On real number labelings and graph invertibility

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A R T I C L E   I N F O

Article history:
Received 27 July 2011
Received in revised form 3 May 2012
Accepted 17 May 2012
Available online 19 June 2012

Keywords:
Distance-constrained labeling
Self-complementary graphs
Kneser graphs
λ -invertible
λ r 1 -labeling
λ s 1 -labeling

A B S T R A C T

For non-negative real x0 and simple graph G, λ x0,1(G) is the minimum span over all labelings that assign real numbers to the vertices of G such that adjacent vertices receive labels that differ by at least x0 and vertices at distance two receive labels that differ by at least 1. In this paper, we introduce the concept of λ -invertibility: G is λ -invertible if and only if for all positive x, λ x,1(G) = xλ 1,1(G'). We explore the conditions under which a graph is λ -invertible, and apply the results to the calculation of the function λ x,1(G) for certain λ -invertible graphs G. We give families of λ -invertible graphs, including certain Kneser graphs, line graphs of complete multipartite graphs, and self-complementary graphs. We also derive the complete list of all λ -invertible graphs with maximum degree 3.

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1. Introduction

For simple finite graph G and non-negative real numbers r 1, r 2, . . . , r k, an L(r 1, r 2, . . . , r k)-labeling of G is a function L from the vertex set V (G) into the non-negative reals such that, for 1 ≤ i ≤ k, the absolute difference between L(u) and L(v) is at least ri if u and v are at distance i. The span of L, denoted as sp(L), is the absolute difference between the largest and smallest labels assigned by L, and the smallest such span over all L(r 1, r 2, . . . , r k)-labelings of G is called the λ r 1, . . . , r k-number of G, denoted as λ r 1, . . . , r k(G).

Commonly called distance-constrained vertex labelings, such functions arose in the literature as the graph-theoretic outgrowth of Hale’s channel assignment problem [14] in which we seek the shortest possible interval of frequencies (represented by vertex labels) from which to make assignments to transmitters (represented by vertices) subject to the interference-reducing constraint that the smaller the distance between two transmitters, the greater the minimum difference between their assigned transmission frequencies. It is thus natural that the most common considerations of distance-constrained vertex labelings of G have required r 1 ≥ r 2 ≥ . . . ≥ r k, and have focussed on the derivation of λ r 1, . . . , r k(G). It is equally natural that the history of results on distance-constrained labelings is reflective of increasingly general conditions on r 1, r 2, . . . , r k.

The seminal paper of Griggs and Yeh [13] dealt particularly with L(2r, r)-labelings for positive real r, which are equivalent to L(2, 1)-labelings due to the fact that L is an L(r 1, r 2, . . . , r k)-labeling with span sp(L) if and only if L is an L(c r 1, c r 2, . . . , c r k)-labeling with span c · sp(L). Therein, the authors derived λ 2,1(G) for G in certain classes of graphs and
cited various results useful in the establishment of $\lambda_{2,1}(G)$, including relationships between $\lambda_{2,1}(G)$ and $\Delta(G)$. Continuing the study of $L(2,1)$-labelings, Georges, Mauro, and Whittlesey [8] considered the relationship between $\lambda_{2,1}(G)$ and the path-covering number $q(G')$ of $G'$, showing in particular that $\lambda_{2,1}(G) = |V(G)| + q(G') - 2$ if $q(G') \geq 2$. Shortly thereafter, Georges and Mauro [5] studied the more general $L(j,k)$-labelings for integers $1 \leq k \leq j$, deriving a canonical form for $\lambda_{j,k}(G)$ and the $\lambda_{j,k}$-number of graphs in certain classes including cycles, paths, joins, and products. They and other authors have since considered the $\lambda_{j,k}$-number of $G$ in additional classes as well as the behavior of $\lambda_{j,k}(G)$ as it relates to graph invariants including clique-covering number, packing number, domination number, maximum degree, diameter, chromatic number, and chromatic index; for surveys, see [1,12,19].

Extending the depth to which distance constraints are set, various authors have considered particular $L(r_1, r_2, \ldots, r_k)$-labelings for which $k = \text{dia}(G)$ and $r_i$ is the integer $\text{dia}(G) - i + 1$. Such a labeling is called a radio labeling (see [2]). Yet recent works have relaxed even these conditions, specifying only that $r_1, r_2, \ldots, r_k$ are arbitrary non-negative reals and $2 \leq k \leq \text{dia}(G)$. Under these general conditions, Gregg and Jin [9] have shown that as a function of $(r_1, r_2, \ldots, r_k)$, $\lambda_{r_1, r_2, \ldots, r_k}(G)$ is continuous, non-decreasing, and piecewise linear with non-negative integer coefficients, thereby enabling new strategies for the derivation of minimum spans. These results have inspired investigations of $\lambda_{x,1}(G)$ for positive real $x$ and various $G$ (see [7,10,11,16]). Clearly, by the results in [9], such functions are non-decreasing, continuous, piecewise linear functions of the single variable $x$ on $[0, \infty)$.

This paper continues the development of techniques for the derivation of $\lambda_{x,1}$-numbers for arbitrary positive real $x$. Because the case $x < 1$ has been less well explored than the case $x \geq 1$, we aim to explore graphs $G$ for which there is a formulaic relationship between $\lambda_{x,1}(G)$ and $\lambda_{\frac{1}{x},1}(G')$, since knowledge of the latter for $\frac{1}{x} > 1$ will provide knowledge of the former for $x < 1$. To motivate the discussion, we will begin with a brief look at the manner in which a distance-constrained labeling $L$ of $G$ also serves as a distance-constrained labeling of $G'$. Suppose that $G$ is a connected graph that is not complete and suppose that $L$ is an $L(x,1)$-labeling of $G$. Since $V(G) = V(G')$, then $L$ is also an assignment of reals to the vertices of $G'$; moreover, if $r$ is the smallest difference $|L(v) - L(w)|$ such that $v$ and $w$ are distinct and not adjacent in $G$, then $L$ is an $L(r,x,x,\ldots,x)$-labeling of $G'$ where the vector $(r, x, x, \ldots, x)$ has length equal to the greatest diameter among the components of $G'$. Thus $\frac{1}{x}L$ is an $L(\frac{1}{x},1,1,\ldots,1)$-labeling of $G'$. We also note that if $G'$ is connected, $r \geq 1$, and $L$ is a $\lambda_{x,1}$-labeling of $G$ with span $\text{sp}(L)$, then $L$ is a $\lambda_{\frac{1}{x},x,x,x,\ldots,x}$-labeling of $G'$, implying $\lambda_{x,1}(G) = \lambda_{\frac{1}{x},1}(G')$. (To see this, suppose that some $L(r,x,x,\ldots,x)$-labeling $L'$ of $G'$ exists with span smaller than $\text{sp}(L)$. Then $L'$ is an $L(x,1)$-labeling of $G$ with span smaller than $\text{sp}(L)$, a contradiction.) We additionally note that if $G$ has diameter $2$, then $r \geq 1$.

The foundational definition of this paper is that of $\lambda$-invertible graph, as follows.

**Definition 1.1.** The simple graph $G$ is $\lambda$-invertible if and only if $\lambda_{x,1}(G) = x\lambda_{\frac{1}{x},1}(G')$ for all positive real $x$. □

We observe that $G$ is $\lambda$-invertible if and only if $G'$ is $\lambda$-invertible. Moreover, the two $\lambda$-invertible graphs with smallest orders are the self-complementary graphs $K_1$ and $P_4$ (since $\lambda_{x,1}(P_4) = x + 1$ for all positive real $x$, implying $\lambda_{x,1}(P_4) = x + 1 = x(\lambda_{\frac{1}{x},1}(P_4) = \lambda_{\frac{1}{x},1}(P_4)$; see [10]). In fact, $K_1$ is the only $\lambda$-invertible complete graph, which implies that if $G$ is a $\lambda$-invertible graph with diameter $d$, then $d \geq 2$ or $G$ is $K_1$. As we shall see later in Theorem 2.2, the Petersen graph is also $\lambda$-invertible.

In Section 2 of this paper, we will develop general properties of $\lambda$-invertible graphs, while Section 3 considers $\lambda$-invertibility of graphs of diameter 2. In Section 4, we study the relationship between $\lambda$-invertibility and maximum vertex degree, identifying all $\lambda$-invertible graphs of maximum degree 3. Sections 5 and 6 give additional examples of non-self-complementary and self-complementary invertible graphs, respectively.

Finally, we note that the continuity of $\lambda_{x,1}(G)$ on $[0, \infty)$ implies that for any interval of non-negatives $[a, b]$, $\lambda_{x,1}(G)$ on $[a, b]$ is completely determined by $\lambda_{j,k}(G)$ for positive integers $j, k$ such that $\frac{1}{k} \in (a, b)$. Thus, throughout the paper, we will move freely between $\lambda_{x,1}$-numbers and $\lambda_{j,k}$-numbers as clarity dictates.

2. General results on invertible graphs

**Theorem 2.1.** Let $G$ be a $\lambda$-invertible graph with order $n$. Then

1. $\lambda_{1,1}(G) = \lambda_{1,1}(G')$;
2. $G$ and $G'$ are connected;
3. $G$ has diameter 2 if and only if $G'$ has diameter 2;
4. the diameters of $G$ and $G'$ are each at most 3. Moreover, $\text{dia}(G) = 3$ if and only if $\text{dia}(G') = 3$.

**Proof.** (1) This follows immediately from Definition 1.1.

(2) Since it cannot be the case that both $G$ and $G'$ are disconnected, then with no loss of generality we assume to the contrary that $G$ is connected and $G'$ is disconnected. Since the disconnectedness of $G'$ implies that $G$ has diameter 2, it follows that $\lambda_{1,1}(G) = n - 1$. Moreover, since $G'$ is disconnected, $\lambda_{1,1}(G')$ is bounded from above by one fewer than the order of the largest component of $G'$, contradicting (1).
Theorem 3.1. Suppose $G$ is a graph with order $n$ and diameter $d$. Then $\lambda_{1,1}(G) = n - 1$. Hence by (1), $\lambda_{1,1}(G') = n - 1$, implying by (2) that $G'$ has diameter at most 2. But $G'$ cannot be the complete graph $K_n$ since no complete graph $K_n$ has diameter 2. A symmetric argument based on the $\lambda$-invertibility of $G'$ proves the result.

(4) With no loss of generality, suppose that $G$ has diameter at least 4, implying $n \geq 5$. Then the diameter of $G'$ is at most 2 (see [18, p. 76]). But since $G'$ is connected (by (2)) and $G'$ cannot be the complete graph $K_n$, it follows that $\text{diam}(G') = 2$, contradicting (3). Thus $\text{diam}(G) \leq 3$ and by a symmetric argument $\text{diam}(G') \leq 3$. But by (2) and the fact neither $G$ nor $G'$ can be the complete graph $K_n$, it follows that $G$ and $G'$ each have diameter 2 or 3. The result now follows from (3). □

**Theorem 2.2.** Let $G$ be a graph such that $G$ and $G'$ have diameter at most 2. Then $G$ is $\lambda$-invertible.

**Proof.** Select arbitrary $x > 0$ and let $L$ be a $\lambda_{x,1}$-labeling of $G$ with span $\text{sp}(L)$. Then due to the diameter assumptions, $L$ is an $L(1, x)$-labeling of $G'$ with span $\text{sp}(L)$, implying that $\frac{1}{x}L$ is an $L(1, \frac{1}{x})$-labeling of $G'$ with span $\text{sp}(L)$. Thus $\lambda_{1,1}(G') \leq \frac{\text{sp}(L)}{x}$.

Now suppose that $\lambda_{1,1}(G')$ is strictly less than $\frac{\text{sp}(L)}{x}$. Then we may find an $L(1, \frac{1}{x})$-labeling $L'$ of $G'$ with span $\lambda' s < \frac{\text{sp}(L)}{x}$. But by the diameter assumptions, $L'$ is an $L(1, \frac{1}{x})$-labeling of $G'$ with span $\lambda'$, implying that $xL'$ is an $L(x, 1)$-labeling of $G'$ with span $\lambda' s < \text{sp}(L)$, a contradiction. So $\lambda_{1,1}(G') = \frac{\text{sp}(L)}{x}$. We therefore have $\text{sp}(L) = \lambda_{x,1}(G') = x \frac{\text{sp}(L)}{x} = \lambda_{1,1}(G')$. □

In [Theorem 2.1], we established a necessary condition for the $\lambda$-invertibility of $G$ based upon the $\lambda_{1,1}$-number of $G$. In [Theorem 2.4], we establish a necessary condition for the $\lambda$-invertibility of $G$ based upon the $\lambda_{0.1}$-number of $G$.

**Definition 2.3.** Let $G$ be a graph. Then the clique-covering number $\omega(G)$ (respectively path-covering number $q(G)$) of $G$ is the cardinality of the smallest set $S$ of vertex-disjoint cliques (paths) such that each vertex of $G$ is incident to some clique (path) in $S$. □

**Theorem 2.4.** Let $G$ be a $\lambda$-invertible graph with order $n$. Then $\lambda_{0.1}(G) = \chi(G') - 1 = \omega(G) - 1$.

**Proof.** By the piecewise linearity results of Griggs and Jin [9], there exist real $x_0$ and non-negative integers $a$, $b$ such that for $x \geq x_0$, $\lambda_{1,1}(G') = ax + b$. Furthermore, due to the continuity results of [9], $\lambda_{0.1}(G) = \lim_{x \to 0} \lambda_{1,1}(G') = \lim_{x \to 0} x \lambda_{0.1}(G') = \lim_{x \to 0} \lambda_{0.1}(G') = \lim_{x \to 0} x \lambda_{1,1}(G').$ But by [5, see Theorems 2.10 and 5.5], $x(\chi(G') - 1) \leq \lambda_{0.1}(G') \leq x(\chi(G') - 1) + c$ where $c$ is a particular graph invariant (constant). So $x(\chi(G') - 1) \leq ax + b \leq x(\chi(G') - 1) + c$ for all $x \geq x_0$, implying $a = \chi(G') - 1$.

The remainder of the result follows since $\omega(G) = \chi(G')$. □

3. On invertible graphs with diameter 2

If $G$ is a graph with order $n$ and diameter 2, and if $L$ is an $L(j, k)$-labeling of $G$, then no two distinct vertices of $G$ receive the same label under $L$. Thus, $L$ induces a strictly increasing sequence of labels $a_1, a_2, a_3, \ldots, a_n$ such that $a_{i+1} - a_i \geq \min(j, k)$ for $1 \leq i \leq n - 1$. Moreover, we say that $L$ induces the sequence of vertices $s = <x_1, x_2, \ldots, x_n>$ if and only if $L(x_i) = a_i$ for each $i$, $1 \leq i \leq n$.

**Theorem 3.1.** Suppose $G$ is a graph with order $n$ and diameter 2 such that $G'$ has path-covering number $q \geq 1$. Then

1. $\lambda_{j,k}(G) \geq (q - 1)j + (n - q)k$ for $1 \leq \frac{j}{k} \leq 1$, and
2. $\lambda_{j,k}(G) \geq (q - 1)j + (n - q)k$ for $1 \leq \frac{j}{k} \leq 2$.

**Proof.** (1) Let $L$ be an $L(j, k)$-labeling of $G$ and let $s = <x_1, x_2, \ldots, x_n>$ be the sequence of vertices induced by $L$. Let $q_{ij}$ denote the number of distinct pairs $(x_i, x_{i+1})$ of consecutive components of $s$ that are not adjacent in $G'$. Since $a_{i+1} - a_i \geq j$ for such pairs of vertices and $a_{i+1} - a_i \geq k$ otherwise, then the span of $L$ is at least $q_{ij} + (n - 1 - q_{ij})k$. But $q_{ij} \geq q - 1$ and $\frac{j}{k} \geq 1$, implying the result.

(2) By (1), it suffices to find an $L(j, k)$-labeling of $G$ with span $(q - 1)j + (n - q)k$. Let $s = <x_1, x_2, \ldots, x_n>$ be a sequence of the $n$ distinct vertices of $G$ such that the number of distinct pairs $(x_i, x_{i+1})$ of consecutive components of $s$ that are not adjacent in $G'$ is $q - 1$. Let $L$ be defined recursively as

$$L(x_{i+1}) = \begin{cases} 0 & \text{if } i = 0, \\ L(x_i) + j & \text{if } i > 0 \text{ and } x_i \text{ and } x_{i+1} \text{ are not adjacent in } G', \\ L(x_i) + k & \text{if } i > 0 \text{ and } x_i \text{ and } x_{i+1} \text{ are adjacent in } G'. \end{cases}$$

Then since $\frac{j}{k} \leq 2$, it is easily checked that $L$ is an $L(j, k)$-labeling of $G$ with span $(q - 1)j + (n - q)k$, establishing (2). □
Corollary 3.2. Suppose $G$ is a graph with order $n$ such that $G$ and $G'$ have diameter 2. Suppose also that the path covering numbers of $G$ and $G'$ are $q_G$ and $q_{G'}$ respectively. Then

$$
\lambda_{x,1}(G) = \begin{cases} (n - q_G)x + q_G - 1 & \text{if } \frac{1}{2} \leq x \leq 1 \\ (q_{G'} - 1)x + n - q_{G'} & \text{if } 1 \leq x \leq 2. \end{cases}
$$

Proof. By Theorem 3.1, we have that for $1 \leq \frac{1}{2} \leq 2$, $\lambda_{x,1}(G)$ and $\lambda_{x,1}(G')$ are respectively $(q_G - 1)j + (n - q_G)k$ and $(q_{G'} - 1)j + (n - q_{G'})k$, implying that for $1 \leq x \leq 2$, $\lambda_{x,1}(G)$ and $\lambda_{x,1}(G')$ are respectively $(q_G - 1)x + n - q_G$ and $(q_{G'} - 1)x + n - q_{G'}$.

By Theorem 2.2, $G$ is $\lambda$-invertible, implying that for $\frac{1}{2} \leq x \leq 1$, $\lambda_{x,1}(G) = x\lambda_{\frac{1}{2}}(G) = x((q_G - 1)\frac{1}{x} + (n - q_G)) = (q_G - 1) + (n - q_G)x$. \(\square\)

Definition 3.3. Let $G$ be a non-trivial graph and let $E' \subseteq E(G)$. Let $p$ be a non-negative integer. Then a $p$-subdivision of $G$ along the edges of $E'$, denoted as $G*_{p}E'$, is the graph obtained by inserting $p$ new vertices along each edge in $E'$.

It can be readily checked that each member of the following four classes of graphs and its complement has diameter 2:

: $K_{m,n} *_{1} [e]$ for $2 \leq m \leq n$ and $e$ an arbitrary edge of $K_{m,n}$;

: $K_{n} *_{2} [e]$ for $n \geq 3$ and $e$ an arbitrary edge of $K_{n}$;

: $K_{n} *_{3} E(v)$ for $n \geq 3$ and $E_v$ the set of edges incident to arbitrary vertex $v$;

: $(K_{n} \times K_{n}) *_{1} [s]$ for $n \geq 2$ and $s$ an edge joining corresponding vertices in the two copies of $K_{n}$.

We note that $C_{5}$ is isomorphic to both $K_{3} *_{2} [e]$ and $K_{2,2} *_{1} [e]$, and that more generally, $K_{n} *_{2} [e]$ is isomorphic to $(K_{2,n-1} *_{1} [e])^{\frac{1}{2}}$. By Theorem 2.2, we have the following.

Corollary 3.4. The graphs in the four classes given above are $\lambda$-invertible. \(\square\)

Theorem 3.5. For every positive integer except $n = 2, 3$, there exists a graph $G$ with order $n$ such that $G$ is $\lambda$-invertible.

Proof. We exhibit $K_{1}$ and $P_{4}$ as $\lambda$-invertible graphs with orders 1 and 4. In the case $n \geq 5$, we exhibit $K_{2,n-3} *_{1} [e]$ for any edge $e$ in $E(K_{2,n})$ as a $\lambda$-invertible graph with order $n$. \(\square\)

We conclude this section by deriving $\lambda_{x,1}(K_{2,n-3} *_{1} [e])$, a graph of order $n$, for $n \geq 6$. To facilitate our discussion of labelings, it will be convenient to denote $K_{2,n-3} *_{1} [e]$ by $Y_n$ and to characterize that graph as follows: Let $X = \{x_1, x_2, x_3, x_4\}$ and let $Y = \{y_1, y_2, \ldots, y_{n-4}\}$ where $X$ and $Y$ are disjoint. Then $Y_n$ is the graph with vertex set $X \cup Y$ such that the subgraph of $Y_n$ induced by $X$ is isomorphic to the path $x_1, x_2, x_3, x_4$ and each $y_i$ has neighborhood set $\{x_1, x_4\}$.

We observe that the subgraphs of $Y_n^c$ induced by both $Y \cup \{x_2\}$ and $Y \cup \{x_3\}$ are isomorphic to $K_{n-3}$, and that for $n \geq 6$, $\chi(Y_n^c) = n - 3$.

Theorem 3.6. For every $n \geq 6$,

$$
\lambda_{x,1}(Y_n) = \begin{cases} n - 4 + x & \text{if } 0 \leq x \leq \frac{1}{2} \\ n - 6 + 5x & \text{if } \frac{1}{2} < x \leq 1 \\ n - 1 & \text{if } 1 \leq x \leq 2 \\ n - 3 + x & \text{if } 2 \leq x \leq n - 3 \\ 2x & \text{if } x \geq n - 3. \end{cases}
$$

Proof. (i) $0 \leq x \leq \frac{1}{2}$. We show $\lambda_{x,1}(Y_n) = n - 4 + x$ for $0 < x \leq \frac{1}{2}$, from which the result will follow by the continuity of $\lambda_{x,1}(Y_n)$ at 0. Since $Y_n$ is $\lambda$-invertible, it suffices to consider $\lambda_{j,k}(Y_n^c)$ for $\frac{1}{2} \geq 2$. We first observe that the labeling $L$ is an $L(j, k)$-labeling of $Y_n^c$ where $L(y_i) = (i - 1)j, L(x_1) = k, L(x_2) = (n - 4)j + k, L(x_3) = (n - 4)j + k, L(x_4) = (n - 4)j + k$. Hence $\lambda_{j,k}(Y_n^c) \leq (n - 4)j + k$. Next we consider the subgraph $U_n$ of $Y_n^c$ induced by $Y \cup \{x_2, x_3\}$. It is easily seen that $U_n$ is isomorphic to the join of $K_{n-4}$ with $2K_2$. By [5], $\lambda_{j,k}(U_n) = (n - 4)j + k$. Hence, $\lambda_{j,k}(Y_n^c) = (n - 4)j + k$, which implies $\lambda_{x,1}(Y_n^c) = (n - 4)x + 1$. Since $Y_n^c$ is $\lambda$-invertible, the result follows.

(ii) $\frac{1}{2} \leq x \leq 1$. As in (i), it suffices to consider $\lambda_{j,k}(Y_n^c)$ for $1 \leq \frac{1}{2} \leq 2$. It is easily checked that $Y_n$ has path-covering number $\gamma = n - 5$. So by Theorem 3.1 (2), $\lambda_{j,k}(Y_n^c) = (n - 6)j + 5k$, implying that $\lambda_{x,1}(Y_n^c) = (n - 6)x + 5$. The result follows since $Y_n^c$ is $\lambda$-invertible.

(iii) $1 \leq x \leq 2$. Since $Y_n^c$ has hamilton path $x_1, x_3, y_1, y_2, \ldots, y_{n-4}, x_2, x_4$, the path-covering number of $Y_n^c$ is 1, and hence by Theorem 3.1, $\lambda_{j,k}(Y_n) = (n - 1)k$ for $1 \leq \frac{1}{k} \leq 2$. The result thus follows.
Theorem 4.1. \[ \forall \text{observethatif} \]

4. Relating \( \lambda \)

Corollary 3.4.

Proof. Corollary 3.7. Thus the labelsofatmosttwoelementsof \( n \)

\[ L(\nu) = \begin{cases} 
  ik & \text{if } \nu = y_1 \\
  j + (n-4)k & \text{if } \nu = x_1 \\
  0 & \text{if } \nu = x_2 \\
  (n-3)k & \text{if } \nu = x_3 \\
  j + (n-3)k & \text{if } \nu = x_4. 
\end{cases} \]

So it suffices to show that no \( L(j, k) \)-labeling of \( Y_n \) exists with span \( j + (n-3)k - 1 \). Assume to the contrary that \( L' \) is such a labeling. For each \( i, 0 \leq i \leq n-3 \), let \( \Pi_i \) denote the partition of \([0, j + (n-3)k - 1]\) given by the union of the three sets

\[ \Pi_{i,1} = \left\{ \left[ h, (h + 1)k - 1 \right] | 0 \leq h \leq i - 1 \right\}, \]

\[ \Pi_{i,2} = \left\{ \left[ h + j, (h + 1)k + j - 1 \right] | i \leq h \leq n - 4 \right\}, \]

and

\[ \Pi_{i,3} = \left\{ \left[ ik, ik + j - 1 \right] \right\}. \]

Note that we are defining \( n - 2 \) distinct partitions of \([0, j + (n-3)k - 1]\) and that each partition \( \Pi_i \) has \( n - 3 \) elements of length \( k - 1 \) and one element of length \( j - 1 \). Now fix \( i, 0 \leq i \leq n-3 \). Since \( Y_n \) has diameter 2, then every element of \( \Pi_i \) of length \( k - 1 \) contains the label of at most one vertex of \( Y_n \), implying that the element \([ik, ik + j - 1]\) contains the labels of at least three vertices. Furthermore, since no three vertices of \( X \) form an independent set, then \([ik, ik + j - 1]\) contains the labels of at most two elements of \( X \) and thus the label of at least one element of \( Y \). As a result, neither \( x_1 \) nor \( x_4 \) has label \([ik, ik + j - 1]\) since each is adjacent to every element of \( Y \). Because \( i \) was arbitrarily fixed, then neither \( x_1 \) nor \( x_4 \) has label in \( \bigcup_{i=0}^{n-3}[ik, ik + j - 1] = [0, (n-3)k + j - 1] \), contradicting the span of \( L' \). Therefore \( \lambda_{j,k}(Y_n) = (n-3)k + j \), giving the result.

(v) \( n - 3 \leq x \). It is readily seen that the following is an \( L(j, k) \)-labeling of \( Y_n \) with span 2\( j \):

\[ L(\nu) = \begin{cases} 
  ik & \text{if } \nu = y_1 \\
  j + (n-4)k & \text{if } \nu = x_1 \\
  0 & \text{if } \nu = x_2 \\
  j & \text{if } \nu = x_3 \\
  2j & \text{if } \nu = x_4. 
\end{cases} \]

Thus \( \lambda_{j,k}(Y_n) \leq 2j \). Since \( \lambda_{j,k}(C_5) = 2j \) (see [5,10]) and \( Y_n \) has a subgraph isomorphic to \( C_5 \), then \( \lambda_{j,k}(Y_n) = 2j \). \( \Box \)

Corollary 3.7. For every \( n \geq 6 \),

\[ \lambda_{x,1}(K_{n-2} \ast_2 \{e\}) = \begin{cases} 
  2 & \text{if } 0 \leq x \leq \frac{1}{n-3} \\
  (n-3)x + 1 & \text{if } \frac{1}{n-3} \leq x \leq \frac{1}{2} \\
  (n-1)x & \text{if } \frac{1}{2} \leq x \leq 1 \\
  (n-6)x + 5 & \text{if } 1 \leq x \leq 2 \\
  (n-4)x + 1 & \text{if } x \geq 2. 
\end{cases} \]

Proof. It is easily checked that \( Y'_n \) is isomorphic to \( K_{n-2} \ast_2 \{e\} \), from which the result follows by Theorem 3.6 and Corollary 3.4. \( \Box \)

4. Relating \( \lambda \)-invertibility to the maximum degree of graphs

Let \( I_\Delta(n) \) denote the collection of \( \lambda \)-invertible graphs with maximum degree \( \Delta \) and order \( n \). Let \( I_\Delta \) denote \( \bigcup_{n=1}^{\infty} I_\Delta(n) \). We observe that if \( G \in I_\Delta \), then by Theorem 2.1, both \( G \) and \( G^* \) are connected. Thus \( I_0 = I_0(1) = \{K_1\} \), and for \( \Delta \geq 1 \), \( I_\Delta(m) = \phi \) for all \( m \leq \Delta + 1 \).

Theorem 4.1. Let \( G \) be a graph with order \( n \geq 2 \). Then:

(1) \( G \) has diameter 2 if both \( n \geq 2\Delta(G) + 1 \) and \( G \) is \( \lambda \)-invertible, and

(2) \( G \) is not \( \lambda \)-invertible if \( n \geq \Delta^2(G) + 2 \).
Proof. (1) Since $\Delta(G) \leq \frac{n-1}{2}$, then $\delta(G^c) = (n - 1) - \Delta(G) \geq \frac{n-1}{2}$. Thus, for arbitrary vertices $v, w$ in $V(G^c)$, the distance between $v$ and $w$ in $G^c$ is at most 2, implying that $G^c$ has diameter at most 2. Since there are no $\lambda$-invertible graphs of diameter 1, the result follows from Theorem 2.1.

(2) Each vertex $v$ in $V(G)$ is at most distance two from at most $\Delta^2(G) + 1$ other vertices (including $v$). Since $n \geq \Delta^2(G) + 2 \geq 2\Delta(G) + 1$, there is a vertex in $V(G)$ at distance 3 or more from $v$. The results follows from (1). □

Corollary 4.2. If $G$ is a $k$-regular graph with diameter at least 3, then $G$ is not $\lambda$-invertible.

Proof. Suppose to the contrary that $G$ is $\lambda$-invertible. By Theorem 2.1, the diameters of $G$ and $G^c$ are each 3. Thus, since $G^c$ is $\lambda$-invertible, Theorem 4.1(1) implies that $n \leq 2\Delta(G) = 2k$ and $n \leq 2\Delta(G^c) = 2(n-k-1)$. Hence we have the contradiction $n \leq 2k$ and $n \geq 2k + 2$. (We also note that if $G$ is a $k$-regular graph, then either $G$ or $G^c$ has diameter at most 2. So if $G$ has diameter at least 3, then $G$ and $G^c$ cannot have equal diameters, implying that $G$ is not $\lambda$-invertible.) □

If $G \in I_\Delta$, then by Theorem 4.1(2), the order of $G$ is less than $\Delta^2(G) + 2$, thus implying that $I_\Delta$ is finite. It readily follows from Theorems 2.1 and 4.1 that $I_1 = \phi$ and $I_2 = \{P_4, C_5\}$. We next determine $I_3$.

If $G$ has order $n$ and $G \in I_3(n)$, then by Theorem 4.1(2) and the immediately preceding observation, it follows that $5 \leq n \leq 10$. Moreover, by Theorem 4.1(1), it also follows that for $7 \leq n \leq 10$, $G$ has diameter 2. We observe particularly that if $8 \leq n \leq 10$, then $\delta(G) = 3$. (A vertex in $G$ with degree 2 would be within distance two of at most 6 other vertices, contradicting the diameter of the graph.) But the three-regular graphs with diameter 2 and orders 8, 9, or 10 are known and given in Fig. 1 as $H_1$ (the Petersen graph) and $H_2$, each of which has a complement of diameter 2. Thus by Theorem 2.2, $I_3(8) = \{H_2\}$, $I_3(9) = \phi$, and $I_3(10) = \{H_1\}$.

If $G \in I_3(7)$, then $G$ and $G^c$ have diameter 2 and hence $\delta(G) = 2$. Noting that the number of vertices of degree 2 in $G$ is either one, three, or five, it can be checked that the only such graphs in $I_3(7)$ are $K_{3,3} \ast_1 \{e\}$, $(K_3 \times K_2) \ast_1 \{s\}$, and $K_4 \ast_1 E(v)$, giving $I_3(7) = \{K_{3,3} \ast_1 \{e\}, (K_3 \times K_2) \ast_1 \{s\}, K_4 \ast_1 E(v)\}$.

Deferring consideration of $I_3(6)$ for the moment, suppose $G \in I_3(5)$. Then by Theorem 2.1, both $G$ and $G^c$ are connected, implying that the size of $G$ is 4, 5 or 6. By the compendium of order 5 graphs in [15], it thus follows that $G$ is among the six graphs pictured in Fig. 2. We show that none of these graphs is $\lambda$-invertible, giving $I_3(5) = \phi$.

It is easily checked that $\lambda_{2,1}(J_1) = 2$ and $\lambda_{2,1}(J_1^c) = 5$. Thus $J_1$ cannot be $\lambda$-invertible. Moreover, since $J_1^c = J_6$, $J_6$ is not $\lambda$-invertible.

Since $\lambda_{2,1}(J_2) = 4$ and $\lambda_{2,1}(J_2^c) = \frac{5}{2}$, $J_2$ is not $\lambda$-invertible. Moreover, since $J_2^c = J_3$, $J_3$ is not $\lambda$-invertible.

The graph $J_4$ is not $\lambda$-invertible by Theorem 2.1, since $J_4$ has diameter 2 and $J_4^c$ (isomorphic to $P_5$) does not have diameter 2. Noting that $J_5$ is self-complementary, we have $\lambda_{0,1}(J_5) = 1$, and $\chi(J_5^c) - 1 = \chi(J_5) - 1 = 3$. Thus, by Theorem 2.4, $J_5$ cannot be $\lambda$-invertible.

Our investigation of $I_3(6)$ is made more complicated by the existence of precisely 27 non-isomorphic connected graphs of order 6 and maximum degree 3 (see [15]). Only one of these graphs, $K_{3,3}$ has a disconnected complement, leaving 26
graphs for further consideration. Our approach is to partition the set \( S \) of these 26 graphs according to \( \delta(G) = 1, 2, \) and 3; particularly, for \( \delta = 1, 2, 3, S_\delta \) will denote the elements of \( S \) that have minimum degree \( \delta \). Noting that the sole element of \( S_3 \) is the 3-prism, we give the elements of \( S_2 \) and \( S_1 \) in Figs. 3 and 4, respectively.

By Theorem 2.1, the 3-prism \( G \) of \( S_3 \) is not \( \lambda \)-invertible, for \( G \) and \( G' \) have unequal diameters.

The eight graphs in \( S_2 \) contain four complementary pairs: particularly, for \( 1 \leq i \leq 4 \), graphs \( A_i \) and \( B_i \) are complementary. We observe that for \( i = 1, 2 \), \( \lambda_{1,1}(A_i) = 4 \) and \( \lambda_{1,1}(B_i) = 3 \), implying that neither \( A_i \) nor \( B_i \) is \( \lambda \)-invertible. We also note that neither \( A_3 \) nor \( B_3 \) is \( \lambda \)-invertible since \( A_3 \) and \( B_3 \) have unequal diameters, 2 and 3 respectively. On the other hand, both \( A_4 \) and \( B_4 \) have diameter 2, and are thus \( \lambda \)-invertible. (Note that \( B_4 \) is isomorphic to \( Y_6 \), defined in the preceding section.)

There are 17 graphs in \( S_1 \), which we organize in Fig. 4 according to the number of vertices of degree 2 (necessarily 0, 2, 4). None of these graphs is \( \lambda \)-invertible, for reasons given in Table 1. We have thus shown the following:

**Theorem 4.3.** There are precisely 7 graphs in \( I_3 \); two of order 6, three of order 7, one of order 8, and one of order 10. Each graph has diameter 2. \( \square \)

5. Some examples of \( \lambda \)-invertible graphs

In this section, we consider two families of graphs, the Kneser graphs and the line graphs of complete multipartite graphs, within each of which we identify subfamilies of graphs that are \( \lambda \)-invertible. In the former subfamily (a subfamily of Kneser graphs discussed in section (i)), we use existing results, together with the properties of \( \lambda \)-invertibility, to obtain formulas for \( \lambda_{x,1}(G) \) where \( G \) is the line graph of a complete graph. For the latter subfamily (a subfamily of line graphs of complete
multipartite graphs discussed in section (ii), we pay particular attention to $L(G)$ where $G$ is complete bipartite. Noting that $K_{a_1} \times K_{a_2}$ is isomorphic to $L(K_{a_1,a_2})$, we then use the properties of $\lambda$-invertibility, together with new and existing results on the $\lambda_{x,1}$-number of $K_{a_1} \times K_{a_2}$, to obtain $\lambda_{x,1}(K_{a_1} \times K_{a_2})$ and $\lambda_{x,1}(L(K_{a_1} \times K_{a_2}))$ for all non-negative $x$.

(i) Kneser graphs and line graphs of complete graphs.

For positive integers $n$ and $m$ such that $n > 2m \geq 4$, Kneser graph $K(n, m)$ is the $\binom{n-m}{m}$-regular graph on $\binom{n}{m}$ vertices such that (1) $V(K(n, m))$ is the set of all $m$-element subsets of $\{1, 2, 3, \ldots, n\}$, and (2) two vertices of $V(K(n, m))$ are adjacent if and only if the two vertices are disjoint. It is shown in [17] that the diameter of $K(n, m)$ is $\lceil \frac{m-1}{n-2m} \rceil + 1$. Thus the diameter of $K(n,m)$ is $2$ precisely for $n \geq 3m - 1$. Furthermore, $K(n, m)^c$ has diameter $2$, giving the following result by \textbf{Theorem 2.2} and Corollary 4.2.

\textbf{Theorem 5.1.} $K(n, m)$ is $\lambda$-invertible if and only if $n \geq 3m - 1$. \hfill $\square$

The Kneser graph $K(n, 2)$ is therefore $\lambda$-invertible for $n \geq 5$. Erman et al. [3] established the following result for $r \geq 2$:

\[
\lambda_{x,1}(K(2r + 1, 2)) = \begin{cases} 
    2r + (r - 1)x & \text{if } 0 \leq x \leq \frac{1}{r} \\
    (2r^2 + r - 1)x & \text{if } \frac{1}{r} \leq x \leq 1 \\
    2r^2 + r - 1 & \text{if } 1 \leq x \leq 3 \\
    (2r - 2)x + 2r^2 - 5r + 5 & \text{if } 3 \leq x 
\end{cases}
\]

\[
\lambda_{x,1}(K(2r + 2, 2)) = \begin{cases} 
    2r + 3x & \text{if } 0 \leq x \leq \frac{1}{r} \\
    (2r^2 + 3r)x & \text{if } \frac{1}{r} \leq x \leq 1 \\
    2r^2 + 3r & \text{if } 1 \leq x \leq 3 \\
    (2r - 1)x + 2r^2 - 3r + 3 & \text{if } 3 \leq x 
\end{cases}
\]

They also note that $K(n, 2)$ is isomorphic to $L(K_n)^c$ (the complement of the line graph of $K_n$). \textbf{Theorem 5.1} thus implies the complete behavior of $\lambda_{x,1}(L(K_n))$ for $n \geq 5$.

\textbf{Corollary 5.2.} For $r \geq 2$,

\[
\lambda_{x,1}(L(K_{2r+1})) = \begin{cases} 
    (2r - 2) + (2r^2 - 5r + 5)x & \text{if } 0 \leq x \leq \frac{1}{3} \\
    (2r^2 + r - 1)x & \text{if } \frac{1}{3} \leq x \leq 1 \\
    2r^2 + r - 1 & \text{if } 1 \leq x \leq r \\
    (r - 1) + 2rx & \text{if } r \leq x 
\end{cases}
\]
(ii) The line graphs of complete multipartite graphs.

For integer \( p \geq 2 \), let \( a_1, a_2, \ldots, a_p \) denote positive integers such that \( a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_p \) and \( \sum_{i=1}^{p} a_i = n \). We denote the complete \( p \)-partite graph with parts of sizes \( a_1, a_2, \ldots, a_p \) by \( K_{a_1, a_2, \ldots, a_p} \).

Let \( G = (V, E) \) be isomorphic to \( K_{a_1, a_2, \ldots, a_p} \). It is easily argued that \( L(G) \) has diameter at most 2 as follows: If \( |E| = 1 \), then \( G \) is isomorphic to \( K_2 \) and \( L(K_2) \) is isomorphic to \( K_1 \) which has diameter 0. Thus suppose \( |E| \geq 2 \). Let \( \{a, b\} \) and \( \{c, d\} \) be distinct edges in \( E \). If \( \{a, b\} \cap \{c, d\} = \emptyset \), then vertices \( \{a, b\} \) and \( \{c, d\} \) in \( L(G) \) are adjacent in \( L(G) \). And if \( \{a, b\} \cap \{c, d\} = \emptyset \), then either \( \{a, c\} \) or \( \{a, d\} \) is an edge in \( G \), implying the existence of a path of length two in \( L(G) \) from vertex \( \{a, b\} \) to vertex \( \{c, d\} \). However, since \( L(K_{2, 3}) \) is isomorphic to \( C_6 \), it holds that not all complete \( p \)-partite graphs have line graphs with complements of diameter at most 2. We thus will derive necessary and sufficient conditions on \( a_1, a_2, \ldots, a_p \) under which \( L(K_{a_1, a_2, \ldots, a_p}) \) has diameter at most 2, which in turn will allow an appeal to Theorem 2.2, for the establishment of \( \lambda \)-invertibility.

Select \( G = (V, E) \) isomorphic to \( K_{a_1, a_2, \ldots, a_p} \), not \( K_{1,1,1} \). Then \( |E| \geq 2 \), and hence we may select two distinct vertices \( \{a, b\} \) and \( \{c, d\} \) of \( L(G) \). If \( \{a, b\} \cap \{c, d\} = \emptyset \), then vertices \( \{a, b\} \) and \( \{c, d\} \) of \( L(G) \) are not adjacent in \( L(G) \), implying that they are of \( \lambda \)-adjacent in \( L(G) \). On the other hand, if \( \{a, b\} \cap \{c, d\} = \emptyset \), then with no loss of generality, \( a = c \) and the distinct vertices \( a, b, d \) of \( G \) are incident to 2 or 3 parts of \( G \). We show the existence of a path of length 2 from \( \{a, b\} \) to \( \{a, d\} \) in \( L(G) \) by considering the cases \( p = 2, p = 3, p = 4, \) and \( p \geq 5 \).

Case 1. \( p \geq 5 \). Then there exist (at least) 2 distinct parts of \( G \) neither of which contains \( a, b, \) or \( d \). Thus there exist distinct vertices \( g \) and \( h \) in these two respective parts, implying that \( \{g, h\} \) is a vertex of \( L(G) \) adjacent to neither \( \{a, b\} \) nor \( \{a, d\} \) in \( L(G) \). Therefore \( \{a, b\}, \{g, h\}, \{a, d\} \) is a path of length 2 in \( L(G) \).

Case 2. \( p = 4 \). We show that a path of length 2 exists from \( \{a, b\} \) to \( \{a, d\} \) in \( L(G) \) if and only if \( a_1 + a_2 + a_3 \geq 4 \).

\( \iff \) If \( a_1 + a_2 + a_3 \geq 4 \), then \( a_1 \geq 1, a_2 \geq 1, a_3 \geq 2 \), and \( a_4 \geq 2 \), implying the existence of vertices \( g \) and \( h \) in distinct parts of \( G \) such that neither \( g \) nor \( h \) is in \( \{a, b, d\} \). Thus the vertex \( \{g, h\} \) in \( L(G) \) is adjacent to neither \( \{a, b\} \) nor \( \{a, d\} \) in \( L(G) \), which concludes the argument as in the case \( p \geq 5 \).

\( \implies \) Now suppose that \( a_1 + a_2 + a_3 < 4 \). Then \( a_1 = a_2 = a_3 = 1 \) and we may suppose that \( X, Y \) and \( Z \) are distinct parts of \( G \) with cardinality one and respective elements \( x, y \) and \( z \). Then in \( L(G) \), each vertex is adjacent to at least one of the adjacent vertices \( \{x, y\} \) and \( \{x, z\} \), implying that in \( L(G) \), no vertex is adjacent to both of the non-adjacent vertices \( \{x, y\} \) and \( \{x, z\} \). Thus \( L(G) \) cannot have diameter at most 2.

Case 3. \( p = 3 \). We show that a path of length 2 exists from \( \{a, b\} \) to \( \{a, d\} \) in \( L(G) \) if and only if \( a_1 + a_2 \geq 4 \).

\( \iff \) If \( a_1 + a_2 \geq 4 \), then either \( a_1 = 1 \) and \( 3 \leq a_2 \leq a_3 \) or \( 2 \leq a_1 \leq a_2 \leq a_3 \). In either case, we may select vertices \( g \) and \( h \) in distinct parts of \( G \) so that neither \( g \) nor \( h \) is in \( \{a, b, d\} \), which concludes the argument as in the case \( p \geq 5 \).

\( \implies \) Now suppose \( a_1 + a_2 < 4 \). Then \( a_1 = 1 \) and \( a_2 = a_3 = 1 \). Then \( a_2 = a_3 = 1 \) or \( 2 \). If \( a_2 = 1 \), we let distinct parts \( X \) and \( Y \) of \( G \) have cardinality 1 with respective elements \( x \) and \( y \). It is then easily checked that \( \{x, y\} \) is a vertex of \( L(G) \). On the other hand, if \( a_3 = 2 \), then we let distinct parts \( X \) and \( Y \) of \( G \) have cardinality 1 and 2 respectively such that \( X = \{x\} \) and \( Y = \{y_1, y_2\} \). Then in \( L(G) \), each vertex is adjacent to at least one of the adjacent vertices \( \{x, y_1\} \) and \( \{x, y_2\} \), implying that in \( L(G) \), no vertex is adjacent to both of the non-adjacent vertices \( \{x, y_1\} \) and \( \{x, y_2\} \). Thus \( L(G) \) cannot have diameter at most 2.

Case 4. \( p = 2 \). We show that a path of length 2 exists from \( \{a, b\} \) to \( \{a, d\} \) in \( L(G) \) if and only if \( a_1 \geq 3 \).

\( \iff \) If \( a_1 \geq 3 \), then \( a_2 \geq 3 \), and we may therefore select vertices \( g \) and \( h \) in distinct parts of \( G \) so that neither \( g \) nor \( h \) is in \( \{a, b, d\} \). Thus the vertex \( \{g, h\} \) in \( L(G) \) is adjacent to neither \( \{a, b\} \) nor \( \{a, d\} \) in \( L(G) \), which concludes the argument as in the case \( p \geq 5 \).

\( \implies \) Now suppose that \( a_1 < 3 \). If \( a_1 = 1 \), then \( G \) is isomorphic to \( K_{1, a_2} \), which it is easily seen that \( L(G) \) is isomorphic to \( a_2K_1 \) of infinite diameter. And if \( a_1 = 2 \), then we may suppose that \( G \) has parts \( X = \{x_1, x_2\} \) and \( Y = \{y_1, y_2, \ldots, y_{a_2}\} \). It then follows that \( \{x_1, y_1\} \) and \( \{x_2, y_1\} \) are adjacent vertices in \( L(G) \) and that every vertex in \( L(G) \) is adjacent to either \( \{x_1, y_1\} \) or \( \{x_2, y_1\} \). Thus \( \{x_1, y_1\} \) and \( \{x_2, y_1\} \) are non-adjacent vertices in \( L(G) \) and no vertex in \( L(G) \) is adjacent to both \( \{x_1, y_1\} \) and \( \{x_2, y_1\} \). Therefore \( L(G) \) does not have diameter 2.

We have thus shown that if \( G \) is isomorphic to the complete \( p \)-partite graph \( K_{a_1, a_2, \ldots, a_p} \) where \( a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_p \), then \( L(G) \) has diameter 2 and \( L(G)^c \) has diameter 2 under particular conditions on \( a_1, a_2, a_3 \) and \( p \). We therefore invoke Theorem 2.2 as follows:
Theorem 5.3. Suppose $G$ is isomorphic to $K_{a_1,a_2,...,a_p}$, where $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_p$. Then $L(G)$ is $\lambda$-invertible if and only if either

: $p \geq 5$, or
: $p = 4$ and $a_1 + a_2 + a_3 \geq 4$, or
: $p = 3$ and $a_1 + a_2 \geq 4$, or
: $p = 2$ and $a_1 \geq 3$. □

We now turn our attention to $\lambda_{x,1}(L(K_{a_1,a_2}))$ where $a_1 \geq 3$. We note that since $L(K_{a_1,a_2})$ is isomorphic to $K_{a_1} \times K_{a_2}$, it suffices to consider $\lambda_{x,1}(K_{a_1} \times K_{a_2})$. (In what follows, we may represent $K_{a_1} \times K_{a_2}$ as an $a_1 \times a_2$ array of vertices such that the vertices in each row are pairwise adjacent and the vertices in each column are pairwise adjacent.) We begin with three results, the first two of which were proved in [6].

Theorem 5.4 ([6]). For integers $2 \leq a_1 < a_2$,

\[
\lambda_{j,k}(K_{a_1} \times K_{a_2}) = \begin{cases} 
(a_1 a_2 - 1) k & \text{if } 1 \leq \frac{j}{k} \leq a_1 \\
(a_2 - 1) j + (a_1 - 1) k & \text{if } \frac{j}{k} \geq a_1.
\end{cases}
\]

Theorem 5.5 ([6]). For integers $2 \leq a_1 = a_2$,

\[
\lambda_{j,k}(K_{a_1} \times K_{a_1}) = \begin{cases} 
(a_1^2 - 1) k & \text{if } 1 \leq \frac{j}{k} \leq a_1 - 1 \\
(a_1 - 1) j + (2a_1 - 2) k & \text{if } \frac{j}{k} \geq a_1 - 1.
\end{cases}
\]

Lemma 5.6. Let $L$ be an $L(j,k)$-labeling of $K_{a_1} \times K_{a_2}$ with span $sp(L)$ where $0 < \frac{j}{k} \leq \frac{1}{2}$ and $a_1 \leq a_2$. Also let $l = (l_0, l_1, l_2, \ldots, l_{(a_2-1)})$ be the strictly increasing sequence of labels assigned to the vertices of $K_{a_1} \times K_{a_2}$ by $L$. Then

(1) $l_{i+1} - l_i \geq j$ for $0 \leq i \leq a_1 a_2 - 2$;

(2) in any subsequence $(l_h, l_{h+1}, \ldots, l_{h+a_2})$ of $l$, there exist components $l_h$ and $l_{h+2}$ such that $l_{h+2} - l_h \geq k$;

(3) $sp(L) \geq (a_1(a_2 - 2) + 1) j + (a_1 - 1) k$.

Proof. Let $s = (v_0, v_1, v_2, \ldots, v_{a_1 a_2 - 1})$ be the sequence of vertices of $K_{a_1} \times K_{a_2}$ such that $l_i = L(v_i)$ for $0 \leq i \leq a_1 a_2 - 1$.

To prove (1), we merely note that $j < k$ and $v_i$ is at most distance two apart from $v_{i+1}$.

To prove (2) and (3), we fix arbitrary subsequence $l' = (l_h, l_{h+1}, \ldots, (h+a_2))$ of $l$ and let $s' = (v_{h}, v_{h+1}, \ldots, v_{h+a_2})$.

Proof of (2): It suffices to show the existence of three consecutive components $v_x, v_{x+1}, v_{x+2}$ of $s'$ that are not mutually adjacent. To that end, we observe that if vertices $v_x, v_{x+1}, v_{x+2}$ are mutually adjacent for all $x, h \leq x \leq h + a_2 - 2$, then $v_h, v_{h+1}, v_{h+2}$ are in the same row(column) of $K_{a_1} \times K_{a_2}$, and that similarly $v_{h+1}, v_{h+2}, v_{h+3}$ are in the same row(column) as well. Hence $v_h, v_{h+1}, v_{h+2}$ and $v_{h+3}$ are in the same row(column). Proceeding inductively, we see that all $a_2 + 1$ vertices of $s$ are in the same row(column), a contradiction of the dimensions of $K_{a_1} \times K_{a_2}$.

Proof of (3): By (1) and (2), there exists $h, 0 \leq h \leq h + a_2 - 2$, such that $l_i - l_{i-1} \geq j$ for $h + 1 \leq i \leq x, l_{h} - l_{h+2} \geq k$, and $l_{h+2} - l_{h+1} \geq j$ for $x+3 \leq i \leq h+a_2$. Thus $l_{h+2} - l_{h} \geq (a_2-2)j+k$. Since $h'$ was arbitrary, we have $l_{h'+2} - l_{h'+1} \geq (a_2-2)j+k$ for $1 \leq p \leq a_1-1$, giving $l_{(a_1-1)} - l_0 \geq (a_1-1)(a_2-2)j+k = (a_1-1)(a_2-2)j+(a_1-1)k$. Moreover, by (1), $l_{(a_1-1)} - l_0 \geq (a_2-1)j$. Thus $sp(L) \geq (a_1 a_2 - 2) j + (a_1-1)k + (a_1-1)(a_2-2)j + (a_1-1)k$. □

We now state our main result of this subsection.

Theorem 5.7. For integers $3 \leq a_1 \leq a_2$,

\[
\lambda_{x,1}(K_{a_1} \times K_{a_2}) = \begin{cases} 
(a_1 a_2 - 2a_1 + 1)x + a_1 - 1 & \text{if } 0 < x \leq \frac{1}{2} \\
(a_1 a_2 - 1) x & \text{if } \frac{1}{2} \leq x \leq 1 \\
a_1^2 - 1 & \text{if } 1 \leq x < a_1 - 1 \text{ and } a_1 = a_2 \\
a_1 a_2 - 1 & \text{if } 1 \leq x < a_1 \text{ and } a_1 < a_2 \\
(a_1 - 1)x + 2a_1 - 2 & \text{if } x \geq a_1 - 1 \text{ and } a_1 = a_2 \\
(a_2 - 1)x + a_1 - 1 & \text{if } x \geq a_1 \text{ and } a_1 < a_2.
\end{cases}
\]

Proof. If $x \geq 1$, $\lambda_{x,1}(K_{a_1} \times K_{a_2})$ follows immediately from Theorems 5.4 and 5.5.
To establish $\lambda_{x,1}(K_{a_1} \times K_{a_2})$ for $\frac{1}{2} \leq x \leq 1$, we appeal to the $\lambda$-invertibility of $(K_{a_1} \times K_{a_2})^c$, Theorem 3.1, and the fact that $K_{a_1} \times K_{a_2}$ has path-covering number 1.

Now suppose $0 < x \leq \frac{1}{2}$. By Lemma 5.6, it suffices to find an $L(j, k)$-labeling of $K_{a_1} \times K_{a_2}$ with span $(a_1(a_2 - 2) + 1)j + (a_1 - 1)k$ for $0 < \frac{x}{k} \leq \frac{1}{2}$. To that end, we denote $K_{a_1} \times K_{a_2}$ as the indicated $a_1 \times a_2$ array of vertices $v_{p,q}, 0 \leq p \leq a_1 - 1, 0 \leq q \leq a_2 - 1$:

\[
\begin{array}{cccccccc}
v_{0,0} & v_{0,1} & v_{0,2} & \cdots & v_{0,a_2-2} & v_{0,a_2-1} \\
v_{1,0} & v_{1,1} & v_{1,2} & \cdots & v_{1,a_2-2} & v_{1,a_2-1} \\
v_{2,0} & v_{2,1} & v_{2,2} & \cdots & v_{2,a_2-2} & v_{2,a_2-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{a_1-1,0} & v_{a_1-1,1} & v_{a_1-1,2} & \cdots & v_{a_1-1,a_2-2} & v_{a_1-1,a_2-1} \\
\end{array}
\]

Now consider the hamilton path through $K_{a_1} \times K_{a_2}$, commencing at $v_{0,0}$, as indicated:

\[
\begin{align*}
v_{0,0} & \rightarrow v_{0,1} \rightarrow v_{0,2} \rightarrow \cdots \rightarrow v_{0,a_2-2} \rightarrow v_{0,a_2-1} \\
v_{1,0} & \rightarrow v_{1,1} \rightarrow v_{1,2} \rightarrow \cdots \rightarrow v_{1,a_2-2} \rightarrow v_{1,a_2-1} \\
v_{2,0} & \rightarrow v_{2,1} \rightarrow v_{2,2} \rightarrow \cdots \rightarrow v_{2,a_2-2} \rightarrow v_{2,a_2-1} \\
v_{3,0} & \rightarrow v_{3,1} \rightarrow v_{3,2} \rightarrow \cdots \rightarrow v_{3,a_2-2} \rightarrow v_{3,a_2-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{a_1-1,0} & \rightarrow v_{a_1-1,1} \rightarrow v_{a_1-1,2} \rightarrow \cdots \rightarrow v_{a_1-1,a_2-2} \rightarrow v_{a_1-1,a_2-1} \\
\end{align*}
\]

Denoting this path by $w_0$, $w_1$, $w_2$, \ldots, $w_{a_1-1}$, we observe that we form an $L(j, k)$-labeling $L$ of $K_{a_1} \times K_{a_2}$ if

\begin{itemize}
\item $L(w_i)$ is a strictly increasing function of $i$;
\item $L(w_{i+1}) - L(w_i) = j$ whenever $w_i$ and $w_{i+1}$ are on the same row;
\item $L(w_{i+1}) - L(w_i) = k - j$ whenever $w_i$ and $w_{i+1}$ are not on the same row.
\end{itemize}

Requiring the additional condition $L(w_0) = L(v_{0,0}) = 0$, such a labeling is given by

\[
L(v_{p,q}) = \begin{cases} 
(p(a_2 - 2) + q)j + pk & \text{if } p \text{ is even} \\
(p(a_2 - 2) + a_2 - 1 - q)j + pk & \text{if } p \text{ is odd}
\end{cases}
\]

But if $a_1 - 1$ is even (resp. odd), then the largest label assigned by $L$ occurs at $(p, q) = (a_1 - 1, a_2 - 1)$ (resp. $(p, q) = (a_1 - 1, 0)$) and equals $(a_1(a_2 - 2) + 1)j + (a_1 - 1)k$ in each case, concluding the proof. \quad \square

**Corollary 5.8.** For integers $3 \leq a_1 \leq a_2$,

\[
\lambda_{x,1}(K_{a_1} \times K_{a_2})^c = \begin{cases} 
(a_1a_2 - 2a_2 + 1) + (a_1 - 1)x & \text{if } 2 \leq x \\
(a_1a_2 - 1) & \text{if } 1 \leq x \leq 2 \\
(a_1^2 - 1)x & \text{if } \frac{1}{a_1 - 1} \leq x \leq 1 \text{ and } a_1 = a_2 \\
(a_1a_2 - 1)x & \text{if } \frac{1}{a_1} \leq x \leq 1 \text{ and } a_1 < a_2 \\
(a_1 - 1) + (2a_1 - 2)x & \text{if } 0 < x \leq \frac{1}{a_1 - 1} \text{ and } a_1 = a_2 \\
(a_2 - 1) + (a_1 - 1)x & \text{if } 0 \leq x \leq \frac{1}{a_1} \text{ and } a_1 < a_2. \quad \square
\end{cases}
\]

6. **On self-complementary graphs**

We next turn our attention to non-trivial self-complementary graphs, which are necessarily of diameter 2 or diameter 3. Examples include $C_5$, $P_4$, and $J_5$ of Fig. 2. (For $\lambda_{x,1}(C_5)$, see [10].)

By Theorem 2.2, all self-complementary graphs of diameter 2 are $\lambda$-invertible. We note that if $G$ is such a graph, then the clique-covering number of $G$ is equal to $\chi(G)$. Moreover, $\lambda_{x,1}(G) = x\lambda_{x,1,2}(G)$, implying that $\lambda_{x,1}(G)$ can be determined for all positive $x$ if $\lambda_{x,1}(G)$ is known for either $0 < x < 1$ or $1 < x$. For illustration, we will establish the $\lambda_{x,1}$-number of each

\[
\begin{align*}
\end{align*}
\]
member of the infinite family $M_1, M_2, M_3, M_4, \ldots$ of self-complementary hamiltonian graphs of diameter 2, where $M_p$ is defined as follows: consider the pairwise disjoint sets

\begin{align*}
A_1 &= \{a_{1,1}, a_{1,2}, \ldots, a_{1,p}\}; \\
A_2 &= \{a_{2,1}, a_{2,2}, \ldots, a_{2,p}\}; \\
B_1 &= \{b_{1,1}, b_{1,2}, \ldots, b_{1,p}\}; \\
B_2 &= \{b_{2,1}, b_{2,2}, \ldots, b_{2,p}\}; \\
C &= \{c\}.
\end{align*}

Then $M_p$ is the smallest graph with vertex set $A_1 \cup A_2 \cup B_1 \cup B_2 \cup C$ and edge set such that:

1. the subgraph induced by $A_1 \cup A_2$ is isomorphic to $K_{2p}$;
2. each vertex in $B_1$ is adjacent to each vertex in $A_1$;
3. each vertex in $B_2$ is adjacent to each vertex in $A_2$;
4. $c$ is adjacent to each vertex in $B_1 \cup B_2$.

Noting that $M_p$ is otherwise known as the $C_5$-join of $(K_p, K_p^c, K_1, K_1^c, K_p)$ (see [4] for a discussion of such joins), we observe that $M_1$ is isomorphic to $C_5$ and we illustrate $M_2$ in Fig. 5.

To establish $\lambda_{k,1}(M_p)$, we first give some supporting notation. Suppose $L$ is an $L(j, k)$-labeling of $M_p$. Since $M_p$ has diameter 2, the labels assigned by $L$ must be distinct, forming a strictly increasing sequence $l = (l_1, l_2, \ldots, l_{4p+1})$ where $l_1 = 0$. Let $l_0 = (l_{b_1}, l_{b_2}, \ldots, l_{b_{2p}})$ be the strictly increasing subsequence of labels that are assigned to the vertices in $A_1 \cup A_2$ and let $l_{b,C}$ denote the strictly increasing subsequence of labels that are assigned to vertices in $B_1 \cup B_2 \cup C$. Denote the intervals

\[ [0, l_0), (l_0, l_1), (l_1, l_2), (l_2, l_3), \ldots, (l_{2p-1}, l_{2p}), (l_{2p}, \text{sp}(L)) \]

by $l_0, l_1, \ldots, l_{2p}$, respectively. (Note that $l_0$ and $l_{2p}$ may be empty.) Clearly if $l_2$ is a component of $l_{b,C}$, then $l_2 \in l_h$ for some $h$.

**Lemma 6.1.** Let $2 \leq \frac{j}{k} \leq 3$. Then $\lambda_{j,k}(M_p) \geq (2p - 3)j + 6k$.

**Proof.** Suppose to the contrary that $L$ is an $L(j, k)$-labeling with span $\text{sp}(L)$ less than $(2p - 3)j + 6k$ and let $l, l_0$, and $l_{b,C}$ be defined as above. Observing that the vertices in $A_1 \cup A_2$ are mutually adjacent, we note that for $1 \leq h \leq 2p - 1$, $l_{b_{h+1}} - l_{b_h} \geq j$, and therefore $l_{b_{2p}} - l_{b_0} \geq (2p - 1)j$.

We first claim that neither $l_0$ nor $l_{2p}$ contains a label from $l_{b,C}$. If $l_0$ contains $m \geq 2$ labels of $l_{b,C}$, then $l_{b_1} \geq mk$ by the distance two condition, implying by our initial observation that $l_{b_{2p}} \geq mk + (2p - 1)j = (2p - 3)j + 2j + mk \geq (2p - 3)j + 4k + mk \geq (2p - 3)j + 6k$. But this contradicts our assumed span of $L$. Therefore $l_0$ contains at most one label of $l_{b,C}$. (A similar argument shows that $l_{2p}$ contains at most one label of $l_{b,C}$.) Now, if $l_0$ and $l_{2p}$ each contain precisely one label of $l_{b,C}$, then $l_{b_1} \geq k$, $l_{b_{2p}} - l_{b_0} \geq (2p - 1)j$, and $l_{b_{2p+1}} - l_{b_{2p}} \geq k$, giving $\text{sp}(L) = l_{b_{2p+1}} - l_1 \geq (2p - 1)j + 2k \geq (2p - 3)j + 6k$, again providing a contradiction of the assumed span of $L$. We thus have that $l_0 \cup l_{2p}$ contains at most one label of $l_{b,C}$. But if $l_0$ contains such a label and $l_{2p}$ does not, then by the pigeonhole principle, there exists $l_h$, $1 \leq h \leq 2p - 1$, such that $l_h$ contains at least two labels of $l_{b,C}$. Thus, by the distance conditions, $l_{b_1} \geq k$, $l_{b_h} - l_{b_1} \geq (h - 1)j$, $l_{b_{h+1}} - l_{b_h} \geq 3k$, and $l_{b_{2p}} - l_{b_{2p+1}} \geq (2p - h - 1)j$. This gives $\text{sp}(L) = l_{b_{2p+1}} \geq (2p - 2)j + 4k \geq (2p - 3)j + 6k$, another contradiction. (A similar argument shows the impossibility of the case in which $l_{2p}$ contains precisely one label of $l_{b,C}$ and $l_0$ contains none.)

Since neither $l_0$ nor $l_{2p}$ contains a label from $l_{b,C}$, the $2p + 1$ labels of $l_{b,C}$ are contained within the union of the $2p - 1$ sets $l_1, l_2, \ldots, l_{2p-1}$. We argue that no set $l_h$ in the union contains $m \geq 3$ labels of $l_{b,C}$. For if the contrary is true, then by the distance conditions, $l_{b_{h+1}} - l_{b_h} \geq (m + 1)k$, implying $\text{sp}(L) = l_{b_{2p}} \geq (2p - 2)j + (m + 1)k \geq (2p - 3)j + 6k$, a contradiction. Thus, for $1 \leq h \leq 2p - 1$, $l_h$ contains at most two labels of $l_{b,C}$. So by the pigeonhole principle, there exist sets $l_0$ and $l_{b'}$ in the union each of which contains precisely two labels of $l_{b,C}$. The distance conditions then imply $l_{b_{h+1}} - l_{b_h} \geq 3k$ and $l_{b'_{h+1}} - l_{b'_h} \geq 3k$. Thus we have the contradiction $\text{sp}(L) = l_{b_{2p}} \geq (2p - 3)j + 6k$. □
Theorem 6.2. For integer \( p \geq 3 \),

\[
\lambda_{x,1}(M_p) = \\
\begin{cases}
2p - 1 & \text{if } 0 < x \leq \frac{1}{3} \\
2p - 3 + 6x & \text{if } \frac{1}{3} \leq x \leq \frac{1}{2} \\
4px & \text{if } \frac{1}{2} \leq x \leq 1 \\
4p & \text{if } 1 \leq x \leq 2 \\
(2p - 3)x + 6 & \text{if } 2 \leq x \leq 3 \\
(2p - 1)x & \text{if } x \geq 3.
\end{cases}
\]

Proof. Since \( M_p \) is self-complementary and \( \lambda \)-invertible, it suffices to give \( \lambda_{x,1}(M_p) \) for \( x \geq 1 \). We consider the cases \( 1 \leq x \leq 2 \), \( 2 \leq x \leq 3 \), and \( x \geq 3 \).

Case 1. \( 1 \leq x \leq 2 \). It can be easily checked that \( M_p \) has a hamilton path and hence has path-covering number equal to one. By Theorem 3.1, \( \lambda_{j,k}(M_p) = 4pk \) and the result therefore follows.

Case 2. \( 2 \leq x \leq 3 \). By Lemma 6.1, it suffices to demonstrate an \( L(j,k) \)-labeling of \( M_p \) with span \( (2p - 3)j + 6k \). It is easily checked that \( L \) is such a labeling where \( L(c) = (p - 2)j + 4k \) and

\[
\begin{align*}
L(a_{1,i}) &= \begin{cases}
(i - 2)j + 3k & \text{if } 2 \leq i \leq p \\
0 & \text{if } i = 1
\end{cases} \\
L(a_{2,i}) &= \begin{cases}
(2p - 1 - i)j + 3k & \text{if } 2 \leq i \leq p \\
(2p - 3)j + 6k & \text{if } i = 1
\end{cases} \\
L(b_{1,i}) &= \begin{cases}
(2p - 1 - i)j + 4k & \text{if } 2 \leq i \leq p \\
(2p - 3)j + 5k & \text{if } i = 1
\end{cases} \\
L(b_{2,i}) &= \begin{cases}
(i - 3)j + 4k & \text{if } 3 \leq i \leq p \\
i & \text{if } 1 \leq i \leq 2.
\end{cases}
\end{align*}
\]

Case 3. \( x \geq 3 \). We recall that the subgraph induced by \( A_1 \cup A_2 \) is isomorphic to \( K_{2p} \). It follows that \( \lambda_{x,1}(M_p) \geq \lambda_{x,1}(K_{2p}) = (2p - 1)x \). It thus suffices to show a \( \lambda_{j,k} \)-labeling of \( M_p \) with span \( (2p - 3)j \) for \( \frac{j}{k} \geq 3 \). It is easily checked that \( L \) is such a labeling where \( L(c) = (p - 1)j + k \), \( L(a_{1,i}) = (i - 1)j \), \( L(a_{2,i}) = (2p - i)j \) for \( 1 \leq i \leq p \), and

\[
\begin{align*}
L(b_{1,i}) &= \begin{cases}
(2p - 2)j + 2k & \text{if } i = 1 \\
(2p - i)j + k & \text{if } 2 \leq i \leq p
\end{cases} \\
L(b_{2,i}) &= \begin{cases}
(i - 2)j + k & \text{if } 3 \leq i \leq p \\
i & \text{if } 1 \leq i \leq 2.
\end{cases}
\end{align*}
\]

We remark that the sequences of vertices induced by the strictly increasing sequences of labels assigned to the vertices by the labelings of Cases 2 and 3 of Theorem 6.2 are identical. Particularly, this sequence \( s \) is the catenation of the following sequences:

\[
s_1 = \langle a_{1,1}, b_{2,1}, b_{2,2}, a_{1,2} \rangle \\
s_2 = \langle b_{2,3}, a_{1,3}, b_{2,4}, a_{1,4}, \ldots, b_{2,p-1}, a_{1,p-1}, b_{2,p}, a_{1,p} \rangle \\
s_3 = \langle c \rangle \\
s_4 = \langle a_{2,p}, b_{1,p}, a_{2,p-1}, b_{1,p-1}, \ldots, a_{2,4}, b_{1,4}, a_{2,3}, b_{1,3} \rangle \\
s_5 = \langle a_{2,2}, b_{1,2}, b_{1,1}, a_{2,1} \rangle.
\]

We now complete our investigation of \( M_p \) by establishing \( \lambda_{x,1}(M_2) \). We will be using the notation and terms defined immediately prior to Lemma 6.1.

Lemma 6.3. Let \( 2 \leq \frac{j}{k} \) and \( p = 2 \). Then \( \lambda_{j,k}(M_p) \geq 2j + 4k \).

Proof. Let \( L \) be an \( L(j,k) \)-labeling of \( M_p \) with span less than \( 2j + 4k \) and let \( \tau = \{\alpha, \beta, \gamma\} = \{1, 2, 3\} \). Using arguments identical to those given in Lemma 6.1, we have that no label of \( I_{\beta,C} \) is in \( I_0 \cup I_k \), and that no \( m \geq 3 \) distinct labels of \( I_{\beta,C} \) are in \( I_h \), \( h \in \tau \). Thus with no loss of generality \( I_\alpha \) contains labels of two distinct vertices \( b_{i,x} \) and \( b_{i,y} \) in \( B_1 \cup B_2 \) and \( I_\beta \) contains the labels of two distinct vertices in \( B_1 \cup B_2 \cup C \). Now, either \( i = i' \) or \( i \neq i' \). If the latter, then the length of \( I_\alpha \) is at least \( j + k \) since at least one endpoint of \( I_\alpha \) is assigned to a vertex adjacent to either \( b_{i,x} \) or \( b_{i,y} \). Similarly, the length of \( I_\beta \) is at least \( 3k \) and the length of \( I_\gamma \) is at least \( j \). Since these minimum lengths imply the contradiction \( \text{sp}(L) \geq 2j + 4k \), we have
that \( I_\rho \) contains the labels of two vertices \( b_{1,1}, b_{1,2} \) in the same set \( B_i \). With no loss of generality, suppose \( i = 1 \). Then either \( I_\beta \) contains the labels of \( b_{2,1} \) and \( b_{2,2} \) or \( I_\rho \) contains the labels of \( b_{2,x} \) and \( c \). If the latter, then \( I_\rho \) has length at least \( j + 2k \), \( I_\rho \) has length at least \( 3k \), and \( I_\rho \) has length at least \( j \), implying a contradiction to the assumed span of \( L \) as above. Thus we assume that \( I_\rho \) contains the labels of \( b_{2,1} \) and \( b_{2,2} \) and, by implication, \( I_\rho \) contains the label of \( c \).

If at least one endpoint of \( I_\rho \) (resp. \( I_\beta \)) is the label of a vertex in \( A_1 \) (resp. \( A_2 \)), then the length of \( I_\rho \) (resp. \( I_\beta \)) is at least \( j + 2k \). Since this leads to the immediately previous contradiction, we have that the endpoints of \( I_\rho \) are the labels of vertices in \( A_2 \) and the endpoints of \( I_\beta \) are the labels of vertices in \( A_1 \). So with no loss of generality

\[
0 = L(a_{2,1}) < L(b_{1,1}) < L(b_{1,2}) < L(a_{2,2}) < L(c) < L(a_{1,1}) < L(b_{2,1}) < L(b_{2,2}) < L(a_{1,2}) = \text{sp}(L).
\]

But \( L(b_{1,1}) - L(a_{2,1}) \geq 2k \), \( L(c) - L(b_{1,2}) \geq j \), \( L(b_{2,1}) - L(c) \geq j \), and \( L(a_{1,1}) - L(b_{2,1}) \geq 2k \), implying the contradiction \( \text{sp}(L) \geq 2j + 4k \). \( \square \)

**Theorem 6.4.** For integer \( p = 2 \),

\[
\lambda_{x,1}(M_p) = \lambda_{x,1}(M_2) = \begin{cases} 
3 & \text{if } 0 \leq x \leq \frac{1}{4} \\
4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\
8x & \text{if } \frac{1}{2} \leq x \leq 1 \\
8 & \text{if } 1 \leq x \leq 2 \\
2x + 4 & \text{if } 2 \leq x \leq \frac{3}{2} \\
3x & \text{if } x \geq 4.
\end{cases}
\]

**Proof.** As in **Theorem 6.2**, it suffices to establish \( \lambda_{x,1}(M_2) \) for \( x \geq 1 \).

Case 1. \( 1 \leq x \leq 2 \). We appeal to **Theorem 3.1** since the path-covering number of (self-complementary) \( M_2 \) is 1.

Case 2. \( 2 \leq x \leq 4 \). By **Lemma 6.3**, it suffices to find an \( L(j, k) \)-labeling of \( M_2 \) with span \( 2j + 4k \). Such a labeling is given in **Table 2** in the two subcases \( 2 \leq x \leq 3 \) and \( 3 \leq x \leq 4 \). We note that the sequences of vertices induced by each of these labelings are each equal to \( (a_{1,1}, b_{1,1}, b_{2,2}, a_{1,2}, c, a_{2,2}, b_{1,2}, b_{1,1}, a_{2,1}) \), which equals \( s \) (defined following **Theorem 6.2**).

Case 3. \( x \geq 4 \). Similar to our strategy in Case 3 of **Theorem 6.2**, we observe that the subgraph induced by \( A_1 \cup A_2 \) is isomorphic to \( K_4 \). Since \( \lambda_{x,1}(M_2) \geq \lambda_{x,1}(K_4) = 3x \), it suffices to show a \( \lambda_{j,k} \)-labeling of \( M_2 \) with span \( 3j \) for \( \frac{1}{k} \geq 4 \). Such a labeling is given in **Table 2**. We note that the sequence of vertices induced by this labeling is equal to \( s \). \( \square \)

To close this section, we point out that the \( P_4 \)-join of \( (K_2^c, K_p, K_p, K_p^c) \) (the graph \( M_p - \{c\} \)) is a family of self-complemental graphs of diameter 3 of which \( P_4 \) is the only \( \lambda \)-invertible graph. It can be easily checked that for \( p \geq 2 \), \( \chi(M_p - \{c\}) \) equals \( 2p \) and that \( \lambda_{0,1}(M_p - \{c\}) = 2p - 2 \). (We are aware of self-complemental graphs \( G \) of diameter 3 with \( \chi(G) = 1 = \lambda_{0,1}(G) \), yet \( G \) is not \( \lambda \)-invertible.)

In closing, we give the following:

**Conjecture.** The only self-complemental \( \lambda \)-invertible graph with diameter 3 is \( P_4 \).

More generally, we have been unable to find any other diameter 3 graph that is \( \lambda \)-invertible.

**Acknowledgments**

The authors wish to thank the referees for their constructive comments that resulted in an improved paper.

**References**


