operatorname{diag}(0.3, 0.5)$. Clearly Ω satisfies (2.1) with $\rho(|T|)$ replaced by $\rho(T)$. Then

$$T_\Omega = I - \Omega + \Omega T = \begin{pmatrix} 30.7 & -29.7015 \\ 50.5 & -49.5 \end{pmatrix}$$

and $\rho(T_\Omega) = 9.4 + \sqrt{108.08425} > 19$. This illustrates that T_Ω is divergent even though $\rho(T) < 1$ and $0 < \Omega < I$.

DEFINITION 2.1. Let T be a complex matrix. We define the matrix set for T as follows:

$$\Omega(T) = \{A : |A| = |T|\},$$

$$\Lambda(T) = \{A : A \hat{\sim} T\},$$

$$\Lambda(\Omega(T)) = \{A : A \in \Lambda(B),\ B \in \Omega(T)\},$$

$$\Omega(\Lambda(T)) = \{A : A \in \Omega(B),\ B \in \Lambda(T)\}.$$

$\Omega(T)$ is called the set of the equimodular matrices associated with T [12, 15], $\Lambda(T)$ is called the set of the diagonally similar matrices associated with T, and $\Lambda(\Omega(T))$ and $\Omega(\Lambda(T))$ are called the set of the equimodularized diagonally similar matrices associated with T, since $\Lambda(\Omega(T)) = \Omega(\Lambda(T))$ from the following lemma.

LEMMA 2.1. *Let T be a complex matrix. Then:*

(i) $T \in \Lambda(T) \subset \Omega(\Lambda(T))$, $T \in \Omega(T) \subset \Lambda(\Omega(T))$.
(ii) $\Omega(\Lambda(T)) = \Lambda(\Omega(T))$.
(iii) *For* $T^* \in \Omega(\Lambda(T))$, $|T^*| \hat{\sim} |T|$ *and* $\rho(|T^*|) = \rho(|T|)$.
(iv) *For* $T^* \in \Omega(\Lambda(T))$, $|I - \Omega| + \Omega|T^*| \hat{\sim} |I - \Omega| + \Omega|T|$ *and* $\rho(|I - \Omega| + \Omega|T^*|) = \rho(|I - \Omega| + \Omega|T|)$, *where* $\Omega = \operatorname{diag}\Omega > 0$.

Proof. (i) follows directly from Definition 2.1. To prove (ii), observe that $T^0 \in \Omega(\Lambda(T))$ iff there exists some $Q = \operatorname{diag} Q > 0$ satisfying

$$|T^0| = |Q^{-1}TQ| \tag{2.4}$$

and that $T^* \in \Lambda(\Omega(T))$ iff there exists some $Q = \operatorname{diag} Q > 0$ satisfying

$$|QT^*Q^{-1}| = |T|; \qquad \text{equivalently,} \quad |T^*| = Q^{-1}|T|Q. \tag{2.5}$$

Clearly $|Q^{-1}TQ| = Q^{-1}|T|Q$ when $Q = \operatorname{diag} Q > 0$. From (2.4) and (2.5) it is easy to prove that (ii) follows.

Next, for any $T^* \in \Omega(\Lambda(T))$, from (2.5), $|T^*| \hat{\sim} |T|$ and $\rho(|T^*|) = \rho(|T|)$, proving (iii). Let $|T^*| = Q^{-1}|T|Q$ if $T^* \in \Omega(\Lambda(T))$, where $Q = \operatorname{diag} Q > 0$. Then $Q^{-1}(|I - \Omega| + \Omega|T|)Q = |I - \Omega| + \Omega|T^*|$, and (iv) follows immediately. ∎

LEMMA 2.2 [5]. *Suppose that the complex matrix $T = (t_{ij})$ satisfies $\|T\| < 1$ [$\|T^T\| < 1$] and $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n)$ satisfies*

$$0 < \omega_i < \frac{2}{1 + \sum_j |t_{ij}|} \qquad \forall i$$

$$\left[0 < \omega_j < \frac{2}{1 + \sum_i |t_{ij}|} \qquad \forall j\right].$$

Then it follows that (2.2) *holds and, moreover, we have*

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) \leqslant \| |I - \Omega| + \Omega|T| \| < 1 \tag{2.6}$$

$$\left[\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) \leqslant \| |I - \Omega| + \Omega|T^T| \| < 1\right]. \tag{2.6'}$$

Proof. Refer to the proof of the necessity of Theorem 2.8 below. Also see the proof of Theorem 1 in [5]. ∎

DEFINITION 2.2. Suppose that the complex matrix T satisfies $\rho(|T|) < 1$. Define matrix sets as follows:

$$W_r(T) = \{A : A \hat{\sim} T \text{ with } \|A\| < 1\},$$

$$W_c(T) = \{A : A \hat{\sim} T \text{ with } \|A^T\| < 1\}.$$

Clearly, $W_r(T) \subset \Omega(\Lambda(T))$ and $W_c(T) \subset \Omega(\Lambda(T))$.

From the Basic Lemma, for any complex matrix T with $\rho(|T|) < 1$, we have $W_r(T) \neq \emptyset$ and $W_c(T) \neq \emptyset$.

DEFINITION 2.3. For the complex matrix $T = (t_{ij})$ satisfying $\|T\| < 1$ [$\|T^T\| < 1$], define the point set

$$S(T) = \left\{(\omega_1, \omega_2, \dots, \omega_n) \in R^n : 0 < \omega_i < \frac{2}{1 + \sum_j |t_{ij}|} \ \forall i\right\}$$

$$\left[S(T) = \left\{(\omega_1, \omega_2, \dots, \omega_n) \in R^n : 0 < \omega_j < \frac{2}{1 + \sum_i |t_{ij}|} \ \forall j\right\}\right].$$

THEOREM 2.2. *Suppose that the complex matrix T satisfies $\rho(|T|) < 1$, and $\Omega = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_n)$ satisfies*

$$(\omega_1, \omega_2, \dots, \omega_n) \in \bigcup_{\tilde{T} \in W_r(T) \cup W_c(T)} S(\tilde{T}) =: S_T.$$

Then it follows that (2.2) holds.

Proof. For $(\omega_1, \omega_2, \dots, \omega_n) \in S_T$ there exists some $\tilde{T} \in W_r(T) \cup W_c(T)$ such that $(\omega_1, \omega_2, \dots, \omega_n) \in S(\tilde{T})$. Then from Lemma 2.2

$$\rho(\tilde{T}_\Omega) \leqslant \rho(|\tilde{T}_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|\tilde{T}|) < 1, \tag{2.7}$$

where $\tilde{T}_\Omega = I - \Omega + \Omega\tilde{T}$ and $\Omega = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_n)$. Since $\tilde{T} \in W_r(T) \cup W_c(T)$ and $W_r(T) \cup W_c(T) \subset \Omega(\Lambda(T))$, we have $\tilde{T} \in \Omega(\Lambda(T))$. From Lemma 2.1 and (2.7)

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) = \rho(|I - \Omega| + \Omega|\tilde{T}|) < 1,$$

that is, (2.2) holds. ∎

REMARK. Theorem 2.2 includes the results of Theorem 2.1 and Lemma 2.2, and theoretically unifies them. For Lemma 2.2 this is clear, and for Theorem 2.1 it is true from the following lemma.

LEMMA 2.3. *Suppose that the hypotheses of Theorem* 2.1 *hold. Then there exists some* $T^* \in W_r(T)$ *such that*

$$(\omega_1, \omega_2, \ldots, \omega_n) \in S(T^*)$$

and consequently

$$\left\{(\omega_1, \omega_2, \ldots, \omega_n) : 0 < \omega_i < \frac{2}{1 + \rho(|T|)} \ \forall i\right\} \subset S_T.$$

Proof. Following the proof of Theorem 2.1, take ε satisfying (2.3′). Then, from the Basic Lemma, there exists some $T^* \overset{\wedge}{\sim} T$ such that $\|T^*\| < \rho(|T|) + \varepsilon < 1$ and then $T^* \in W_r(T)$, and from (2.3) we have

$$\omega_i \leqslant \max_i \omega_i < \frac{2}{1 + \rho(|T|) + \varepsilon} < \frac{2}{1 + \|T^*\|} \leqslant \frac{2}{1 + \sum_j |t_{ij}^*|} \qquad \forall i,$$

where $T^* = (t_{ij}^*)$, that is, $(\omega_1, \omega_2, \ldots, \omega_n) \in S(T^*)$ proving the lemma. ■

For the irreducible case we have some better results.

LEMMA 2.4 [5]. *Suppose that the complex matrix* $T = (t_{ij})$ *is irreducible with* $\|T\| \leqslant 1$ [$\|T^T\| \leqslant 1$] *and* $\rho(|T|) < 1$, *and that* $\Omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n)$ *satisfies*

$$0 < \omega_i \leqslant \frac{2}{1 + \sum_j |t_{ij}|} \qquad \forall i$$

$$\left[0 < \omega_j \leqslant \frac{2}{1 + \sum_i |t_{ij}|} \qquad \forall j\right]$$

while there exists at least one i $[j]$ such that

$$\sum_j |t_{ij}| < 1 \quad \textit{and} \quad \omega_i < \frac{2}{1 + \sum_j |t_{ij}|}$$

$$\left[\sum_i |t_{ij}| < 1 \quad \textit{and} \quad \omega_j < \frac{2}{1 + \sum_i |t_{ij}|} \right]$$

hold simultaneously. Then it follows tha' (2.2) *holds, and moreover we have* (2.6) [(2.6′)] *or the following*:

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) < \||I - \Omega| + \Omega|T|\| \leqslant 1 \quad (2.8)$$

$$\left[\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) < \||I - \Omega| + \Omega|T^T|\| \leqslant 1 \right]. \quad (2.8')$$

Proof. Refer to the proof of the necessity of Theorem 2.9 below or see the proof of Theorem 2 in [5]. ■

DEFINITION 2.4. Suppose that the complex matrix T is irreducible with $\rho(|T|) < 1$. Define the matrix sets as follows:

$$W_r^*(T) = \{A : A \stackrel{\wedge}{\sim} T \text{ with } \|A\| \leqslant 1\},$$

$$W_c^*(T) = \{A : A \stackrel{\wedge}{\sim} T \text{ with } \|A^T\| \leqslant 1\}.$$

Clearly, $W_r(T) \subset W_r^*(T) \subset \Omega(\Lambda(T))$ and $W_c(T) \subset W_c^*(T) \subset \Omega(\Lambda(T))$ if T is irreducible with $\rho(|T|) < 1$.

DEFINITION 2.5. Let the complex matrix $T = (t_{ij})$ be irreducible with $\|T\| \leqslant 1$ $[\|T^T\| \leqslant 1]$ and $\rho(|T|) < 1$, and define the point set

$$S^*(T) = \left\{ (\omega_1, \omega_2, \ldots, \omega_n) \in R^n : 0 < \omega_i \leqslant 2 \Big/ \left(1 + \sum_j |t_{ij}|\right) \ \forall i \right.$$
$$\text{with strict inequality for at least one } i \text{ for which}$$
$$\left. \text{the row sum of } |T| \text{ is less than one simultaneously} \right\}$$

$$\left[S^*(T) = \left\{(\omega_1, \omega_2, \dots, \omega_n) \in R^n : 0 < \omega_j \leqslant 2\Big/\left(1 + \sum_i |t_{ij}|\right) \ \forall j \right.\right.$$

with strict inequality for at least one j for which

$$\left.\left. \text{the column sum of } |T| \text{ is less than one simultaneously}\right\}\right].$$

Clearly $S(T) \subset S^*(T)$ if T is irreducible with $\|T\| < 1$ $[\|T^T\| < 1]$ and $\rho(|T|) < 1$.

THEOREM 2.3. *Suppose that the complex matrix T is irreducible with $\rho(|T|) < 1$, and $\Omega = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_n)$ satisfies*

$$(\omega_1, \omega_2, \dots, \omega_n) \in \bigcup_{\tilde{T} \in W_r^*(T) \cup W_c^*(T)} S^*(\tilde{T}) =: S_T^*.$$

Then it follows that (2.2) *holds.*

Proof. Similarly to the proof of Theorem 2.2, from Lemmas 2.1 and 2.3 the result follows. ∎

REMARK. For the irreducible case with $\rho(|T|) < 1$. Theorem 2.3 includes the results of Lemma 2.3 and Theorem 2.2, and so those of Lemma 2.2 and Theorem 2.1, since in that case $S(T) \subset S^*(T)$ and then $S_T \subset S_T^*$.

LEMMA 2.5. *Suppose that the complex matrix T satisfies $\rho(|T|) < 1$. Then*

$$S_{\tilde{T}} = S_T \qquad \text{for} \quad \tilde{T} \in \Omega(\Lambda(T)),$$

and if, moreover, T is irreducible, then

$$S_{\tilde{T}}^* = S_T^* \qquad \text{for} \quad \tilde{T} \in \Omega(\Lambda(T)).$$

Proof. Let $\tilde{T} \in \Omega(\Lambda(T))$. Then we have

$$|\tilde{T}| = Q^{-1}|T|Q, \tag{2.9}$$

where $Q = \operatorname{diag} Q > 0$. Thus if $A \in W_r(T)$ then $\|A\| < 1$ and $A = Q_1^{-1}TQ_1$, where $Q_1 = \operatorname{diag} Q_1 > 0$. Let $\tilde{A} = (Q_1Q^{-1})^{-1}\tilde{T}(Q_1Q^{-1})$. Then from (2.9),

$$|\tilde{A}| = (Q_1Q^{-1})^{-1}|\tilde{T}|(Q_1Q^{-1}) = Q_1^{-1}|T|Q_1 = |A|,$$

and hence $\tilde{A} \in W_r(\tilde{T})$ and $S(\tilde{A}) = S(A)$. Conversely, for $\tilde{B} \in W_r(\tilde{T})$ there exists necessarily $B \in W_r(T)$ such that $|B| = |\tilde{B}|$ and then $S(B) = S(\tilde{B})$. Clearly there exists also a one-to-one correspondence between $W_c(\tilde{T})$ and $W_c(T)$, and consequently the result is proved. Similarly we can prove $S_{\tilde{T}}^* = S_T^*$. ∎

REMARK. From Lemma 2.5 we see that applying Theorems 2.2 and 2.3 to the matrix $\tilde{T} \in \Omega(\Lambda(T))$, we get necessarily the same result as applying them to T.

THEOREM 2.4. *Suppose that the hypotheses of any one of Theorems 2.1–2.3 hold, and let $T_\Omega^* = I - \Omega + \Omega T^*$, where $T^* \in \Omega(\Lambda(T))$. Then it follows that*

$$\rho(|I - \Omega| + \Omega|T^*|) = \rho(|I - \Omega| + \Omega|T|)$$

and

$$\rho(T_\Omega^*) \leqslant \rho(|T_\Omega^*|) \leqslant \rho(|I - \Omega| + \Omega|T|) < 1. \tag{2.10}$$

In particular, if the hypotheses of Lemma 2.2 hold and $T^* \in \Omega(\Lambda(T))$, then we have

$$\rho(T_\Omega^*) \leqslant \rho(|T_\Omega^*|) \leqslant \rho(|I - \Omega| + \Omega|T|) \leqslant \||I - \Omega| + \Omega|T|\| < 1 \tag{2.11}$$

$$\left[\rho(T_\Omega^*) \leqslant \rho(|T_\Omega^*|) \leqslant \rho(|I - \Omega| + \Omega|T|) \leqslant \||I - \Omega| + \Omega|T^T|\| < 1\right], \tag{2.11'}$$

and if the hypotheses of Lemma 2.4 *hold and* $T^* \in \Omega(\Lambda(T))$, *then we have* (2.11) [(2.11′)] *or the following*:

$$\rho(T_\Omega^*) \leqslant \rho(|T_\Omega^*|) \leqslant \rho(|I-\Omega| + \Omega|T|) < \||I-\Omega| + \Omega|T|\| \leqslant 1 \quad (2.12)$$

$$\left[\rho(T_\Omega^*) \leqslant \rho(|T_\Omega^*|) \leqslant \rho(|I-\Omega| + \Omega|T|) < \||I-\Omega| + \Omega|T^T|\| \leqslant 1\right]. \quad (2.12')$$

Proof. From Lemma 2.1 and Theorems 2.1–2.3 we have (2.10) immediately. Moreover, from (2.10), (2.6), and (2.8) [(2.6′), and (2.8′)], we have (2.11) and (2.12) [(2.11′) and (2.12′)]. ∎

REMARK. Theorems 2.1–2.3 and Lemmas 2.2 and 2.4 are special cases of Theorem 2.4.

To illustrate we give the following example.

EXAMPLE 2.2. Consider the nonnegative matrix

$$T = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \geqslant 0$$

with

$$\rho(T) < 1; \qquad \text{equivalently,} \quad bc < 1.$$

Consider the diagonal matrix $Q = \operatorname{diag}(k, 1)$, where $k > 0$, and the similar matrix

$$\tilde{T} = Q^{-1}TQ = \begin{pmatrix} 0 & b/k \\ ck & 0 \end{pmatrix}$$

with

$$\|\tilde{T}\| < 1; \qquad \text{equivalently,} \quad b < k < 1/c.$$

Then, from Theorem 2.2, for $\Omega = \operatorname{diag}(\omega_1, \omega_2)$ satisfying

$$0 < \omega_1 < \frac{2}{1 + b/k}, \qquad 0 < \omega_2 < \frac{2}{1 + ck},$$

we have that (2.2) holds. With a small additional computational effort we have

$$S_T = \bigcup_{\tilde{T} \in W_r(T) \cup W_c(T)} S(\tilde{T}) = \bigcup_{\tilde{T} \in W_r(T)} S(\tilde{T})$$

$$= \left\{ (\omega_1, \omega_2) : 0 < \omega_2 < \frac{2(\omega_1 - 2)}{(1 - bc)\omega_1 - 2}, 0 < \omega_1, \omega_2 < \frac{2}{1 + bc} \right\}$$

$$= \left\{ (\omega_1, \omega_2) : 0 < \omega_2 < \frac{2(\omega_1 - 2)}{[1 - \rho^2(T)]\omega_1 - 2}, \right.$$

$$\left. 0 < \omega_1, \omega_2 < \frac{2}{1 + \rho^2(T)} \right\}.$$

If $b = 0.9$ and $c = 0.4$, then $\rho(T) = 0.6$ and

$$S_T = \left\{ (\omega_1, \omega_2) : 0 < \omega_2 < \frac{2 - \omega_1}{1 - 0.32\omega_1}, 0 < \omega_1, \omega_2 < \frac{1}{0.68} \right\}. \quad (2.13)$$

Then, from Theorems 2.2 and 2.4, for $\Omega = \operatorname{diag} \Omega$ satisfying $(\omega_1, \omega_2) \in S_T$ we have that (2.2) and (2.10) hold.

Also, noticing that T is irreducible when $bc \neq 0$, we can determine

$$S_T^* = \left\{ (\omega_1, \omega_2) : 0 < \omega_2 < \frac{2(\omega_1 - 2)}{[1 - \rho^2(T)]\omega_1 - 2}, \right.$$

$$\left. 0 < \omega_1, \omega_2 \leqslant \frac{2}{1 + \rho^2(T)} \right\}.$$

When $b = 0.9$ and $c = 0.4$,

$$S_T^* = \left\{ (\omega_1, \omega_2) : 0 < \omega_2 < \frac{2 - \omega_1}{1 - 0.32\,\omega_1}, 0 < \omega_1, \omega_2 \leqslant \frac{1}{0.68} \right\}. \quad (2.14)$$

Then, from Theorems 2.3 and 2.4, for $\Omega = \operatorname{diag} \Omega$ satisfying $(\omega_1, \omega_2) \in S_T^*$ we have that (2.2), (2.10), (2.11), (2.11′), (2.12), and (2.12′) hold.

Clearly, in general, $S_T^* \supset S_T$ with $S_T^* \neq S_T$.

To prove both the sufficient and the necessary conditions given in Section 3, we give some results as follows.

THEOREM 2.5. *Suppose that the matrices* $T \geqslant 0$, $B \geqslant 0$, *and* $T_B = I - B + BT \geqslant 0$. *Then* $\rho(T) < 1$ *if* $\rho(T_B) < 1$.

Proof. Since $T_B \geqslant 0$ and $\rho(T_B) < 1$, from Theorem 3.8 in [11], $(I - T_B)^{-1}$ exists and $(I - T_B)^{-1} \geqslant 0$. Since $I - T_B = B - BT = B(I - T)$, we have $(I - T_B)^{-1}B(I - T) = I$, implying that $(I - T)^{-1}$ exists and $(I - T)^{-1} = (I - T_B)^{-1}B \geqslant 0$, since $B \geqslant 0$ and $(I - T_B)^{-1} \geqslant 0$. Combining with Theorem 3.8 in [11], we have $\rho(T) < 1$, since $T \geqslant 0$. ■

THEOREM 2.6. *Suppose that the matrices* $T \geqslant 0$, $\Omega = \operatorname{diag}\Omega > 0$, *and* $T_\Omega = I - \Omega + \Omega T \geqslant 0$. *Then* $\rho(T_\Omega) < 1$ *iff* $\rho(T) < 1$.

Proof. From Theorem 2.5, $\rho(T) < 1$ if $\rho(T_\Omega) < 1$. Conversely, if $\rho(T) < 1$ then $(1 - T_\Omega)^{-1} = (\Omega - \Omega T)^{-1} = (I - T)^{-1}\Omega^{-1} \geqslant 0$, since $T \geqslant 0$ and $\Omega > 0$ and thus $\Omega^{-1} > 0$. Combining with $T_\Omega \geqslant 0$, we have $\rho(T_\Omega) < 1$. ■

COROLLARY 2.6.1. *Let the matrices* $T \geqslant 0$ *and* $T_\omega = (1 - \omega)I + \omega T \geqslant 0$ *with* $\omega > 0$. *Then* $\rho(T_\omega) < 1$ *iff* $\rho(T) < 1$.

THEOREM 2.7. *Let* $T_\Omega = I - \Omega + \Omega T$, *where* $\Omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n)$. *Then*

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega|T|) < 1 \quad \textit{and} \quad \Omega > 0$$

iff

$$\rho(|T|) < 1 \quad \textit{and} \quad (\omega_1, \omega_2, \ldots, \omega_n) \in S_T.$$

Proof. Suppose that $\rho(|I - \Omega| + \Omega|T|) < 1$ and $\Omega > 0$. From the Basic Lemma, for the nonnegative matrix $|I - \Omega| + \Omega|T|$ there exists some $Q = \operatorname{diag} Q > 0$ such that all row sums of the matrix

$$Q^{-1}(|I - \Omega| + \Omega|T|)Q = |I - \Omega| + \Omega Q^{-1}|T|Q = |I - \Omega| + \Omega|\tilde{T}|,$$

where $\tilde{T} = (\tilde{t}_{ij}) = Q^{-1}TQ$, are less than one. Then for $\omega_i \leqslant 1$ we have

$$|1 - \omega_i| + \omega_i \sum_j |\tilde{t}_{ij}| = 1 - \omega_i + \omega_i \sum_j |\tilde{t}_{ij}| < 1, \tag{2.15}$$

implying

$$\sum_j |\tilde{t}_{ij}| < 1, \tag{2.16}$$

and for $\omega_i > 1$ we have

$$|1 - \omega_i| + \omega_i \sum_j |\tilde{t}_{ij}| = \omega_i - 1 + \omega_i \sum_j |\tilde{t}_{ij}| < 1, \tag{2.17}$$

implying

$$\sum_j |\tilde{t}_{ij}| < \frac{2 - \omega_i}{\omega_i} < 1 \quad \text{and} \quad \omega_i < \frac{2}{1 + \sum_j |\tilde{t}_{ij}|}. \tag{2.18}$$

Thus from (2.16) and (2.18) we have

$$\rho(|\tilde{T}|) \leqslant \|\tilde{T}\| < 1 \quad \text{and} \quad (\omega_1, \omega_2, \ldots, \omega_n) \in S(\tilde{T}),$$

and then

$$\rho(|T|) = \rho\big(Q|\tilde{T}|Q^{-1}\big) = \rho(|\tilde{T}|) < 1$$

and

$$\tilde{T} \in W_r(T) \quad \text{and} \quad (\omega_1, \omega_2, \dots, \omega_n) \in S_T.$$

So the necessity is proved. From Theorem 2.2 the sufficiency is clear, and the theorem is proved. ∎

COROLLARY 2.7.1. *Let $T_\omega = (1-\omega)I + \omega T$, where ω is a real number. Then*

$$\rho(T_\omega) \leqslant \rho(|T_\omega|) \leqslant |1-\omega| + \omega\rho(|T|) < 1 \quad \textit{and} \quad \omega > 0$$

iff

$$\rho(|T|) < 1 \quad \textit{and} \quad 0 < \omega < \frac{2}{1+\rho(|T|)}.$$

THEOREM 2.8. *Let T be a complex matrix and $T_\Omega = I - \Omega + \Omega T$, where $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n)$. Then it follows that*

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I-\Omega| + \Omega|T|) \leqslant \||I-\Omega| + \Omega|T|\| < 1 \quad \textit{and} \quad \Omega > 0$$

iff

$$\rho(|T|) \leqslant \|T\| < 1 \quad \textit{and} \quad 0 < \omega_i < \frac{2}{1+\sum_j |t_{ij}|} \quad \forall i$$

and that

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I-\Omega| + \Omega|T|) \leqslant \||I-\Omega| + \Omega|T^T|\| < 1 \quad \textit{and} \quad \Omega > 0$$

iff

$$\rho(|T|) \leqslant \|T^T\| < 1 \quad \textit{and} \quad 0 < \omega_j < \frac{2}{1+\sum_i |t_{ij}|} \quad \forall j.$$

Proof. If $\| \, |I - \Omega| + \Omega |T| \, \| < 1$, then every row sum of $|I - \Omega| + \Omega |T|$ is less than one; that is, for $\omega_i \leqslant 1$, similar to (2.15) and (2.16), we have $\sum_j |t_{ij}| < 1$, and for $\omega_i > 1$, similar to (2.17) and (2.18), we have also $\sum_j |t_{ij}| < 1$. In short, $\|T\| < 1$, and the necessity is proved. From Lemma 2.2 the sufficiency follows. Similarly we can prove the result for $\|T^T\| < 1$. ■

THEOREM 2.9. *Let T be an irreducible complex matrix and $T_\Omega = I - \Omega + \Omega T$, where $\Omega = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n)$. Then it follows that*

$$\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega |T|) \overset{\leqslant}{<} \| \, |I - \Omega| + \Omega |T| \, \| \overset{<}{\leqslant} 1 \quad \textit{and} \quad \Omega > 0$$

iff

$$\rho(|T|) \overset{\leqslant}{<} \|T\| \overset{<}{\leqslant} 1 \quad \textit{and} \quad 0 < \omega_i \leqslant \frac{2}{1 + \sum_j |t_{ij}|} \quad \forall i$$

$$\Big[\rho(T_\Omega) \leqslant \rho(|T_\Omega|) \leqslant \rho(|I - \Omega| + \Omega |T|) \overset{\leqslant}{<} \| \, |I - \Omega| + \Omega |T^T| \, \| \overset{<}{\leqslant} 1$$

iff

$$\rho(|T|) \overset{\leqslant}{<} \|T^T\| \overset{<}{\leqslant} 1 \quad \textit{and} \quad 0 < \omega_j \leqslant \frac{2}{1 + \sum_i |t_{ij}|} \quad \forall j \Big],$$

while there exists at least one i $[j]$ such that

$$\sum_j |t_{ij}| < 1 \quad \textit{and} \quad \omega_i < \frac{2}{1 + \sum_j |t_{ij}|}$$

$$\left[\sum_i |t_{ij}| < 1 \quad \textit{and} \quad \omega_j < \frac{2}{1 + \sum_i |t_{ij}|} \right]$$

hold simultaneously.

Proof. If $\rho(|I-\Omega|+\Omega|T|)<1$ then, from Theorem 2.7,

$$\rho(|T|)<1. \tag{2.19}$$

There exist two cases:

(i) If $\rho(|I-\Omega|+\Omega|T|)\leqslant \| \, |1-\Omega|+\Omega|T| \, \|<1$, then, from the inequalities similar to (2.15) to (2.18), we have

$$\rho(|T|)\leqslant \|T\|<1 \quad \text{and} \quad 0<\omega_i<\frac{2}{1+\sum_j |t_{ij}|} \quad \forall i. \tag{2.20}$$

The necessity is proved.

(ii) If $\rho(|I-\Omega|+\Omega|T|)<\| \, |I-\Omega|+\Omega|T| \, \|\leqslant 1$ then we have the results similar to (2.15) to (2.18):

$$1-\omega_i+\omega_i\sum_j |t_{ij}|\leqslant 1 \text{ and } \sum_j |t_{ij}|\leqslant 1 \qquad \text{for} \quad \omega_i\leqslant 1, \tag{2.21}$$

$$\omega_i-1+\omega_i\sum_j |t_{ij}|\leqslant 1 \text{ and } \sum_j |t_{ij}|\leqslant \frac{2-\omega_i}{\omega_i}<1 \qquad \text{for} \quad \omega_i>1. \tag{2.22}$$

Thus we have

$$\|T\|\leqslant 1 \quad \text{and} \quad 0<\omega_i\leqslant \frac{2}{1+\sum_j |t_{ij}|} \quad \forall i. \tag{2.23}$$

Observe that

$$\omega_i=\frac{2}{1+\sum_j |t_{ij}|} \quad \text{implies} \quad \omega_i\geqslant 1$$

(otherwise $\omega_i < 1$ and $\sum_j |t_{ij}| > 1$, contradicting $\|T\| \leqslant 1$), and that

$$\omega_i = \frac{2}{1 + \sum_j |t_{ij}|} \quad \text{implies} \quad \omega_i - 1 + \omega_i \sum_j |t_{ij}| = 1. \tag{2.24}$$

Thus if $\omega_i = 2/(1 + \sum_j |t_{ij}|)$ $\forall i$, then all row sums of $|I - \Omega| + \Omega|T|$ are equal to one and we have $\rho(|I - \Omega| + \Omega|T|) = 1$, contradicting the assumption of $\rho(|I - \Omega| + \Omega|T|) < 1$. So the set $J = \{i : \omega_i < 2/(1 + \sum_j |t_{ij}|)\}$ is nonempty.

There exist two subcases: (1) If there exists some $\omega_i > 1$ with $i \in J$, then, from (2.22), $\sum_j |t_{ij}| < 1$. Combining (2.23) and the irreducibility of T, from Lemma 2.5 in [11] we have $\rho(|T|) < \|T\| \leqslant 1$, and the necessity is proved immediately. (2) If $\omega_i \leqslant 1$ $\forall i \in J$, then there necessarily exists some $i \in J$ such that $\sum_i |t_{ij}| < 1$. Otherwise

$$1 - \omega_i + \omega_i \sum_j |t_{ij}| = 1 \qquad \forall i \in J,$$

and then, combining with (2.24), we have that all row sums of $|I - \Omega| + \Omega|T|$ are equal to one, contradicting $\rho(|I - \Omega| + \Omega|T|) < 1$ again, which proves the necessity immediately.

From Lemma 2.4 we have the sufficiency. Similarly we can prove the result for T^T. ■

3. CONVERGENCE OF THE GJ AND GAOR METHODS

Applying the results on the generalized extrapolation principle to the GJ and GAOR iterative methods, we have some further new results.

Throughout this section we assume that $A = D - B = D_A - B_A$, where $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ and $D_A = \operatorname{diag} A$ are nonsingular, is a complex coefficient matrix for the linear system $Ax = b$. Then $D_A^{-1}B_A$ and $D^{-1}B$ $(D \neq D_A)$ are respectively the iteration matrices of the Jacobi and GJ methods for A.

First we discuss the convergence of the GJ method.

Basic Theorem A. *The GJ method for A is a generalized extrapolation of the Jacobi method for A with the extrapolation-parameter diagonal matrix $D^{-1}D_A$.*

Proof. This follows from the equality

$$D^{-1}B = D^{-1}(D - A) = D^{-1}(D - D_A + B_A) = 1 - D^{-1}D_A$$
$$+ (D^{-1}D_A)(D_A^{-1}B_A). \qquad \blacksquare \quad (3.1)$$

REMARK. When $D^{-1}D_A = \omega I$, where ω is a real number, $D^{-1}B$ is the iteration matrix of the JOR method. So the JOR method is a special case of the GJ method.

Also, from (3.1) we have

$$|D^{-1}B| = |I - D^{-1}D_A| + |D^{-1}B_A|. \tag{3.2}$$

From Basic Theorem A and Theorems 2.1–2.4 we can obtain some important results for the convergence of the GJ method, of which most are new.

THEOREM 3.1. *Let $A = D - B$ be a nonsingular complex H-matrix, and suppose that D satisfies*

$$0 < D^{-1}D_A < \frac{2}{1 + \rho(|D_A^{-1}B_A|)} I.$$

Then it follows that

$$\rho(D^{-1}B) \leqslant \rho(|D^{-1}B|) = \rho(|I - D^{-1}D_A| + |D^{-1}B_A|) < 1, \tag{3.3}$$

and consequently the GJ method for A converges.

Proof. Letting $\Omega = D^{-1}D_A$ and $T = D_A^{-1}B_A$ and observing (3.1) and $\rho(|T|) < 1$, since A is a nonsingular H-matrix, the proof follows from Basic Theorem A, Theorem 2.1, and (3.2).

THEOREM 3.2. *Let $A = D - B$ be a nonsingular complex H-matrix, and suppose that $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ satisfies*

$$\left(\frac{a_{11}}{d_1}, \frac{a_{22}}{d_2}, \ldots, \frac{a_{nn}}{d_n}\right) \in \bigcup_{\tilde{T} \in W_r(D_A^{-1}B_A) \cup W_c(D_A^{-1}B_A)} S(\tilde{T}) =: S_{D_A^{-1}B_A}.$$

Then (3.3) *holds; consequently, the GJ method for A converges.*

Proof. It is similar to the proof of Theorem 3.1 and follows from Theorem 2.2 and Basic Theorem A. ■

From Lemma 2.2 we have immediately

COROLLARY 3.2.1. *Let $A = D - B$ be a strictly diagonally dominant complex matrix by rows [by columns], and suppose that $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ satisfies*

$$0 < \frac{a_{ii}}{d_i} < \frac{2|a_{ii}|}{\sum_j |a_{ij}|} \qquad \forall i$$

$$\left[0 < \frac{a_{jj}}{d_j} < \frac{2|a_{jj}|}{\sum_i |a_{ij}|} \qquad \forall j\right].$$

Then (3.3) *holds, and moreover we have*

$$\rho(D^{-1}B) \leqslant \rho(|D^{-1}B|) = \rho\big(|I - D^{-1}D_A| + |D^{-1}B_A|\big)$$

$$\leqslant \big\||I - D^{-1}D_A| + |D^{-1}B_A|\big\| < 1, \tag{3.3'a}$$

$$\rho(D^{-1}B) \leqslant \rho(|D^{-1}B|) = \rho\big(|I - D^{-1}D_A| + |D^{-1}B_A|\big)$$

$$\leqslant \big\||I - D^{-1}D_A| + |D^{-1}B_A^T|\big\| < 1, \tag{3.3'b}$$

and consequently the GJ method for A converges.

REMARK. The result of Corollary 3.2.1 includes that of Theorem 4 in [2] for the strict diagonal dominance, and so the latter is a special case of Theorem 3.2. Also, since Theorem 2.1 is a special case of Theorem 2.2, Theorem 3.1 is also a special case of Theorem 3.2.

THEOREM 3.3. *Let $A = D - B$ be a nonsingular irreducible complex H-matrix, and suppose that $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ satisfies*

$$\left(\frac{a_{11}}{d_1}, \frac{a_{22}}{d_2}, \dots, \frac{a_{nn}}{d_n}\right) \in \bigcup_{\tilde{T} \in W_r^*(D_A^{-1}B_A) \cup W_c^*(D_A^{-1}B_A)} S^*(\tilde{T}) =: S^*_{D_A^{-1}B_A}.$$

Then (3.3) *holds, and consequently the GJ method for A converges.*

Proof. Similarly, it follows from Theorem 2.3 and Basic Theorem A. ■

From Lemma 2.4 we have immediately

COROLLARY 3.3.1. *Let $A = D - B$ be an irreducibly diagonally dominant complex matrix by rows [by columns], and suppose that $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ satisfies*

$$0 < \frac{a_{ii}}{d_i} \leqslant \frac{2|a_{ii}|}{\sum_j |a_{ij}|} \qquad \forall i$$

$$\left[0 < \frac{a_{jj}}{d_j} \leqslant \frac{2|a_{jj}|}{\sum_i |a_{ij}|} \qquad \forall j\right],$$

while there exists at least one i [j] such that

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}| \quad \text{and} \quad \frac{a_{ii}}{d_i} < \frac{2|a_{ii}|}{\sum_j |a_{ij}|}$$

$$\left[\sum_{i \neq j} |a_{ij}| < |a_{jj}| \quad \text{and} \quad \frac{a_{jj}}{d_j} < \frac{2|a_{jj}|}{\sum_i |a_{ij}|}\right]$$

hold simultaneously. Then (3.3) *holds, and moreover we have* (3.3′a) [(3.3′b)] *or the following*:

$$\rho(D^{-1}B) \leqslant \rho(|D^{-1}B|) = \rho\big(|I - D^{-1}D_A| + |D^{-1}B_A|\big)$$
$$< \big\| |I - D^{-1}D_A| + |D^{-1}B_A| \big\| \leqslant 1$$
$$\Big[\, \rho(D^{-1}B) \leqslant \rho(|D^{-1}B|) = \rho\big(|I - D^{-1}D_A| + |D^{-1}B_A|\big)$$
$$< \big\| |I - D^{-1}D_A| + |D^{-1}B_A^T| \big\| \leqslant 1 \Big],$$

and consequently the GJ method for A converges.

REMARK. The result of Corollary 3.3.1 includes that of Theorem 4 in [2] for the irreducible diagonal dominance, and the latter is a special case of Theorem 3.3. Thus the results of Theorem 4 in [2] are included in those of Theorems 3.2 and 3.3. For the case when A is both irreducibly and strictly diagonally dominant, the results of Theorem 3.3 include the ones given by Theorems 3.1 and 3.2, since in that case

$$\left\{ (\omega_1, \omega_2, \dots, \omega_n) \in R^n : 0 < \omega_i < \frac{2}{1 + \rho\big(|D_A^{-1}B_A|\big)} \ \forall i \right\}$$
$$\subset S_{D_A^{-1}B_A} \subset S^*_{D_A^{-1}B_A}.$$

LEMMA 3.1. *Let* $\tilde{A} \in \Omega(\Lambda(A))$. *Then* $D_{\tilde{A}}^{-1}B_{\tilde{A}} \in \Omega(\Lambda(D_A^{-1}B_A))$ *and* $\rho(|D_{\tilde{A}}^{-1}B_{\tilde{A}}|) = \rho(|D_A^{-1}B_A|)$.

Proof. Since $\tilde{A} \in \Omega(\Lambda(A))$, by Lemma 2.1 there exists some $Q = \operatorname{diag} Q > 0$ such that $|\tilde{A}| = Q^{-1}|A|Q$, and then the result follows from the equality

$$|D_{\tilde{A}}^{-1}B_{\tilde{A}}| = \big(Q^{-1}|D_A^{-1}|Q\big)\big(Q^{-1}|B_A|Q\big) = Q^{-1}|D_A^{-1}B_A|Q. \quad \blacksquare \tag{3.4}$$

THEOREM 3.4. *Suppose that the hypotheses of any one of Theorems 3.1–3.3 hold. Let* $\tilde{A} \in \Omega(\Lambda(A))$, $\tilde{A} = \tilde{D} - \tilde{B} = D_{\tilde{A}} - B_{\tilde{A}}$, *where* $\tilde{D} = \operatorname{diag} \tilde{D}$ *is nonsingular and satisfies*

$$\tilde{D}^{-1}D_{\tilde{A}} = D^{-1}D_A. \tag{3.5}$$

Then

$$\rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho(|D^{-1}B|) = \rho\left(|I - \tilde{D}^{-1}D_{\tilde{A}}| + |\tilde{D}^{-1}B_{\tilde{A}}|\right)$$

$$= \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right) \tag{3.6}$$

and

$$\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right) < 1, \tag{3.7}$$

and consequently the GJ method for $\tilde{A}$ *converges. In addition, if the hypotheses of Corollary* 3.2.1 *hold and* $\tilde{A} = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ *with* (3.5), *then we have*

$$\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right)$$

$$\leqslant \left\| |I - D^{-1}D_A| + |D^{-1}B_A| \right\| \leqslant 1 \tag{3.8}$$

$$\Big[\, \rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right)$$

$$\leqslant \left\| |I - D^{-1}D_A| + |D^{-1}B_A^T| \right\| < 1 \Big], \tag{3.8$'$}$$

and if the hypotheses of Corollary 3.3.1 *hold and* $\tilde{A} = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ *with* (3.5), *then we have* (3.8) [(3.8′)] *or the following*:

$$\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right)$$

$$< \left\| |I - D^{-1}D_A| + |D^{-1}B_A| \right\| \leqslant 1 \tag{3.9}$$

$$\Big[\, \rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right)$$

$$< \left\| |I - D^{-1}D_A| + |D^{-1}B_A^T| \right\| \leqslant 1 \Big]. \tag{3.9$'$}$$

Proof. From (3.5) we have

$$\tilde{D}^{-1}\tilde{B} = \tilde{D}^{-1}(\tilde{D} - \tilde{A}) = \tilde{D}^{-1}\left(\tilde{D} - D_{\tilde{A}} + B_{\tilde{A}}\right)$$

$$= I - \tilde{D}^{-1}D_{\tilde{A}} + \left(\tilde{D}^{-1}D_{\tilde{A}}\right)\left(D_{\tilde{A}}^{-1}B_{\tilde{A}}\right)$$

$$= I - D^{-1}D_A + \left(D^{-1}D_A\right)\left(D_{\tilde{A}}^{-1}B_{\tilde{A}}\right).$$

Since $\tilde{A} \in \Omega(\Lambda(A))$, we have (3.4), and then, noticing that the diagonal elements of $B_{\tilde{A}}$ and B_A are all zero,

$$|\tilde{D}^{-1}\tilde{B}| = |I - \tilde{D}^{-1}D_{\tilde{A}}| + \left|\left(\tilde{D}^{-1}D_{\tilde{A}}\right)\left(D_{\tilde{A}}^{-1}B_{\tilde{A}}\right)\right|$$

$$= |I - D^{-1}D_A| + \left|\left(D^{-1}D_A\right)\left(D_{\tilde{A}}^{-1}B_{\tilde{A}}\right)\right|$$

$$= |I - D^{-1}D_A| + |D^{-1}D_A|Q^{-1}|D_A^{-1}B_A|Q$$

$$= Q^{-1}\left[|I - D^{-1}D_A| + \left|\left(D^{-1}D_A\right)\left(D_A^{-1}B_A\right)\right|\right]Q$$

$$= Q^{-1}|I - D^{-1}D_A + D^{-1}B_A|Q = Q^{-1}|D^{-1}B|Q. \quad (3.10)$$

Hence

$$\rho\left(|\tilde{D}^{-1}\tilde{B}|\right) = \rho\left(|I - \tilde{D}^{-1}D_{\tilde{A}}| + |\tilde{D}^{-1}B_{\tilde{A}}|\right)$$

$$= \rho\left(|I - D^{-1}D_A| + |D^{-1}B_A|\right) = \rho\left(|D^{-1}B|\right).$$

Combining with Theorem 2.4, the result follows. ■

REMARK. As Theorems 2.1–2.3 are special cases of Theorem 2.4, Theorems 3.1–3.3 are likewise special cases of Theorem 3.4.

Letting $D^{-1}D_A = \omega I$ where ω is a real number, we can obtain the following result, slightly different from the corresponding result in [12].

COROLLARY 3.4.1. *Let* $A = D_A - B_A$ *be a nonsingular complex H-matrix and* $\tilde{B}_\omega = (1 - \omega)I + \omega D_{\tilde{A}}^{-1}B_{\tilde{A}}$, *where* $\tilde{A} \in \Omega(\Lambda(A))$. *Then for* $0 < \omega < 2/[1 + \rho(|D_A^{-1}B_A|)]$

$$\rho(\tilde{B}_\omega) \leqslant \rho(|\tilde{B}_\omega|) \leqslant |1-\omega| + \omega\rho(|D_{\tilde{A}}^{-1}B_{\tilde{A}}|) < 1.$$

The following example shows that our results are better than the similar previous results (see [2] and [8], for example).

EXAMPLE 3.1. Suppose that the matrices $A = D - B = D_A - B_A$, $\bar{A} = \bar{D} - \bar{B} = D_{\bar{A}} - B_{\bar{A}}$, and $\tilde{A} = \tilde{D} - \tilde{B} = D_{\tilde{A}} - B_{\tilde{A}}$ are defined, respectively, by

$$A = \begin{pmatrix} 2 & -1.8 \\ -1.2 & 3 \end{pmatrix} \hat{\sim} \bar{A} = \begin{pmatrix} 2 & -3.6 \\ -0.6 & 3 \end{pmatrix}, \qquad \tilde{A} \in \Omega(\Lambda(A)).$$

Then from Lemma 3.1 we have

$$D_A^{-1}B_A = \begin{pmatrix} 0 & 0.9 \\ 0.4 & 0 \end{pmatrix} \hat{\sim} D_{\bar{A}}^{-1}B_{\bar{A}} = \begin{pmatrix} 0 & 1.8 \\ 0.2 & 0 \end{pmatrix} \hat{\sim} |D_{\tilde{A}}^{-1}B_{\tilde{A}}|$$

and

$$\rho(D_A^{-1}B_A) = \rho(D_{\bar{A}}^{-1}B_{\bar{A}}) = \rho(|D_{\tilde{A}}^{-1}B_{\tilde{A}}|) = 0.6.$$

Then A is both strictly and irreducibly diagonally dominant, and, from Theorem 3.1 (also Theorem 4 in [2]), for A the GJ method converges, that is, $\rho(D^{-1}B) < 1$ for $D > 0.8D_A$. Also, from Theorem 3.1, if $\bar{A}$ and $\tilde{A}$ are H-matrices, the GJ method converges respectively for $\bar{D} > 0.8D_{\bar{A}}$ and $0 < \tilde{D}^{-1}D_{\tilde{A}} < (2/1.6)I$. But from [8] the same results are obtained only for $D \geqslant D_A$, $\bar{D} \geqslant D_{\bar{A}}$, and $|\tilde{D}| \geqslant |D_{\tilde{A}}|$ (equivalently, $0 < \tilde{D}^{-1}D_{\tilde{A}} \leqslant I$), with $\tilde{A} \in \Omega(A)$. Moreover, from Theorem 3.4 alone, we have that for any $\tilde{A} \in \Omega(\Lambda(A))$ the GJ method converges for $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \tilde{d}_2)$ satisfying $(\tilde{a}_{11}/\tilde{d}_1, \tilde{a}_{22}/\tilde{d}_2) \in S_{D_A^{-1}B_A}$, where the point set $S_{D_A^{-1}B_A}$ has been determined in Example 2.2 and is given by (2.13). In other words, for any complex matrix $\tilde{A} \in \Omega(\Lambda(A))$ the GJ method converges for $\tilde{D} = \operatorname{diag}(\tilde{a}_{11}/\omega_1, \tilde{a}_{22}/\omega_2)$ where $(\omega_1, \omega_2) \in S_T$, defined in (2.13), or indeed where $(\omega_1, \omega_2) \in S_T^*$, defined in (2.14), since $\tilde{A}$ is irreducible.

Secondly, we discuss the problem for the convergence of the GAOR method. The iteration matrix

$$L_{\gamma\omega}(D, E) = (D - \gamma E)^{-1}[(1-\omega)D + (\omega - \gamma)E + \omega F],$$

as has been pointed out in Section 1, uniformly expresses the AOR and GAOR iteration matrices for $A = D - E - F$. Also, in our argument it is only needed to assume that

$$|E + F| = |E| + |F|$$

without the assumption that E and F are triangular matrices.

DEFINITION 3.1. Let $A = D - B$ with $\rho(|D^{-1}B|) = \rho$. For A define the point set $S = S(A, D) = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_0 = \left\{(\gamma, \omega) : -\frac{1-\rho}{2\rho} < \gamma < 0, 0 < \omega < 2\frac{1+\rho\gamma}{1+\rho}\right\},$$

$$S_1 = \left\{(\gamma, \omega) : 0 \leqslant \gamma, \omega < \frac{2}{1+\rho}, \omega \neq 0\right\},$$

$$S_2 = \left\{(\gamma, \omega) : 1 \leqslant \omega \leqslant \gamma, 0 < \frac{2\gamma - \omega}{2-\omega}\rho < 1\right\},$$

$$S_3 = \left\{(\gamma, \omega) : 0 < \omega \leqslant \gamma, \omega \leqslant 1, \frac{2\gamma - \omega}{\omega}\rho < 1\right\},$$

$$S_4 = \left\{(\gamma, \omega) : 0 \leqslant \gamma < \frac{1+\rho}{2\rho}, \omega > 0, \frac{2\gamma - \omega}{\omega}\rho \geqslant 1\right\}.$$

It is easy to show that if $\rho < 1$, then S is a hexagonal region which has vertices $(-(1-\rho)/(2\rho), 0)$, $(-(1-\rho)/(2\rho), 1)$, $(0, 2/(1+\rho))$, $(2/(1+\rho), 2/(1+\rho))$, $((1+\rho)/(2\rho), 1)$, and $((1+\rho)/(2\rho), 0)$. Clearly,

$$\{(\gamma, \omega) : 0 \leqslant \gamma, \omega < 2/(1+\rho), \omega \neq 0\} \subset S(A, D). \qquad (3.11)$$

We first give a basic result.

BASIC THEOREM B. *Let the complex matrix* $A = D - B = D_A - B_A$ *with* $D = \operatorname{diag} D$ *nonsingular. And let* $\tilde{A} \in \Omega(\Lambda(A))$, $\tilde{A} = \tilde{D} - \tilde{B} = D_{\tilde{A}} -$

$B_{\tilde{A}}$, *with* $\tilde{D} = \operatorname{diag} \tilde{D}$ *nonsingular and* $\tilde{B} = \tilde{E} + \tilde{F}$ *satisfying*

$$\tilde{D}^{-1}D_{\tilde{A}} = D^{-1}D_A \quad \text{and} \quad |\tilde{B}| = |\tilde{E}| + |\tilde{F}|. \tag{3.12}$$

If $\rho(|D^{-1}B|) = \rho < 1$, *then for any* $(\gamma, \omega) \in S(A, D)$

$$\rho\big(L_{\gamma\omega}(\tilde{D}, \tilde{E})\big) \leqslant \rho\big(\big|L_{\gamma\omega}(\tilde{D}, \tilde{E})\big|\big) < 1. \tag{3.13}$$

In particular,

$$\rho\big(L_{\gamma\omega}(D, E) \leqslant \rho\big(\big|L_{\gamma\omega}(D, E)\big|\big) < 1; \tag{3.14}$$

consequently, the GAOR method for $\tilde{A} \in \Omega(\Lambda(A))$ (*in particular, for* A) *converges.*

Proof. Since $\tilde{A} \in \Omega(\Lambda(A))$ and $\tilde{D}^{-1}D_{\tilde{A}} = D^{-1}D_A$, from (3.6) we have

$$\rho\big(|\tilde{D}^{-1}\tilde{B}|\big) = \rho(|D^{-1}B|) = \rho < 1. \tag{3.15}$$

Noticing

$$\gamma_M = \sup\{\gamma : (\gamma, \omega) \in S(A, D)\} = \frac{1+\rho}{2\rho}$$

and observing that $0 \leqslant \gamma|\tilde{D}^{-1}\tilde{E}| \leqslant \gamma|\tilde{D}^{-1}\tilde{B}|$ with $\gamma > 0$, we have

$$\rho\big(\gamma|\tilde{D}^{-1}\tilde{E}|\big) \leqslant \gamma_M \, \rho\big(|\tilde{D}^{-1}\tilde{B}|\big) \leqslant \frac{1+\rho}{2\rho}\rho < 1,$$

and then, from Theorem 3.8 in [11],

$$\big(I - \gamma|\tilde{D}^{-1}\tilde{E}|\big)^{-1} \geqslant 0.$$

Hence

$$|L_{\gamma\omega}| \leqslant \big(I - |\gamma|\,|\tilde{D}^{-1}\tilde{E}|\big)^{-1}\Big[|1-\omega|I + |\omega - \gamma|\,|\tilde{D}^{-1}\tilde{E}| + \omega|\tilde{D}^{-1}\tilde{F}|\Big].$$

Let

$$M = \frac{1}{1 - |1 - \omega|}\left(I - |\gamma|\,|\tilde{D}^{-1}\tilde{E}|\right),$$

$$N = \frac{1}{1 - |1 - \omega|}\left(|1 - \omega| I + |\omega - \gamma|\,|\tilde{D}^{-1}\tilde{E}| + \omega|\tilde{D}^{-1}\tilde{F}|\right).$$

Then

$$M - N = I - \frac{1}{1 - |1 - \omega|}\left[(|\gamma| + |\omega - \gamma|)|\tilde{D}^{-1}\tilde{E}| + \omega|\tilde{D}^{-1}\tilde{F}|\right].$$

For $(\gamma, \omega) \in S_3$, we have that $[(2\gamma - \omega)/\omega]\rho < 1$, $0 < \omega \leqslant \gamma$, and $\omega \leqslant 1$, and then

$$0 \leqslant \frac{1}{1 - |1 - \omega|}\left[(|\gamma| + |\omega - \gamma|)|\tilde{D}^{-1}\tilde{E}| + \omega|\tilde{D}^{-1}\tilde{F}|\right]$$

$$= \frac{2\gamma - \omega}{\omega}|\tilde{D}^{-1}\tilde{E}| + |\tilde{D}^{-1}\tilde{F}| \leqslant \frac{2\gamma - \omega}{\omega}|\tilde{D}^{-1}\tilde{B}|$$

and

$$\rho\left(\frac{2\gamma - \omega}{\omega}|\tilde{D}^{-1}\tilde{E}| + |\tilde{D}^{-1}\tilde{F}|\right) \leqslant \frac{2\gamma - \omega}{\omega}\rho < 1.$$

Notice that $|L_{\gamma\omega}| \leqslant M^{-1}N$ and hence, from the convergence theorem for regular splittings [11], we have

$$\rho(L_{\gamma\omega}) \leqslant \rho(|L_{\gamma\omega}|) \leqslant \rho(M^{-1}N) < 1.$$

For $(\gamma, \omega) \in S_4$ there exists $(\gamma, \omega^*) \in S_3$, with $\omega^* > \omega$. Since (γ, ω^*)

$\in S_3$, we have $\rho(|L_{\gamma\omega^*}|) < 1$. Notice that

$$L_{\gamma\omega} = \frac{\omega}{\omega^*} L_{\gamma\omega^*} + \left(1 - \frac{\omega}{\omega^*}\right) I,$$

$$|L_{\gamma\omega}| \leqslant \frac{\omega}{\omega^*} |L_{\gamma\omega^*}| + \left(1 - \frac{\omega}{\omega^*}\right) I;$$

it follows that $\rho(L_{\gamma\omega}) \leqslant \rho(|L_{\gamma\omega}|) < 1$.

Similarly for $(\gamma, \omega) \in S_i$, $i = 0, 1, 2$, we can prove that the last inequality holds, proving the theorem. ■

From Basic Theorem B and Theorems 3.1–3.4 we can obtain further results for the convergence of the GAOR method when A (or $\tilde{A}$) is a complex H-matrix. Here $|D| \geqslant |D_A|$ is not required, and the region of convergence of the GAOR method is improved.

THEOREM 3.5. *Suppose that the hypotheses of any one of Theorems 3.1–3.3 hold. Then for any* $(\gamma, \omega) \in S(A, D)$ *we have that* (3.14) *holds and consequently the GAOR iterative method for* A *converges, provided that* $B = E + F$ *satisfies* $|B| = |E| + |F|$.

REMARK. Theorem 3.5 implies Theorem 3 in [5].

THEOREM 3.6. *Suppose that the hypotheses of Theorem* 3.4 *hold. Then for any* $(\gamma, \omega) \in S(A, D)$ *we have that* (3.13) *holds and consequently the GAOR iterative method for* $\tilde{A} \in \Omega(\Lambda(A))$ *converges, provided that* $\tilde{B} = \tilde{E} + \tilde{F}$ *satisfies* $|\tilde{B}| = |\tilde{E}| + |\tilde{F}|$.

REMARK. Similarly to the remark after Theorem 3.4, Theorem 3.5 is a special case of Theorem 3.6. Also, Theorem 3.6 includes Theorem 5 in [5]. If $D = D_A$, then Theorem 3.6 gives results for the convergence of the AOR method, extending similar previous results (see [1], [8], and [12–15], for example). For comparison we state the following result.

COROLLARY 3.6.1. *Suppose that* A *is a nonsingular* H*-matrix. Then for any* $(\gamma, \omega) \in S(A, D_A)$ *the AOR iterative method converges for* $\tilde{A} \in \Omega(\Lambda(A))$.

REMARK. From (3.11), Corollary 3.6.1 extends the previous results for the AOR method (see e.g. [1] and [10]).

Finally we give a set of sufficient and necessary conditions for H-matrices and diagonally dominant matrices in connection with the GJ and GAOR methods.

THEOREM 3.7. *Let A be a complex matrix, then the following conditions are equivalent:*

(i) *A is a nonsingular H-matrix;*

(ii) $\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho(|I - \tilde{D}^{-1}D_{\tilde{A}}| + |\tilde{D}^{-1}B_{\tilde{A}}|) < 1$ *for any* $\tilde{A} = (\tilde{a}_{ij}) = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ *with any* $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n)$ *satisfying* $(\tilde{a}_{11}/\tilde{d}_1, \tilde{a}_{22}/\tilde{d}_2, \dots, \tilde{a}_{nn}/\tilde{d}_n) \in S_{D_{\tilde{A}}^{-1}B_{\tilde{A}}}$;

(iii) $\rho(D_0^{-1}B_0) \leqslant \rho(|I - D_0^{-1}D_{A_0}| + |D_0^{-1}B_{A_0}|) < 1$ *for some* $A_0 = (a_{ij}^0) = D_0 - B_0 \in \Omega(\Lambda(A))$ *with some* $D_0 = \operatorname{diag}(d_1^0, d_2^0, \dots, d_n^0)$ *satisfying* $a_{ii}^0/d_i^0 > 0\ \forall i$, *in which case* $(a_{11}^0/d_1^0, a_{22}^0/d_2^0, \dots, a_{nn}^0/d_n^0) \in S_{D_A^{-1}B_A}$ *necessarily holds;*

(iv) $\rho(\tilde{D}^{-1}\tilde{B}) < 1$ *for any* $\tilde{A} = (\tilde{a}_{ij}) = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ *with any* $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n)$ *satisfying* $(\tilde{a}_{11}/\tilde{d}_1, \tilde{a}_{22}/\tilde{d}_2, \dots, \tilde{a}_{nn}/\tilde{d}_n) \in S_{D_{\tilde{A}}^{-1}B_{\tilde{A}}}$;

(v) $\rho(D_0^{-1}B_0) < 1$ *for some L-matrix* $A_0 = D_0 - B_0 \in \Omega(\Lambda(A))$ *with some* $D_0 = \operatorname{diag} D_0$ *satisfying* $0 < D_0^{-1}D_{A_0} \leqslant I$;

(vi) $\rho(|\tilde{D}^{-1}\tilde{E}|) < 1$ *and* $\rho(L_{\gamma\omega}(\tilde{D}, \tilde{E})) < 1$ *for any* $\tilde{A} = (\tilde{a}_{ij}) = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ *with any* $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n)$ *satisfying* $(\tilde{a}_{11}/\tilde{d}_1, \tilde{a}_{22}/\tilde{d}_2, \dots, \tilde{a}_{nn}/\tilde{d}_n) \in S_{D_{\tilde{A}}^{-1}B_{\tilde{A}}}$ *and with any* $\tilde{B} = \tilde{E} + \tilde{F}$ *satisfying* $|\tilde{B}| = |\tilde{E}| + |\tilde{F}|$, *and for any* $(\gamma, \omega) \in S(A, D)$ *with D satisfying* $D^{-1}D_A = \tilde{D}^{-1}D_{\tilde{A}}$;

(vii) $\rho(D_0^{-1}E_0) < 1$ *and* $\rho(L_{\gamma\omega}(D_0, E_0)) < 1$ *for some L-matrix* $A_0 = D_0 - B_0 \in \Omega(\Lambda(A))$ *with some* $D_0 = \operatorname{diag} D_0$ *satisfying* $0 < D_0^{-1}D_{A_0} \leqslant I$ *and with* $B_0 = E_0 + F_0$ *satisfying* $E_0 \geqslant 0$ *and* $F_0 \geqslant 0$, *and for some* (γ, ω) *with* $0 \leqslant \gamma \leqslant 1$ *and* $0 < \omega \leqslant 1$.

Proof. Suppose that (i) holds. From Theorem 3.4 it follows that (i) implies (ii) and (iv). Clearly, (ii) and (iv) imply, respectively, (iii) and (v).

Assume that (iii) holds. Then $a_{ii}^0/d_i^0 > 0\ \forall i$ and D_{A_0} is nonsingular. Notice that

$$D_0^{-1}B_0 = I - D_0^{-1}D_{A_0} + \left(D_0^{-1}D_{A_0}\right)\left(D_{A_0}^{-1}B_{A_0}\right). \tag{3.16}$$

Let $\Omega = D_0^{-1}D_{A_0} = \operatorname{diag}(a_{11}^0/d_1^0, a_{22}^0/d_2^0, \dots, a_{nn}^0/d_n^0)$, $T = D_{A_0}^{-1}B_{A_0}$, and $T_\Omega = D_0^{-1}B_0$. Then from Theorem 2.7 we have that (iii) implies

$$\rho(|T|) = \rho\left(|D_{A_0}^{-1}B_{A_0}|\right) < 1 \quad \text{and} \quad \left(\frac{a_{11}^0}{d_1^0}, \frac{a_{22}^0}{d_2^0}, \dots, \frac{a_{nn}^0}{d_n^0}\right) \in S_{D_{A_0}^{-1}B_{A_0}}.$$

Since $A_0 \in \Omega(\Lambda(A))$, from Lemma 3.1 we have

$$D_{A_0}^{-1}B_{A_0} \in \Omega\big(\Lambda\big(D_A^{-1}B_A\big)\big) \tag{3.17}$$

and

$$\rho\big(|D_A^{-1}B_A|\big) = \rho\big(|D_{A_0}^{-1}B_{A_0}|\big) < 1.$$

Then A is a nonsingular H-matrix and, from Lemma 2.5, $(a_{11}^0/d_1^0, a_{22}^0/d_2^0, \dots, a_{nn}^0/d_n^0) \in S_{D_A^{-1}B_A}$, proving that (iii) implies (i). Thus (i), (ii), and (iii) are equivalent.

Suppose that condition (v) holds. Then we have $D_{A_0} > 0$ nonsingular and $D_{A_0}^{-1}B_{A_0} \geqslant 0$, since A_0 is an L-matrix. Let $\Omega = D_0^{-1}D_{A_0}$ and $T = D_{A_0}^{-1}B_{A_0}$. Then, from (3.16) and $0 < D_0^{-1}D_{A_0} \leqslant I$, we have $\Omega > 0$, $T \geqslant 0$, and $D_0^{-1}B_0 \geqslant 0$. Then since $\rho(D_0^{-1}B_0) < 1$, from Theorem 2.6 it follows that $\rho(D_{A_0}^{-1}B_{A_0}) < 1$. Also, since $A_0 \in \Omega(\Lambda(A))$, from Lemma 3.1 we have (3.17) and $\rho(|D_A^{-1}B_A|) = \rho(|D_{A_0}^{-1}B_{A_0}|) = \rho(D_{A_0}^{-1}B_{A_0}) < 1$, since A_0 is an L-matrix. So A is a nonsingular H-matrix, proving that (v) implies (i). Thus (i), (iv), and (v) are equivalent.

Suppose that (i) holds. Let $\tilde{A}$, $\tilde{D}$, $\tilde{B}$, and D be given as in (vi). Since $D^{-1}D_A = \tilde{D}^{-1}D_{\tilde{A}}$ and $(\tilde{a}_{11}/\tilde{d}_1, \tilde{a}_{22}/\tilde{d}_2, \dots, \tilde{a}_{nn}/\tilde{d}_n) \in S_{D_A^{-1}B_A}$, we have $(a_{11}/d_1, a_{22}/d_2, \dots, a_{nn}/d_n) \in S_{D_A^{-1}B_A}$. Combining with Theorem 3.2, we have $\rho(|D^{-1}B|) < 1$. Then from Basic Theorem B, $\rho(L_{\gamma\omega}(\tilde{D}, \tilde{E})) \leqslant \rho(|L_{\gamma\omega}(\tilde{D}, \tilde{E})|) < 1$, since $\tilde{A} = \tilde{D} - \tilde{B} \in \Omega(\Lambda(A))$ satisfies (3.12) and $(\gamma, \omega) \in S(A, D)$. In particular, when $(\gamma, \omega) = (0, 1)$ we have $\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho(|\tilde{D}^{-1}\tilde{B}|) < 1$, and then $\rho(|\tilde{D}^{-1}\tilde{E}|) \leqslant \rho(|\tilde{D}^{-1}\tilde{B}|) < 1$, since $|\tilde{D}^{-1}\tilde{E}| \leqslant |\tilde{D}^{-1}\tilde{E}| + |\tilde{D}^{-1}\tilde{F}| = |\tilde{D}^{-1}\tilde{B}|$, proving that (i) implies (vi). Clearly (vi) implies (vii).

Suppose that (vii) holds. Then $D_0 = \operatorname{diag} D_0 > 0$ and D_0^{-1} exists, since $D_0^{-1}D_{A_0} > 0$ and A_0 is an L-matrix. Notice that for $0 \leqslant \gamma \leqslant 1$ and $0 < \omega \leqslant 1$

$$L_{\gamma\omega}(D_0, E_0) = (1-\omega)I + \omega\big(I - \gamma D_0^{-1}E_0\big)^{-1} \times\big[(1-\gamma)D_0^{-1}E_0 + D_0^{-1}F_0\big] \geqslant 0$$

and

$$\big(I - \gamma D_0^{-1}E_0\big)^{-1}\big[(1-\gamma)D_0^{-1}E_0 + D_0^{-1}F_0\big] \geqslant 0,$$

since $\rho(D_0^{-1}E_0) < 1$ and $D_0^{-1}E_0 \geqslant 0$ with $D_0^{-1}F_0 \geqslant 0$. From Corollary 2.6.1,

$$\rho\Big(\big(I - \gamma D_0^{-1}E_0\big)^{-1}\big[(1-\gamma)D_0^{-1}E_0 + D_0^{-1}F_0\big]\Big) < 1.$$

From the Stein-Rosenberg theorem [7], $\gamma D_0^{-1}E_0 + (1-\gamma)D_0^{-1}E_0 + D_0^{-1}F_0 = D_0^{-1}B_0 \geqslant 0$ is convergent, i.e., $\rho(D_0^{-1}B_0) < 1$. Letting $\Omega = D_0^{-1}D_{A_0}$ and $T = D_{A_0}^{-1}B_{A_0}$, from Theorem 2.6 and (3.16) it follows that $\rho(D_{A_0}^{-1}B_{A_0}) < 1$, since $\Omega > 0$, $T \geqslant 0$, and $D_0^{-1}B_0 \geqslant 0$. From Lemma 3.1 we have $\rho(|D_A^{-1}B_A|) = \rho(|D_{A_0}B_{A_0}|) = \rho(D_{A_0}B_{A_0}) < 1$, since A_0 is an L-matrix, proving that (i) follows. Thus the theorem is proved. ∎

For comparison we give the following results in connection with the Jacobi, JOR, Gauss-Seidel, SOR, and AOR methods. For short, define $J_1 := D_A^{-1}B_A$, $J_\omega[A] := (1-\omega)I + \omega D_A^{-1}B_A$, $\mathscr{L}_1[A] = (D_A - E_A)^{-1}F_A$, $\mathscr{L}_\omega[A] = (D_A - \omega E_A)^{-1}[(1-\omega)D_A + \omega F_A]$.

COROLLARY 3.7.1. *Let* $A = D_A - B_A = D_A - E_A - F_A$ *be a complex matrix where* E_A *and* F_A *are respectively the strictly lower and upper triangular parts of* $-A$. *Then the following conditions are equivalent*:

(i) A *is a nonsingular H-matrix*;

(ii) $\rho(J_\omega[\tilde{A}]) \leqslant |1-\omega| + \omega\rho(|J_1|) < 1$ *for any* $\tilde{A} \in \Omega(\Lambda(A))$ *and any* $\omega \in (0, 2/[1+\rho(|J_1|)])$;

(iii) $\rho(J_\omega[A_0]) \leqslant |1-\omega| + \omega\rho(|J_1|) < 1$ *for some* $A_0 \in \Omega(\Lambda(A))$ *and some* $\omega > 0$, *in which case* $\omega \in (0, 2/[1+\rho(|J_1|)])$ *necessarily holds*;

(iv) $\rho(J_\omega[\tilde{A}] < 1$ *for any* $\tilde{A} \in \Omega(\Lambda(A))$ *and any* $\omega \in (0, 2/[1+\rho(|J_1|)])$;

(v) $\rho(J_\omega[A_0]) < 1$ *for some L-matrix* $A_0 \in \Omega(\Lambda(A))$ *and some* $\omega \in (0, 1]$;

(vi) $\rho(L_{\gamma\omega}(D_{\tilde{A}}, E_{\tilde{A}})) < 1$ *for any* $\tilde{A} \in \Omega(\Lambda(A))$ *and any* $(\gamma, \omega) \in S(A, D_A)$;

(via) $\rho(\mathscr{L}_\omega[\tilde{A}]) < 1$ *for any* $\tilde{A} \in \Omega(\Lambda(A))$ *and any* $\omega \in (0, 2/[1+\rho(|J_1|)])$;

(vii) $\rho(L_{\gamma\omega}(D_{A_0}, E_{A_0})) < 1$ *for some L-matrix* $A_0 \in \Omega(\Lambda(A))$ *and some* (γ, ω) *with* $0 \leqslant \gamma \leqslant 1$ *and* $0 < \omega \leqslant 1$;

(viia) $\rho(\mathscr{L}_1[A_0]) < 1$ *for some L-matrix* $A_0 \in \Omega(\Lambda(A))$.

Proof. Clearly, when $D_0 = (1/\omega)D_{A_0}$, conditions (iii), (v), and (vii) of Theorem 3.7 become those of Corollary 3.7.1, and when $D_0 = D_{A_0}$ and

$\gamma = \omega = 1$, condition (vii) of Theorem 3.7 becomes condition (viia) of Corollary 3.7.1. Then from Theorem 3.7, conditions (i), (iii), (v), (vii), and (viia) of Corollary 3.7.1 are equivalent. Notice that, in Corollary 3.7.1, condition (vi) implies condition (via), which implies condition (viia), while conditions (ii), (iv), and (vi) of Theorem 3.7 imply respectively those of Corollary 3.7.1, which imply respectively (iii), (v), and (vii) of Corollary 3.7.1. From these results with Theorem 3.7 the corollary follows immediately. ∎

REMARK. Theorem 3.7 and its corollary greatly extend the previous results for the GAOR and AOR methods [8, 9, 12, 15].

THEOREM 3.8. *Let A be a complex matrix. Then the following conditions are equivalent*:

(i) *A is a strictly diagonally dominant matrix by rows*;

(ii) $\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho(|\tilde{D}^{-1}\tilde{B}|) \leqslant \| \, |I - \tilde{D}^{-1}D_{\tilde{A}}| + |\tilde{D}^{-1}B_{\tilde{A}}| \, \| < 1$ *for any* $\tilde{A} = (\tilde{a}_{ij}) = \tilde{D} - \tilde{B} \in \Omega(A)$ *with any* $\tilde{D} = \mathrm{diag}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n)$ *satisfying* $0 < \tilde{a}_{ii}/\tilde{d}_i < 2|a_{ii}|/\Sigma_j |a_{ij}| \ \forall i$;

(iia) $\rho(J_\omega[\tilde{A}]) \leqslant \rho(|J_\omega[\tilde{A}]|) \leqslant |1 - \omega| + \omega\|J_1\| < 1$ *for any* $\tilde{A} \in \Omega(A)$ *with any* $\omega \in (0, \min_i 2|a_{ii}|/\Sigma_j |a_{ij}|)$;

(iii) $\rho(D_0^{-1}B_0) \leqslant \rho(|D_0^{-1}B_0|) \leqslant \| \, |I - D_0^{-1}D_{A_0}| + |D_0^{-1}B_{A_0}| \, \| < 1$ *for some* $A_0 = (a_{ij}^0) = D_0 - B_0 \in \Omega(A)$ *with some* $D_0 = \mathrm{diag}(d_1^0, d_2^0, \dots, d_n^0)$ *satisfying* $a_{ii}^0/d_i^0 > 0 \ \forall i$, *in which case* $0 < a_{ii}^0/d_i^0 < 2|a_{ii}|/\Sigma_j |a_{ij}| \ \forall i$ *necessarily holds*;

(iiia) $\rho(J_\omega[A_0]) \leqslant \rho(|J_\omega[A_0]|) \leqslant |1 - \omega| + \omega\|J_1\| < 1$ *for some* $A_0 \in \Omega(A)$ *and some* $\omega > 0$, *in which case* $0 < \omega < \min_i 2|a_{ii}|/\Sigma_j |a_{ij}|$ *necessarily holds.*

Proof. From Theorem 3.4, condition (i) implies condition (ii). Clearly condition (ii) implies conditions (iia) and (iii), while condition (iia) implies condition (iiia).

From Theorem 2.8 conditions (iii) and (iiia) respectively imply $0 < a_{ii}^0/d_i^0 < 2|a_{ii}|/\Sigma_j |a_{ij}| \ \forall i$ or $0 < \omega < \min_i 2|a_{ii}|/\Sigma_j |a_{ij}|$, while $\|D_{A_0}^{-1}B_{A_0}\| < 1$ if $\Omega = D_0^{-1}D_{A_0}$ or ωI, and $T = D_{A_0}^{-1}B_{A_0}$. From $A_0 \in \Omega(A)$ we have $|D_A^{-1}B_A| = |D_{A_0}^{-1}B_{A_0}|$ and then $\|D_A^{-1}B_A\| < 1$, that is, A is a strictly diagonally dominant matrix by rows, proving that conditions (iii) and (iiia) respectively imply condition (i). Then the theorem is proved. ∎

Similarly, from Theorems 3.4 and 2.9 we have the following results with the irreducible diagonal dominance. For short we define the condition (C) for i with b_i:

(C) there exists at least one i such that both

$$b_i < \frac{2|a_{ii}|}{\sum_j |a_{ij}|} \quad \text{and} \quad \sum_{j \neq i} |a_{ij}| < |a_{ii}|$$

hold simultaneously.

THEOREM 3.9. *Let A be an irreducible complex matrix. Then the following conditions are equivalent:*

(i) *A is an irreducibly diagonally dominant matrix by rows;*

(ii) $\rho(\tilde{D}^{-1}\tilde{B}) \leqslant \rho(|\tilde{D}^{-1}\tilde{B}|) \underset{<}{\leqslant} \| |I - \tilde{D}^{-1}D_{\tilde{A}}| + |\tilde{D}^{-1}B_{\tilde{A}}| \| \underset{\leqslant}{<} 1$ *for any* $\tilde{A} = (\tilde{a}_{ij}) = \tilde{D} - \tilde{B} \in \Omega(A)$ *with any* $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n)$ *satisfying* $0 < \tilde{a}_{ii}/\tilde{d}_i \leqslant 2|a_{ii}|/\Sigma_j |a_{ij}|$ $\forall i$ *if the condition (C) for i with $b_i = \tilde{a}_{ii}/\tilde{d}_i$ is satisfied;*

(iia) $\rho(J_\omega[\tilde{A}]) \leqslant \rho(|J_\omega[\tilde{A}]|) \underset{<}{\leqslant} |1 - \omega| + \omega \|J_1\| \underset{\leqslant}{<} 1$ *for any* $\tilde{A} \in \Omega(A)$ *with any* $\omega \in (0, \min_i 2|a_{ii}|/\Sigma_j |a_{ij}|)$ *if the condition (C) for i with $b_i = \omega$ is satisfied;*

(iii) $\rho(D_0^{-1}B_0) \leqslant \rho(|D_0^{-1}B_0|) \underset{<}{\leqslant} \| |I - D_0^{-1}D_{A_0}| + |D_0^{-1}B_{A_0}| \| \underset{\leqslant}{<} 1$ *for some* $A_0 \in \Omega(A)$ *with some* $D_0 = \operatorname{diag}(d_1^0, d_2^0, \ldots, d_n^0)$ *satisfying* $a_{ii}^0/d_i^0 > 0$ $\forall i$, *in which case* $0 < a_{ii}^0/d_i^0 \leqslant 2|a_{ii}|/\Sigma_j |a_{ij}|$ $\forall i$ *necessarily holds if the condition (C) for i with $b_i = a_{ii}^0/d_i^0$ is satisfied;*

(iiia) $\rho(J_\omega[A_0]) \leqslant \rho(|J_\omega[A_0]|) \underset{<}{\leqslant} |1 - \omega| + \omega \|J_1\| \underset{\leqslant}{<} 1$ *for some* $A_0 \in \Omega(A)$ *with some* $\omega > 0$, *in which case* $0 < \omega \leqslant \min_i 2|a_{ii}|/\Sigma_j |a_{ij}|$ *necessarily holds if the condition (C) for i with $b_i = \omega$ is satisfied.*

APPENDIX

THEOREM [6, Theorem 1]. *Let $A = (a_{ij})$ be an $n \times n$ irreducible complex matrix. Then there exists some $\tilde{A} \overset{\sim}{\sim} A$ such that $\Sigma_j |\tilde{a}_{ij}| = \rho(|A|)$ $\forall i$, where $\tilde{A} = (\tilde{a}_{ij})$.*

Proof. Since A is irreducible, from the Perron-Frobenius theorem it follows that for the nonnegative matrix $|A|$ there exists a real positive eigenvector $q^T = (q_1, q_2, \ldots, q_n)$ such that $|A|q = \rho(|A|)q$. Without loss of generality, assume $q^T q = 1$. Let $Q = \mathrm{diag}(q_1, q_2, \dots, q_n)$. Then

$$|A|Q(1,1,\dots,1)^T = |A|q = \rho(|A|)q = \rho(|A|)Q(1,1,\dots,1)^T,$$

where $(1,1,\dots,1)^T$ is an n-dimensional vector whose components are all one. Hence

$$Q^{-1}|A|Q(1,1,\dots,1)^T = \rho(|A|)(1,1,\dots,1)^T.$$

Let $\tilde{A} = (\tilde{a}_{ij}) = Q^{-1}AQ$. Then we have $\sum_j |\tilde{a}_{ij}| = \rho(|A|)$ and the theorem is proved. ■

THEOREM [6, Theorem 2]. *Let $A = (a_{ij})$ be an $n \times n$ complex matrix. Then for any given $\varepsilon > 0$ there exists some $\tilde{A} \hat{\sim} A$ such that $\|\tilde{A}\| < \rho(|A|) + \varepsilon$.*

Proof. When A is irreducible the result trivially holds from Theorem 1 in [6]. Now suppose that A is reducible. First assume that $\varepsilon > 0$ is arbitrarily given. For any matrix $\hat{A} = |A| + \delta I$ where $\delta > 0$ and I is the identity matrix, there exists a permutation matrix P such that the matrix PAP^T is partitioned as follows:

$$P\hat{A}\,P^T = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1} \\ & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ & & \ddots & \vdots \\ 0 & & & \hat{A}_{pp} \end{pmatrix},$$

where the diagonal submatrix $\hat{A}_{tt}$ is an $n_t \times n_t$ irreducible square matrix, $1 \leqslant t \leqslant p$, $\sum_t n_t = n$. From Theorem 1 in [6], it follows that there exist p nonsingular diagonal matrices Q_t, $t = 1, 2, \dots, p$, with positive diagonal elements such that all row sums of the matrix $Q_t^{-1}\hat{A}_{tt}Q_t$ are equal to $\rho(\hat{A}_{tt})$, $t = 1, 2, \dots, p$. Since P is a permutation matrix, we have $P^T = P^{-1}$, $P\hat{A}\,P^T \sim \hat{A}$, $P\hat{A}\,P^T \geqslant 0$, and $\rho(PAP^T) = \rho(\hat{A}) = \max_t \rho(\hat{A}_{tt})$. Letting the diagonal matrix

$$\tilde{Q} = \begin{pmatrix} k^{p-1}Q_1 & & & \\ & k^{p-2}Q_2 & & \\ & & \ddots & \\ & & & Q_p \end{pmatrix},$$

where $k > 0$ is an undetermined coefficient, we have

$$\tilde{Q}^{-1}P\hat{A}\,P^T\tilde{Q} = \begin{pmatrix} Q_1^{-1}\hat{A}_{11}Q_1 & \frac{1}{k}Q_1^{-1}\hat{A}_{12}Q_2 & \cdots & \frac{1}{k^{p-1}}Q_1^{-1}\hat{A}_{1p}Q_p \\ & Q_2^{-1}\hat{A}_{22}Q_2 & \cdots & \frac{1}{k^{p-2}}Q_2^{-1}\hat{A}_{2p}Q_p \\ & & \ddots & \vdots \\ & & & Q_p^{-1}\hat{A}_{pp}Q_p \end{pmatrix}.$$

Let M_{ij} be the sum of the values of all the elements of the matrix $Q_i^{-1}A_{ij}Q_j$, $1 \leqslant i \leqslant j \leqslant p$, and $M = \max_{i,j} M_{ij} + \varepsilon$, where ε is arbitrarily given in advance. Take $k > PM/\varepsilon$, implying $k > 1$. If $U = (u_{ij}) \equiv \tilde{Q}^{-1}P\hat{A}\,P^T\tilde{Q}$, then $U \geqslant 0$ and for any i there exists some t such that u_{ii} is a diagonal element of $Q_t^{-1}\hat{A}_{tt}Q_t$ and

$$\sum_j u_{ij} \leqslant \rho\left(Q_t^{-1}\hat{A}_{tt}Q_t\right) + \frac{pM}{k} < \rho(P\hat{A}\,P^T) + \varepsilon$$

$$= \rho(\hat{A}) + \varepsilon = \rho(|A|) + \delta + \varepsilon \qquad \forall i.$$

Then letting $V = (v_{ij}) = P^TUP$, we have $V \geqslant 0$ and

$$\sum_j v_{ij} < \rho(\hat{A}) + \varepsilon = \rho(|A|) + \delta + \varepsilon \qquad \forall i.$$

Observing that $V = P^TUP = (p^T\tilde{Q}P)^{-1}\hat{A}(P^T\tilde{Q}P)$ and letting the diagonal matrix $Q = P^T\tilde{Q}P$ and the similar matrix $\tilde{A} = (\tilde{a}_{ij}) = Q^{-1}AQ$, we have

$$|\tilde{A}| = (\tilde{a}_{ij}) = Q^{-1}|A|Q = Q^{-1}(\hat{A} - \delta I)Q = V - \delta I$$

and hence $\sum_j |\tilde{a}_{ij}| = \sum_j v_{ij} - \delta < \rho(|A|) + \delta + \varepsilon - \delta = \rho(|A|) + \varepsilon\ \forall i$. The theorem is proved. ∎

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