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Remarks on periodic boundary value problems for functional differential equations☆

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Abstract

We extend some results on existence and approximation of solution for a class of first-order functional differential equations with periodic boundary conditions. We show the validity of the monotone iterative technique under weaker hypotheses and present some examples.

Keywords: Functional differential equation; Periodic boundary value problem; Lower and upper solutions; Monotone iterative method

1. Introduction

Functional differential equations of first-order with periodic boundary conditions are considered by different authors [1–11]. In this paper, we study the periodic boundary value problem:

$$u'(t) = g(t, u(t), u(\theta(t))), \quad t \in I,$$

$$u(0) = u(T) \tag{1}$$

with $g : I \times \mathbb{R}^2 \to \mathbb{R}$ continuous, and $\theta : I \to \mathbb{R}$ continuous verifying

$$0 \leqslant \theta(t) \leqslant t, \quad t \in I.$$

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We say that α, β are lower and upper solutions of (1) if there exist $M, N \geqslant 0$ such that

$$\alpha'(t) \leqslant g(t, \alpha(t), \alpha(\theta(t))) - a(t), \quad t \in I,$$

where

$$a(t) = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T) \end{cases}$$

and

$$\beta'(t) \geqslant g(t, \beta(t), \beta(\theta(t))) + b(t), \quad t \in I$$

with

$$b(t) = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T). \end{cases}$$

The definition of classical lower and upper solutions makes reference to the case $\alpha(0) \leqslant \alpha(T)$ and $\beta(0) \geqslant \beta(T)$.

In [10, Theorem 5.1] we have proved the following result.

Theorem 1.1. *Suppose that $g \in C(I \times \mathbb{R}^2, \mathbb{R})$ and that there exist $\alpha, \beta \in C^1(I)$ lower and upper solutions of problem (1), respectively, satisfying that $\alpha \leqslant \beta$ on I. Moreover, assume that for the constants M, N given above, the following hypotheses are verified:*

(H_3) $NTe^{MT} \leqslant 1$.
(H_4) $N < M$.
(H_5) $g(t, x, y) - g(t, u, v) \geqslant -M(x - u) - N(y - v)$
for every $t \in I$, $\alpha(t) \leqslant u \leqslant x \leqslant \beta(t)$, $\alpha(\theta(t)) \leqslant v \leqslant y \leqslant \beta(\theta(t))$.

Then there exist monotone sequences $\{\alpha_n\} \uparrow \rho$, $\{\beta_n\} \downarrow \gamma$ uniformly on I with $\alpha_0 = \alpha \leqslant \alpha_n \leqslant \beta_n \leqslant \beta_0 = \beta$ for every $n \in \mathbb{N}$. Here ρ, γ are, respectively, the minimal and maximal solutions of problem (1) in

$$[\alpha, \beta] = \{u \in C(I) : \alpha(t) \leqslant u(t) \leqslant \beta(t), \quad t \in I\},$$

that is, if u is a solution of (1) on $[\alpha, \beta]$, then $u \in [\rho, \gamma]$.

In this paper, we extend this result by showing that hypothesis (H_3) can be improved and that condition (H_4) can be eliminated in the development of the monotone method for problem (1).

We start our study by recalling, in Section 2, some helpful results from [10] and then proving some comparison theorems that will be very useful later in our procedure. In Section 3, we give a different proof for the existence theorem related to a linear problem associated to Eq. (1) which yields to the development of the monotone iterative technique for (1) under more general conditions (Section 4). In Section 5, we make some remarks on the possibility of obtaining periodic lower and upper solutions given nonperiodic lower and upper solutions. Finally, in Section 6, we give some examples to illustrate the applications of the new results.

2. Preliminaries

Theorem 2.1 (Nieto and Rodríguez-López [10, Theorem 2.1]). *Let $M > 0$, $N > 0$, and $\theta : I \to I$ continuous with*

$$0 \leqslant \theta(t) \leqslant t, \quad t \in I.$$

Then, the problem

$$u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in I,$$

$$u(0) = u_0 \tag{2}$$

has a unique solution for each $u_0 \in \mathbb{R}$. In addition, the solution has continuous dependence on σ and on the initial condition u_0.

Remark. This theorem is an extension of the well-known result for linear ordinary differential equations ($N = 0$) and can be obtained as a consequence of a more general result from [5, Theorem 1.4].

In what follows, we assume that $\theta : I \to \mathbb{R}$ is continuous and verifies

$$0 \leqslant \theta(t) \leqslant t, \quad t \in I.$$

Theorem 2.2 (Nieto and Rodríguez-López [10, Theorem 3.1]). *Let $u \in C^1(I)$, $M > 0$, $N \geqslant 0$, such that*:

(i) $u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0$, $t \in I$.
(ii) $u(0) \leqslant u(T)$.
(iii) $NT\mathrm{e}^{MT} \leqslant 1$.

Then $u \leqslant 0$ on I.

If (ii) does not hold, the conclusion is not valid in general. In the case $u(0) > u(T)$ we have the following maximum principle.

Theorem 2.3 (Nieto and Rodríguez-López [10, Theorem 3.2]). *Let $u \in C^1(I)$, $M > 0$, $N \geqslant 0$, $u(0) > u(T)$ and $NT\mathrm{e}^{MT} \leqslant 1$. If*

$$u'(t) + Mu(t) + Nu(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}[u(0) - u(T)] \leqslant 0, \quad t \in I$$

then, $u(t) \leqslant 0$, $t \in I$.

We now present a comparison result.

Theorem 2.4. *Let* $u \in C^1(I)$, $M > 0$, $N \geqslant 0$, *such that*

(i) $u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0$, $t \in I$.
(ii) $u(0) \leqslant 0$.
(iii) $N \int_0^T e^{M(t-\theta(t))} \, dt \leqslant 1$.

Then $u \leqslant 0$ *on* I.

Proof. The result is valid for $N = 0$. In consequence, we assume that $N > 0$.

Let $v(t) = e^{Mt}u(t), t \in [0, T]$. Then, using (i),

$$v'(t) = Me^{Mt}u(t) + e^{Mt}u'(t) \leqslant -Ne^{Mt}u(\theta(t)), \quad t \in I$$

or, equivalently,

$$v'(t) \leqslant -Ne^{M(t-\theta(t))}v(\theta(t)), \quad t \in I. \tag{3}$$

The functions u and v have the same sign, so we have to prove that $v \leqslant 0$ on I. If this was false, there would exist $t_1 \in I$ such that $v(t_1) > 0$. Since $v(0) = u(0) \leqslant 0$, then $t_1 \in (0, T]$. Let $t_2 \in [0, t_1)$ such that

$$v(t_2) = \min_{t \in [0, t_1]} v(t) \leqslant 0.$$

Now, if we integrate expression (3) between t_2 and t_1, we obtain that

$$v(t_1) - v(t_2) \leqslant -N \int_{t_2}^{t_1} e^{M(t-\theta(t))} v(\theta(t)) \, dt$$

$$\leqslant -v(t_2) N \int_0^T e^{M(t-\theta(t))} \, dt \leqslant -v(t_2),$$

where we have taken into account that $0 \leqslant \theta(t) \leqslant t$, for $t \in I$ and condition (iii). Thus, we obtain $v(t_1) \leqslant 0$, which is absurd. This proves that $v \leqslant 0$ on I, and so, $u \leqslant 0$ on I. □

If we consider the estimate $N(e^{MT} - 1)/M \leqslant 1$, then (iii) of Theorem 2.4 is true, since

$$N \int_0^T e^{M(t-\theta(t))} \, dt \leqslant N \int_0^T e^{Mt} \, dt = \frac{N(e^{MT} - 1)}{M} \leqslant 1.$$

Thus, this hypothesis (iii) improves that on Theorem 5 in [8] for this kind of inequalities.

It is obvious that we can replace the condition $NTe^{MT} \leqslant 1$ in Theorems 2.2 and 2.3 by condition (iii) of Theorem 2.4.

Theorem 2.5. *Let* $u \in C^1(I)$, $M > 0$, $N \geqslant 0$, *such that*

(i) $u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0$, $t \in I$, *if* $u(0) \leqslant u(T)$.
(ii) $u'(t) + Mu(t) + Nu(\theta(t)) + (Mt + N\theta(t) + 1)/T[u(0) - u(T)] \leqslant 0$, $t \in I$, *if* $u(0) > u(T)$.
(iii) $N \int_0^T e^{M(t-\theta(t))} \, dt \leqslant 1$.

Then $u \leqslant 0$ *on* I.

Proof. In the case of (i), if $u \geqslant 0$ on I, then $u'(t) \leqslant 0$ on I, so u is a nonincreasing function. This fact joint to $u(0) \leqslant u(T)$ produces that u is a constant function, so that $u' \equiv 0$ and also $u \equiv 0$.

Thus, we can consider that u takes some negative value. The proof consists on demonstrate that $u(0) \leqslant 0$ so that we could apply Theorem 2.4 and affirm that $u \leqslant 0$. If $u(0) > 0$, also $u(T) > 0$, and considering again the function v defined by $v(t) = e^{Mt}u(t)$, $t \in [0,T]$, we obtain that $v(0) > 0$, $v(T) > 0$ and $v(t_\star) = \min_{[0,T]} v < 0$, with $t_\star \in (0,T)$. The integration of (3) between $t_\star$ and T yields

$$-v(t_\star) < v(T) - v(t_\star) \leqslant -N \int_{t_\star}^{T} v(\theta(t))e^{M(t-\theta(t))}\,dt$$

$$\leqslant -Nv(t_\star)\int_{t_\star}^{T} e^{M(t-\theta(t))}\,dt \leqslant -Nv(t_\star)\int_{0}^{T} e^{M(t-\theta(t))}\,dt \leqslant -v(t_\star),$$

which is absurd. Then $u(0) \leqslant 0$ and the conclusion follows.

In case (ii), we consider the function $m(t) = u(t) + t/T(u(0) - u(T))$, that verifies $m(0) = m(T)$ and is included in case (i). Thus, $m \leqslant 0$ on I and obviously $u \leqslant 0$ on I. □

3. Existence of solution for the linear problem

Now, we are going to prove the result given in Theorem 4.1 [10], using a sharper estimate on the constants and eliminating the assumption $N < M$, which is replaced by the existence of appropriate lower and upper solutions.

Theorem 3.1. *Let $\sigma \in C(I)$, $M > 0$, $N \geqslant 0$ and consider the problem*:

$$u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in I,$$

$$u(0) = u(T). \tag{4}$$

Suppose that there exist $\alpha, \beta \in C^1(I)$ such that:

(h$_1$) $\alpha \leqslant \beta$ *on* I.
(h$_2$) $\alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \leqslant \sigma(t) - a(t), \quad t \in I,$
$\beta'(t) + M\beta(t) + N\beta(\theta(t)) \geqslant \sigma(t) + b(t), \quad t \in I,$
where

$$a(t) = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$b(t) = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T). \end{cases}$$

(h_3) $N \int_0^T e^{M(t-\theta(t))}\,dt \leqslant 1$.

Then, there exists a unique solution u for problem (4). *Moreover*, $u \in [\alpha, \beta]$.

Proof. We first prove the uniqueness of solution for this problem. If u_1, u_2 are solutions of (4), set $v_1 = u_1 - u_2$ and $v_2 = u_2 - u_1$. Thus,

$$v_1(0) = v_1(T), \quad v_1'(t) + Mv_1(t) + Nv_1(\theta(t)) = 0, \quad t \in I,$$
$$v_2(0) = v_2(T), \quad v_2'(t) + Mv_2(t) + Nv_2(\theta(t)) = 0, \quad t \in I.$$

By Theorem 2.5, we have that $v_1 = u_1 - u_2 \leqslant 0$ and $v_2 = u_2 - u_1 \leqslant 0$, and hence $u_1 = u_2$.

Now, we show that if u is a solution to (4), then $\alpha \leqslant u \leqslant \beta$. Define $m_1 = \alpha - u$ and $m_2 = u - \beta$. We can write that

$$m_1'(t) + Mm_1(t) + Nm_1(\theta(t)) \leqslant 0, \quad t \in I \quad \text{if } m_1(0) \leqslant m_1(T),$$

$$m_1'(t) + Mm_1(t) + Nm_1(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}(m_1(0) - m_1(T))$$
$$\leqslant 0, \quad t \in I \quad \text{if } m_1(0) > m_1(T),$$

$$m_2'(t) + Mm_2(t) + Nm_2(\theta(t)) \leqslant 0, \quad t \in I \quad \text{if } m_2(0) \leqslant m_2(T),$$

$$m_2'(t) + Mm_2(t) + Nm_2(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}(m_2(0) - m_2(T))$$
$$\leqslant 0, \quad t \in I \quad \text{if } m_2(0) > m_2(T).$$

Now, Theorem 2.5 allows to assure that $m_1 = \alpha - u \leqslant 0$ and $m_2 = u - \beta \leqslant 0$.

To prove the existence of solution, we consider the functions

$$\bar{\alpha}(t) = \begin{cases} \alpha(t) & \text{if } \alpha(0) \leqslant \alpha(T), \\ \alpha(t) + \frac{t}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T) \end{cases} \tag{5}$$

and

$$\bar{\beta}(t) = \begin{cases} \beta(t) & \text{if } \beta(0) \geqslant \beta(T), \\ \beta(t) - \frac{t}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T). \end{cases} \tag{6}$$

It is evident that $\alpha \leqslant \bar{\alpha}$ and $\bar{\beta} \leqslant \beta$ on I. Also, $\bar{\alpha}(0) = \alpha(0) \leqslant \bar{\alpha}(T)$ and $\bar{\beta}(0) = \beta(0) \geqslant \bar{\beta}(T)$. Note that, if $\alpha(0) > \alpha(T)$, $\bar{\alpha}$ is T-periodic, and the same for $\bar{\beta}$, if $\beta(0) < \beta(T)$.

We can check that $\bar{\alpha}$ and $\bar{\beta}$ are classical lower and upper solutions, respectively, for problem (4) and that $\bar{\alpha} \leqslant \bar{\beta}$, so that $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$. Indeed,

$$\bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t))$$
$$= \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \leqslant \sigma(t), \quad t \in I \quad \text{if } \alpha(0) \leqslant \alpha(T),$$

$$\bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t))$$
$$= \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) \leqslant \sigma(t), \quad t \in I \quad \text{if } \alpha(0) > \alpha(T)$$

and

$$\bar{\beta}'(t) + M\bar{\beta}(t) + N\bar{\beta}(\theta(t))$$
$$= \beta'(t) + M\beta(t) + N\beta(\theta(t)) \geqslant \sigma(t), \quad t \in I \quad \text{if } \beta(0) \geqslant \beta(T),$$

$$\bar{\beta}'(t) + M\bar{\beta}(t) + N\bar{\beta}(\theta(t))$$
$$= \beta'(t) + M\beta(t) + N\beta(\theta(t)) - \frac{Mt + N\theta(t) + 1}{T}(\beta(T) - \beta(0)) \geqslant \sigma(t), \quad t \in I \quad \text{if } \beta(0) < \beta(T).$$

Thus, $\bar{\alpha}$ is a classical lower solution and $\bar{\beta}$ is a classical upper solution for (4). Now, consider the function $m = \bar{\alpha} - \bar{\beta} \in C^1(I)$. It is easy to prove that

$$m'(t) + Mm(t) + Nm(\theta(t))$$
$$= \bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) - \bar{\beta}'(t) - M\bar{\beta}(t) - N\bar{\beta}(\theta(t))$$
$$\leqslant \sigma(t) - \sigma(t) = 0, \quad t \in I.$$

Also, $m(0) = \bar{\alpha}(0) - \bar{\beta}(0) = \alpha(0) - \beta(0) \leqslant 0$. Using Theorem 2.4, we obtain that $m \leqslant 0$ on I or, equivalently, $\bar{\alpha} \leqslant \bar{\beta}$ on I. In [7, Theorem 3.1], it was proved that if we have well-ordered classical lower and upper solutions χ and ζ for (4) and $(M + N)T < 1$, there exists a unique solution for (4) and it lies between χ and ζ. It is not difficult to show that, in this result, the hypothesis $(M+N)T < 1$ can be replaced by condition (h_3). The proof consists on obtaining a periodic solution for the initial value problem

$$u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in I,$$
$$u(0) = a \tag{7}$$

for some value of a. We know that this problem has a unique solution (denoted by $u(\cdot; a)$) and that the solution continuously depends on the initial data σ and a. If $\chi(0) < \zeta(0)$, we define the continuous function $\psi : [\chi(0), \zeta(0)] \to \mathbb{R}$ by $\psi(a) = u(T; a)$ and prove the existence of a fixed point for ψ. Using Theorem 2.4, it is not difficult to check that if $a \in [\chi(0), \zeta(0)]$ then $u(\cdot; a)$ lies between χ and ζ. Setting $m_1(t) = \chi(t) - u(t;a)$ and $m_2(t) = u(t;a) - \zeta(t)$, we obtain that $m_1, m_2 \in C^1(I)$ and

$$\begin{aligned} & m_1'(t) + Mm_1(t) + Nm_1(\theta(t)) \\ & \quad = \chi'(t) + M\chi(t) + N\chi(\theta(t)) - u'(t;a) - Mu(t;a) - Nu(\theta(t);a) \\ & \quad \leqslant \sigma(t) - \sigma(t) = 0, \quad t \in I, \end{aligned}$$

$$m_1(0) = \chi(0) - u(0;a) = \chi(0) - a \leqslant 0,$$

$$\begin{aligned} & m_2'(t) + Mm_2(t) + Nm_2(\theta(t)) \\ & \quad = u'(t;a) + Mu(t;a) + Nu(\theta(t);a) - \zeta'(t) - M\zeta(t) - N\zeta(\theta(t)) \\ & \quad \leqslant \sigma(t) - \sigma(t) = 0, \quad t \in I, \end{aligned}$$

$$m_2(0) = u(0;a) - \zeta(0) = a - \zeta(0) \leqslant 0.$$

Using assumption (h_3) and applying Theorem 2.4, we get that $m_1 \leqslant 0$ and $m_2 \leqslant 0$ on I, so that $\chi \leqslant u(\cdot;a) \leqslant \zeta$ on I.

Thus, ψ applies the interval $[\chi(0), \zeta(0)]$ into $[\chi(T), \zeta(T)] \subseteq [\chi(0), \zeta(0)]$ and it has a fixed point c. So we have obtained a periodic solution $u(\cdot;c)$ to (7), that is, a solution to (4).

In the case $\chi(0) = \zeta(0)$, $u(\cdot;\chi(0))$ is a solution to (4).

Applying this result to lower and upper solutions $\bar{\alpha}$ and $\bar{\beta}$, we show that there exists a solution to (4) between $\bar{\alpha}$ and $\bar{\beta}$ and, therefore, in $[\alpha, \beta]$. This completes the proof. □

This result could have been proved considering different cases depending on how the values of α and β at the boundary of I are related and taking into account Theorem 2.5. But, in some cases, the proof we have made provides a better localization of the solution. Also, this result improves Theorem 3.1 [7].

4. Monotone method

Theorem 4.1. *Assuming that* $M > 0$, *it is possible to develop the monotone iterative technique as in Theorem* 1.1 [10, Theorem 5.1] *without the hypotheses* (H_3) $NTe^{MT} \leqslant 1$ *and* (H_4) $N < M$ *and considering the estimate* (h_3) $N \int_0^T e^{M(t-\theta(t))}\, dt \leqslant 1$ *instead.*

Proof. If α and β are lower and upper solutions for (1), then the functions $\bar{\alpha}$ and $\bar{\beta}$ defined by (5) and (6) verify that $\bar{\alpha}(0) \leqslant \bar{\alpha}(T)$, $\bar{\beta}(0) \geqslant \bar{\beta}(T)$, $\alpha(t) \leqslant \bar{\alpha}(t)$, $\bar{\beta}(t) \leqslant \beta(t)$, for $t \in I$ and $\bar{\alpha} \leqslant \bar{\beta}$. To prove the last assertion, take $m = \bar{\alpha} - \bar{\beta} \in C^1(I)$. Then $m(0) = \alpha(0) - \beta(0) \leqslant 0$. In the case $\alpha(0) > \alpha(T)$

and $\beta(0) < \beta(T)$, we have, according to (H_5), that

$$\begin{aligned} & m'(t) + Mm(t) + Nm(\theta(t)) \\ & \quad = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) \\ & \qquad -\beta'(t) - M\beta(t) - N\beta(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}(\beta(T) - \beta(0)) \\ & \quad \leqslant g(t, \alpha(t), \alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) \\ & \qquad -g(t, \beta(t), \beta(\theta(t))) - M\beta(t) - N\beta(\theta(t)) \leqslant 0, \quad t \in I. \end{aligned}$$

The validity of this inequality in other cases can be proved analogously. Now, using (h_3) and applying Theorem 2.4, we obtain that $m \leqslant 0$ on I and then $\bar{\alpha}(t) \leqslant \bar{\beta}(t)$ for $t \in I$.

Moreover, $\bar{\alpha}$ and $\bar{\beta}$ are, respectively, classical lower and upper solutions for (1). Indeed, if $\alpha(0) > \alpha(T)$, then

$$\begin{aligned} \bar{\alpha}'(t) &= \alpha'(t) + \frac{1}{T}(\alpha(0) - \alpha(T)) \\ & \leqslant g(t, \alpha(t), \alpha(\theta(t))) - \frac{Mt + N\theta(t)}{T}(\alpha(0) - \alpha(T)). \end{aligned}$$

Since $\alpha \leqslant \bar{\alpha} \leqslant \beta$, according to ($H_5$), we get that

$$g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) - g(t, \alpha(t), \alpha(\theta(t))) \geqslant -M(\bar{\alpha}(t) - \alpha(t)) - N(\bar{\alpha}(\theta(t)) - \alpha(\theta(t)))$$

for $t \in I$, so that

$$\begin{aligned} \bar{\alpha}'(t) & \leqslant g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M(\bar{\alpha}(t) - \alpha(t)) + N(\bar{\alpha}(\theta(t)) - \alpha(\theta(t))) \\ & \quad - \frac{Mt + N\theta(t)}{T}(\alpha(0) - \alpha(T)) = g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + \frac{Mt}{T}(\alpha(0) - \alpha(T)) \\ & \quad + \frac{N\theta(t)}{T}(\alpha(0) - \alpha(T)) - \frac{Mt + N\theta(t)}{T}(\alpha(0) - \alpha(T)) = g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) \end{aligned}$$

and it is trivial when $\alpha(0) \leqslant \alpha(T)$. Thus, $\bar{\alpha}$ is a classical lower solution for (1). Analogously for $\bar{\beta} \in [\alpha, \beta]$ and using (H_5), we get, for the case $\beta(0) < \beta(T)$, that

$$\begin{aligned} \bar{\beta}'(t) &= \beta'(t) - \frac{1}{T}(\beta(T) - \beta(0)) \\ & \geqslant g(t, \beta(t), \beta(\theta(t))) + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0)) \\ & \geqslant g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))) - M(\beta(t) - \bar{\beta}(t)) - N(\beta(\theta(t)) - \bar{\beta}(\theta(t))) \\ & \quad + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0)) = g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))) - \frac{Mt}{T}(\beta(T) - \beta(0)) \\ & \quad - \frac{N\theta(t)}{T}(\beta(T) - \beta(0)) + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0)) = g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))) \end{aligned}$$

and obviously for $\beta(0) \geqslant \beta(T)$. So that $\bar{\beta}$ is a classical upper solution for (1). In [7, Theorem 3.2] it was developed the monotone iterative technique for problem (1) under the restriction $(M+N)T < 1$. Following a similar proof that takes into account Theorem 2.5 instead of Theorem 2.1 [7], it can be proved that there exist two monotone sequences that approximate the extremal solutions of (1) between the lower and upper solutions. This improved version of Theorem 3.2 [7], can be applied to obtain ρ, γ the minimal and maximal solutions in $[\bar{\alpha}, \bar{\beta}]$.

Otherwise, the whole proof of Theorem 5.1 in [10] would serve if we use Theorem 2.5 as our maximum principle and if we justify that the operator $\mathscr{A}$ is well defined.

For each $\eta \in [\bar{\alpha}, \bar{\beta}]$, we consider the problem:

$$u'(t) + Mu(t) + Nu(\theta(t)) = \sigma_\eta(t), \quad t \in I,$$
$$u(0) = u(T), \tag{8}$$

where σ_η is a continuous function given by $\sigma_\eta(t) = g(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t))$. If we show that the hypotheses of Theorem 3.1 are verified, we will get that there exists a unique solution u for (8) and that $\bar{\alpha} \leqslant u \leqslant \bar{\beta}$. In this case, we could define $\mathscr{A} : \eta \in [\bar{\alpha}, \bar{\beta}] \to \mathscr{A}\eta = u \in [\bar{\alpha}, \bar{\beta}]$ and complete the proof in the same way as in the cited theorem. We only have to prove that the functions $\bar{\alpha}$ and $\bar{\beta}$ are lower and upper solutions for problem (8), respectively. This is easy to check, since $\alpha \leqslant \bar{\alpha} \leqslant \eta \leqslant \bar{\beta} \leqslant \beta$ and hypothesis (H_5) implies that

$$\begin{aligned}
&\bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) \\
&\quad \leqslant g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) \\
&\quad \leqslant g(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t)) \\
&\quad = \sigma_\eta(t), \quad t \in I
\end{aligned}$$

and analogously for $\bar{\beta}$.

So we can apply Theorem 3.1 and the proof is complete. □

This result improves Theorem 3.2 [7]. Note that the monotone sequences that we obtain approximate the extremal solutions between $\bar{\alpha}$ and $\bar{\beta}$.

5. Notes on the definition of lower and upper solutions

Thus, as we have seen above, the existence of α, β according to the definition of lower and upper solutions given in the introduction, makes it possible to find $\bar{\alpha}$, $\bar{\beta}$ classical lower and upper solutions, respectively, with $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$. So it suggests that it will be enough to consider classical lower and upper solutions. We also have checked that, if $\alpha(0) > \alpha(T)$ and $\beta(0) < \beta(T)$, then $\bar{\alpha}$ and $\bar{\beta}$ are T-periodic functions. The question now is to know if we are able to find, beginning with a lower solution α with $\alpha(0) < \alpha(T)$, another lower solution $\tilde{\alpha}$ with $\alpha \leqslant \tilde{\alpha} \leqslant \beta$ and $\tilde{\alpha}(0) = \tilde{\alpha}(T)$. Analogously, if β is an upper solution for (1), with $\beta(0) > \beta(T)$, we ask for the existence of another upper solution $\tilde{\beta}$ such that $\alpha \leqslant \tilde{\beta} \leqslant \beta$ and $\tilde{\beta}(0) = \tilde{\beta}(T)$. With this purpose, we set

$$\tilde{\alpha}(t) = \alpha(t) + \frac{T-t}{T}(\alpha(T) - \alpha(0)), \quad t \in I \tag{9}$$

and

$$\tilde{\beta}(t) = \beta(t) - \frac{T-t}{T}(\beta(0) - \beta(T)), \quad t \in I. \tag{10}$$

We will prove that these functions are appropriate to our requirements if we impose some restrictions.

Theorem 5.1. *Under hypotheses* (H_5) *and* $(M+N)T \leqslant 1$, *if* α *and* β *are classical lower and upper solutions for* (1) *with* $\alpha(0) < \alpha(T)$ *and* $\beta(0) > \beta(T)$, *then the functions* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *defined by* (9) *and* (10) *are T-periodic classical lower and upper solutions, respectively, for problem* (1) *and* $\alpha \leqslant \tilde{\alpha} \leqslant \tilde{\beta} \leqslant \beta$ *on* I, *so that* $[\tilde{\alpha}, \tilde{\beta}] \subseteq [\alpha, \beta]$.

Proof. Obviously, $\alpha \leqslant \tilde{\alpha}$ and $\tilde{\beta} \leqslant \beta$ on I, $\tilde{\alpha}(0) = \alpha(T) = \tilde{\alpha}(T)$ and $\tilde{\beta}(0) = \beta(T) = \tilde{\beta}(T)$.

To prove that $\tilde{\alpha} \leqslant \tilde{\beta}$, set $m = \tilde{\alpha} - \tilde{\beta} \in C^1(I)$. Then, using ($H_5$), we get that

$$\begin{aligned}
&m'(t) + Mm(t) + Nm(\theta(t)) \\
&\quad = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) - \beta'(t) - M\beta(t) - N\beta(\theta(t)) \\
&\qquad + \frac{M(T-t) + N(T-\theta(t)) - 1}{T}(\alpha(T) - \alpha(0) + \beta(0) - \beta(T)) \\
&\quad \leqslant g(t, \alpha(t), \alpha(\theta(t))) - g(t, \beta(t), \beta(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) - M\beta(t) \\
&\qquad - N\beta(\theta(t)) + \frac{M(T-t) + N(T-\theta(t)) - 1}{T}(\alpha(T) - \alpha(0) + \beta(0) - \beta(T)) \\
&\quad \leqslant \frac{(M+N)T - 1}{T}(\alpha(T) - \alpha(0) + \beta(0) - \beta(T)) \leqslant 0, \quad t \in I.
\end{aligned}$$

Also, $m(0) = \tilde{\alpha}(0) - \tilde{\beta}(0) = \alpha(T) - \beta(T) \leqslant 0$. Using Theorem 2.4, we obtain that $m \leqslant 0$ on I or, equivalently, $\tilde{\alpha} \leqslant \tilde{\beta}$ on I. Note that $(M+N)T \leqslant 1$ implies that $NTe^{MT} \leqslant 1$ and, therefore, (h_3) $N\int_0^T e^{M(t-\theta(t))}\,dt \leqslant 1$ is true, so that the applicability of Theorem 2.4 is justified.

Finally, we will show that $\tilde{\alpha}$ and $\tilde{\beta}$ are, respectively, lower and upper solutions for (1). Using that $\alpha \leqslant \tilde{\alpha} \leqslant \beta$ and (H_5), we get

$$\begin{aligned}
\tilde{\alpha}'(t) &= \alpha'(t) + \frac{-1}{T}(\alpha(T) - \alpha(0)) \leqslant g(t, \alpha(t), \alpha(\theta(t))) + \frac{-1}{T}(\alpha(T) - \alpha(0)) \\
&\leqslant g(t, \tilde{\alpha}(t), \tilde{\alpha}(\theta(t))) + \frac{M(T-t)}{T}(\alpha(T) - \alpha(0)) \\
&\quad + \frac{N(T-\theta(t))}{T}(\alpha(T) - \alpha(0)) - \frac{1}{T}(\alpha(T) - \alpha(0)) \\
&= g(t, \tilde{\alpha}(t), \tilde{\alpha}(\theta(t))) + (M(T-t) + N(T-\theta(t)) - 1)\frac{\alpha(T) - \alpha(0)}{T} \\
&\leqslant g(t, \tilde{\alpha}(t), \tilde{\alpha}(\theta(t))) + \frac{(M+N)T - 1}{T}(\alpha(T) - \alpha(0)) \\
&\leqslant g(t, \tilde{\alpha}(t), \tilde{\alpha}(\theta(t))), \quad t \in I,
\end{aligned}$$

since $(M+N)T \leqslant 1$. Therefore, $\tilde{\alpha}$ is a T-periodic lower solution for (1).

Following an analogous reasoning for $\tilde{\beta}$,

$$\begin{aligned}\tilde{\beta}'(t) &= \beta'(t) + \frac{1}{T}(\beta(0) - \beta(T)) \geqslant g(t, \beta(t), \beta(\theta(t))) + \frac{1}{T}(\beta(0) - \beta(T)) \\ &\geqslant g(t, \tilde{\beta}(t), \tilde{\beta}(\theta(t))) - \frac{M(T-t)}{T}(\beta(0) - \beta(T)) \\ &\quad - \frac{N(T-\theta(t))}{T}(\beta(0) - \beta(T)) + \frac{1}{T}(\beta(0) - \beta(T)) \\ &= g(t, \tilde{\beta}(t), \tilde{\beta}(\theta(t))) + (-M(T-t) - N(T-\theta(t)) + 1)\frac{\beta(0) - \beta(T)}{T} \\ &\geqslant g(t, \tilde{\beta}(t), \tilde{\beta}(\theta(t))), \quad t \in I.\end{aligned}$$

And so, $\tilde{\beta}$ is a T-periodic upper solution for (1). □

6. Examples

Consider the equation

$$\begin{aligned}&u'(t) = g(t, u(t), u(\theta(t))) = \cos u(t) - 2u(\tfrac{1}{2}t) + e^t, \quad t \in I = [0, \tfrac{1}{3}], \\ &u(0) = u(\tfrac{1}{3}).\end{aligned} \tag{11}$$

In this case,

$$\begin{aligned}g(t, x, y) - g(t, u, v) &= \cos x - 2y + e^t - (\cos u - 2v + e^t) \\ &= \cos x - \cos u - 2(y - v) \geqslant -(x - u) - 2(y - v)\end{aligned}$$

for all $t \in I$, $x, y, u, v \in \mathbb{R}$, $x \geqslant u$, $y \geqslant v$. Then, condition (H_5) is valid for $M = 1$ and $N = 2$, while condition (H_4) fails. The hypothesis (h_3) holds, since $4(e^{1/6} - 1) \leqslant 1$.

It is not difficult to find lower and upper solutions consisting of constant functions. Indeed, let $\alpha(t) = \frac{4}{5}$, $t \in I$ is a lower solution for (11). We have

$$\alpha'(t) = 0 \leqslant g\left(t, \alpha(t), \alpha\left(\frac{1}{2}t\right)\right) = \cos\frac{4}{5} - \frac{8}{5} + e^t, \quad t \in I, \quad \alpha(0) = \alpha\left(\frac{1}{3}\right).$$

The function $\beta(t) = 1$, $t \in [0, \frac{1}{3}]$ is an upper solution for (11):

$$\beta'(t) = 0 \geqslant g\left(t, \beta(t), \beta\left(\frac{1}{2}t\right)\right) = \cos 1 - 2 + e^t, \quad t \in I.$$

Using Theorem 4.1, there exist monotone sequences converging uniformly to the extremal solutions of (11) in the functional interval $[\alpha, \beta]$ (Fig. 1).

Consider the function $\zeta(t) = e^t - \pi/100$, $t \in I$, and ζ is an upper solution for problem (11). Indeed, $\zeta(0) = 1 - \pi/100 < e^{1/3} - \pi/100 = \zeta(\frac{1}{3})$, and

$$\zeta'(t) = e^t \geqslant \cos\left(e^t - \frac{\pi}{100}\right) - 2\left(e^{(1/2)t} - \frac{\pi}{100}\right) + e^t + 3(2t + 1)(e^{1/3} - 1), \quad t \in I.$$

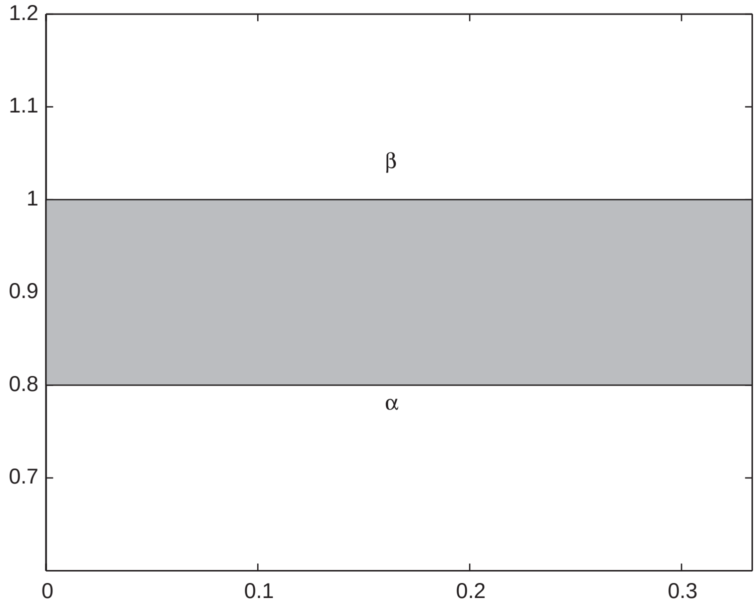

Fig. 1. Functional interval $[\alpha, \beta]$.

To see the validity of this inequality, let

$$\varphi(t) = \cos\left(e^t - \frac{\pi}{100}\right) - 2\left(e^{(1/2)t} - \frac{\pi}{100}\right) + 3(2t+1)(e^{1/3} - 1), \quad t \in I$$

and note that φ has in the interval $[0, \frac{1}{3}]$, a negative maximum value.

Therefore, applying again Theorem 4.1, we obtain the existence of monotone sequences that approximate the extremal solutions of (11) in a functional interval contained in $[\alpha, \zeta]$.

Observe that we cannot find a constant upper solution in the set $[\alpha, \zeta]$.

The function $\chi(t) = e^{-t} - \pi/20$, $t \in I$, is also a lower solution for (11). Note that $\chi(0) > \chi(\frac{1}{3})$.

Thus, there exist extremal solutions for problem (11) in a functional interval contained in the set shown in Fig. 2,

$$\{u \in C^1(I) : \chi(t) \leqslant u(t) \leqslant \zeta(t), \quad t \in I\}.$$

Obviously, there is no constant lower nor constant upper solution between χ and ζ.

Now, consider

$$\chi_1(t) = e^{-t} - \frac{\pi}{20} + \frac{\chi(0) - \chi(1/3)}{1/3}\, t = e^{-t} - \frac{\pi}{20} + 3(1 - e^{-1/3})t, \quad t \in I,$$

that is a periodic lower solution for (11) on $[0, \frac{1}{3}]$.

The function

$$\zeta_1(t) = e^t - \frac{\pi}{100} - \frac{\zeta(1/3) - \zeta(0)}{1/3}\, t = e^t - \frac{\pi}{100} - 3(e^{1/3} - 1)t, \quad t \in I$$

is a periodic upper solution for (11).

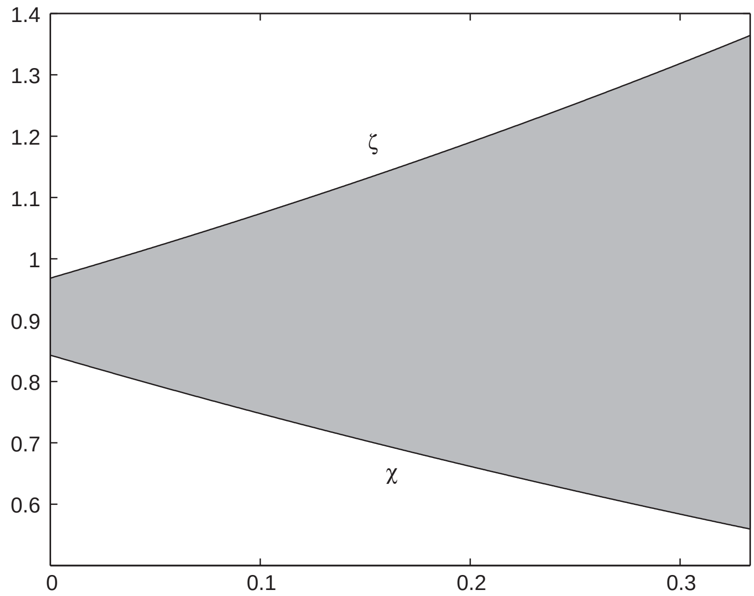

Fig. 2. Functional interval $[\chi, \zeta]$.

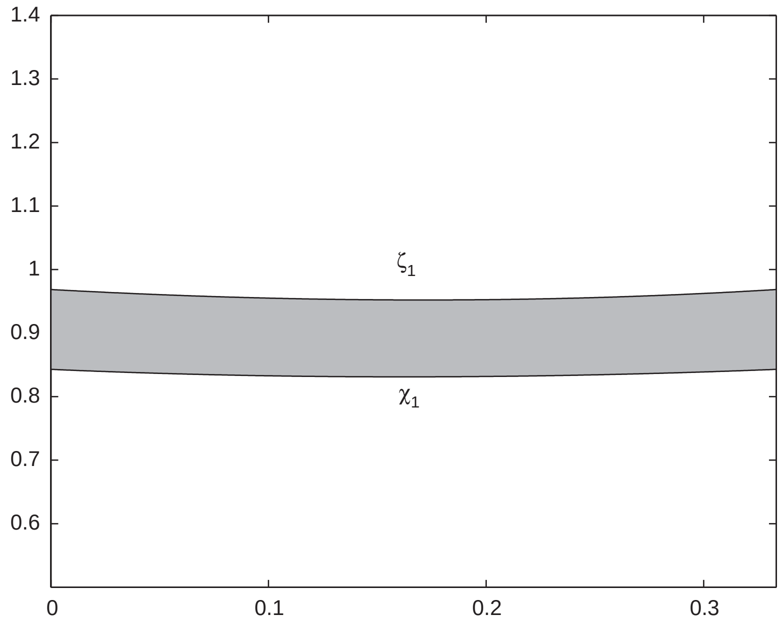

Fig. 3. Functional interval $[\chi_1, \zeta_1]$.

In fact, $\chi_1 \equiv \bar{\chi}$ and $\zeta_1 \equiv \bar{\zeta}$ defined by (5) and (6). What Theorem 4.1 provides is the existence of monotone sequences to approximate the extremal solutions of (11) in the functional interval $[\chi_1, \zeta_1]$ (Fig. 3).

In this example, given a lower and an upper solutions we are always able to construct periodic classical lower and upper solutions, since $(M+N)T=1$.

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References

[1] R. Hakl, I. Kiguradze, B. Půža, Upper and lower solutions of boundary value problems for functional differential equations and theorems on functional differential inequalities, Georgian Math J. 7 (2000) 489–512.
[2] R. Hakl, A. Lomtatidze, B. Půža, On periodic solutions of first-order linear functional differential equations, Nonlinear Anal. 49 (2002) 929–945.
[3] Z. He, J. Yu, Periodic boundary value problem for first-order impulsive functional differential equations, J. Comput. Appl. Math. 138 (2002) 205–217.
[4] D. Jiang, J. Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, Nonlinear Anal. 50 (2002) 885–898.
[5] I. Kiguradze, B. Půža, On boundary value problems for systems of linear functional differential equations, Czechoslovak Math J. 47 (1997) 341–373.
[6] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.
[7] E. Liz, J.J. Nieto, Periodic boundary value problems for a class of functional differential equations, J. Math. Anal. Appl. 200 (1996) 680–686.
[8] J.J. Nieto, Differential inequalities for functional perturbations of first-order ordinary differential equations, Appl. Math. Lett. 15 (2002) 173–179.
[9] J.J. Nieto, N. Álvarez-Noriega, Periodic boundary value problems for nonlinear first order ordinary differential equations, Acta Math. Hungar. 71 (1996) 49–58.
[10] J.J. Nieto, R. Rodríguez-López, Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions, Comput. Math. Appl. 40 (2000) 433–442.
[11] Z. Wan, Y. Chen, J. Chen, Remarks on the periodic boundary value problems for first-order differential equations, Comput. Math. Appl. 37 (8) (1999) 49–55.