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# NOTE

# Homogeneous and Ultrahomogeneous Linear Spaces

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mapping $S'$ onto $S''$. If every isomorphism from $S'$ to $S''$ can be extended to an automorphism of $S$, $S$ is called *ultrahomogeneous*. We give a complete classification of all homogeneous (resp. ultrahomogeneous) linear spaces, without making any finiteness assumption on the number of points of $S$. 

## 1. INTRODUCTION

A *linear space* $S$ is a non-empty set of elements called *points*, provided with a collection of subsets called *lines* such that any pair of points is contained in exactly one line and every line contains at least two points. The line containing two points $x$ and $y$ will be denoted by $xy$. If $S'$ is a non-empty subset of $S$, the *linear structure induced on* $S'$ is the linear space whose points are those of $S'$ and whose lines are the intersections of $S'$ with all the lines of $S$ having at least two points in $S'$.

Given a positive integer $d$, a linear space $S$ is said to be *d-homogeneous* if, whenever the linear structures induced on two subsets $S'$ and $S''$ of cardinality at most $d$ are isomorphic, there is at least one automorphism of $S$ mapping $S'$ onto $S''$; if every isomorphism from $S'$ to $S''$ can be extended to an automorphism of $S$, we shall say that $S$ is *d-ultrahomogeneous*. $S$ is called *homogeneous* (resp. *ultrahomogeneous*) if it is $d$-homogeneous (resp. $d$-ultrahomogeneous) for all positive integers $d$. Note that, in an ultrahomogeneous linear space $S$, every automorphism of the linear structure induced on a finite subset $S'$ extends to an automorphism of $S$.

Historically, the notions of homogeneity and ultrahomogeneity arise from the Helmholtz–Lie principle of "free mobility of rigid bodies", a property



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common to the motion groups of Euclidean and non-Euclidean spaces: in such spaces, every (finite) set of points can be carried by motion onto any congruent one (see for example Freudenthal [6] for more details).

The purpose of this paper is to give a complete classification of all homogeneous (resp. ultrahomogeneous) linear spaces $S$, without making any finiteness assumption on the number of points of $S$. Actually, we will prove a result which is a little stronger:

THEOREM. *Any 6-homogeneous linear space is homogeneous. Any homogeneous linear space is one of the following*:

(1) *a linear space reduced to a single point* (*and no line*) *or to a single line*;

(2) *a linear space all of whose lines have exactly two points*;

(3) *the Desarguesian affine plane* $AG(2, 3)$;

(4) *one of the Desarguesian projective planes* $PG(2, 2)$, $PG(2, 3)$ *or* $PG(2, 4)$.

*Any homogeneous linear space is ultrahomogeneous, except* $AG(2, 3)$ *and* $PG(2, 4)$.

The Desarguesian planes $AG(2, 3)$ and $PG(2, 4)$ are well-known to have several remarkable features (think for example of their connection with the Mathieu groups). It is amazing that they can also be characterized as the only linear spaces which are homogeneous but not ultrahomogeneous. Surprisingly, a computer was needed to prove the homogeneity of $PG(2, 4)$.

Note also that the value $d = 6$ in the first sentence of the above theorem is best possible. Indeed, $PG(2, 5)$ is not homogeneous but it is easy to check, using well-known transitivity properties of the group $PGL(3, 5)$, that $PG(2, 5)$ is 5-homogeneous.

The classification of *finite* 2-homogeneous linear spaces is due to Kantor [8] and Delandtsheer, Doyen, Siemons and Tamburini [4]. However, the proof relies on the classification of finite simple groups. Our theorem does not make such assumptions.

The notions of homogeneity and ultrahomogeneity can be defined more generally in any class of relational structures (see Cameron [2]), for example the class of undirected graphs. Gardiner [7] proved that a finite ultrahomogeneous undirected graph is either a disjoint union of isomorphic complete graphs or a regular complete multipartite graph or the $3 \times 3$ lattice graph or the graph of the pentagon. Ronse [10] showed that the list of finite homogeneous undirected graphs is exactly the same (compare with the situation in our theorem). Finally, Lachlan and Woodrow [9] have determined all countable ultrahomogeneous graphs; one of them is the famous countable random graph discovered by Erdös and Rényi [5] and discussed in detail by Cameron [3].

## 2. PROOF OF THE THEOREM

We will need a few definitions. In a linear space $S$, a *quadrangle* $Q$ is a set of four points, no three of which are collinear. The six lines joining two points of $Q$ are its *diagonals*. A *diagonal point* of $Q$ is a point $p$ of $S$ not belonging to $Q$ such that any line joining $p$ to a point of $Q$ intersects $Q$ in exactly two points. Thus a quadrangle may have 0, 1, 2 or 3 diagonal points.

The linear spaces belonging to the classes (1) and (2) in the statement of the theorem are clearly homogeneous, and even ultrahomogeneous. Let $S$ be a 6-homogeneous linear space which is not in one of these classes, so that every point of $S$ is on at least two lines and some line of $S$ has at least 3 points.

Since the linear structures induced on any two subsets of cardinality 2 are isomorphic and since $S$ is 6-homogeneous, *Aut* $S$ is transitive on the unordered pairs of points of $S$, hence also on the lines of $S$. It follows that all the lines of $S$ have the same size $k \geqslant 3$ (which may be finite or infinite).

We distinguish two cases:

*Case* I: $k \geqslant 4$

Let $L_1, L_2$ be two lines through a point $p$ of $S$ and let $x_1, y_1, z_1$ (resp. $x_2, y_2, z_2$) be three points of $L_1$ (resp. of $L_2$) distinct from $p$. The linear space induced by $S$ on the set $X = \{x_1, y_1, z_1, x_2, y_2, z_2\}$ has two disjoint lines of size 3, all the other lines being of size 2.

Suppose that $S$ contains two disjoint lines $L'_1$ and $L'_2$. Then, if $x'_1, y'_1, z'_1$ (resp. $x'_2, y'_2, z'_2$) are three points of $L'_1$ (resp. of $L'_2$), the linear space induced on $X' = \{x'_1, y'_1, z'_1, x'_2, y'_2, z'_2\}$ is isomorphic to the one induced on $X$. Therefore, some automorphism $\alpha$ of $S$ must map $X$ onto $X'$, a contradiction because $\alpha$ would have to map the two intersecting lines $L_1, L_2$ onto the two disjoint lines $L'_1, L'_2$.

It follows that $S$ has no pair of disjoint lines, and so $S$ is a projective plane with lines of size $k \geqslant 4$.

If $k = 4$, then $S = PG(2, 3)$ and it is not difficult to check, using the group *Aut* $S = PGL(3, 3)$, that $S$ is ultrahomogeneous (hence also homogeneous).

If $k = 5$, then $S = PG(2, 4)$ and it is very tedious to prove by hand that $S$ is indeed homogeneous with respect to the group *Aut* $S = P\Gamma L(3, 4)$. Actually, this was checked by computer and the authors would like to thank Raymond Devillers for his kind and valuable help in this matter. On the other hand, $PG(2, 4)$ is not ultrahomogeneous, as is easily seen by considering the symmetric difference $S' = L_1 \triangle L_2$ of two lines of $PG(2, 4)$: the linear structure induced on $S'$ admits an automorphism $\alpha'$ interchanging two points of $L_1$ and fixing the other 6 points of $S'$, but $\alpha'$ does not extend to an automorphism $\alpha$ of $PG(2, 4)$.

We will now show that no projective plane $S$ with lines of size $k \geqslant 6$ is 6-homogeneous.

Let $\{x, y, z, t\}$ be a quadrangle of $S$ and let $a, b, c$ be its diagonal points, where $a = xy \cap zt$.

Suppose first that $a, b, c$ are not collinear. Then the line $L = bc$ intersects $xy$ and $zt$ in two points $p$ and $q$. Since $k \geqslant 6$, there is a point $u \in L$ distinct from $b, c, p$ and $q$. The linear structure induced on $X = \{a, x, y, z, t, u\}$ consists of two intersecting lines of size 3, all the other lines being of size 2. On the other hand, we claim that there is a point $v$ which does not belong to $L$ nor to the union of the 6 diagonals of the quadrangle $\{x, y, z, t\}$. This is obvious if $k$ is infinite. If $k = n + 1$ is finite, an easy counting argument shows that the number of such points is equal to

$$(n^2 + n + 1) - 4 - 3 - 6(n - 2) - (n - 3) = (n - 3)^2 \geqslant 1$$

as soon as $n \geqslant 4$. The linear structure induced on $X' = \{a, x, y, z, t, v\}$ is isomorphic to the one induced on $X$. However, there is no automorphism $\alpha$ of $S$ mapping $X$ onto $X'$ because $\alpha$ would have to leave $L$ invariant and map $u \in L$ onto $v \notin L$.

Suppose now that $a, b, c$ are collinear and let $L$ be the line containing these three diagonal points. Since $k \geqslant 6$, there is a point $u \in L$ distinct from $a, b$ and $c$. The linear structure induced on $Y = \{a, x, y, z, t, u\}$ consists of two intersecting lines of size 3, all the other lines being of size 2. On the other hand, there is a point $v$ which does not belong to any of the 7 lines of $S$ intersecting the subplane $\{a, b, c, x, y, z, t\}$ in 3 points. This is obvious if $k$ is infinite. If $k = n + 1$ is finite, the number of such points is equal to

$$(n^2 + n + 1) - 7 - 7(n - 2) = (n - 2)(n - 4) \geqslant 1$$

as soon as $n \geqslant 5$. The linear structure induced on $Y' = \{a, x, y, z, t, v\}$ is isomorphic to the one induced on $Y$, but one can show as above that there is no automorphism of $S$ mapping $Y$ onto $Y'$.

*Case* II: $k = 3$

Since the linear structures induced on any two quadrangles of $S$ are isomorphic, *Aut* $S$ acts transitively on the set of all quadrangles. Therefore, any two quadrangles of $S$ have the same number $\delta$ of diagonal points.

Let $L_1, L_2$ be two lines through a point $p \in S$ and let $x_1, y_1$ (resp. $x_2, y_2$) be two points of $L_1$ (resp. of $L_2$) distinct from $p$. Since the quadrangle $\{x_1, y_1, x_2, y_2\}$ has at least one diagonal point, it follows that $\delta = 1, 2$ or 3.

We shall consider these three cases separately (actually, the conclusions obtained below can be derived from a theorem of Buekenhout, Metz and Totten [1] classifying the linear spaces in which all quadrangles have the same number of diagonal points; however, since our proof is very short due to the fact that $k=3$, we will give it to make our paper self-contained).

If $\delta=3$, let $Q=\{x, y, z, t\}$ be a quadrangle of $S$ and let $a=xy\cap zt$, $b=xz\cap yt$ and $c=xt\cap yz$ be its diagonal points. Since the quadrangle $\{x, y, b, c\}$ has also 3 diagonal points, $\{a, b, c\}$ must be a line of $S$, and so $Q$ is contained in a linear space $S'$ consisting of the 7 points $x, y, z, t, a, b, c$ and isomorphic to $PG(2, 2)$. If $S$ contains a point $u\notin S'$, the diagonals $xy$ and $zu$ of the quadrangle $\{x, y, z, u\}$ must intersect in the point $a$, contradicting the fact that $S$ is a linear space since the pair $\{a, z\}$ is contained in two different lines. Therefore $S$ is isomorphic to $PG(2, 2)$ and it is easy to check that $S$ is ultrahomogeneous (hence also homogeneous).

If $\delta=2$, let $Q=\{x, y, z, t\}$ be a quadrangle of $S$ and let $a=xy\cap zt$ and $b=xz\cap yt$ be its diagonal points, the other two diagonals of $Q$ being $\{x, t, c\}$ and $\{y, z, d\}$ with $c\neq d$. The quadrangle $Q'=\{x, z, a, c\}$ must also have two diagonal points, one of which being $t=xc\cap za$. If the diagonals $xa$ and $zc$ of $Q'$ intersect, it is necessarily in the point $y$, but this is impossible since $y$, $z$ and $d$ are already collinear. Therefore, the second diagonal point of $Q'$ must be the intersection of $xz$ and $ac$, which implies that $\{a, b, c\}$ is a line of $S$. A similar argument applied to the quadrangle $Q''=\{z, t, b, d\}$ shows that $\{a, b, d\}$ must be a line of $S$. This contradicts the fact that $S$ is a linear space, since the pair $\{a, b\}$ is contained in two different lines.

If $\delta=1$, let $Q=\{x, y, z, t\}$ be a quadrangle of $S$ and let $a=xy\cap zt$ be its unique diagonal point, the other diagonals of $Q$ being $\{x, z, b\}$, $\{y, t, c\}$, $\{x, t, d\}$ and $\{y, z, e\}$. The quadrangle $Q'=\{x, z, t, c\}$ must also have a unique diagonal point. The diagonals $xz$ and $tc$ are disjoint. If the diagonals $xc$ and $zt$ of $Q'$ intersect, it is necessarily in the point $a$, but this is impossible since $x$, $y$ and $a$ are already collinear. Therefore, the diagonal point of $Q'$ must be the intersection of $xt$ and $zc$, which forces $\{z, c, d\}$ to be a line of $S$. By applying similar arguments successively to the quadrangles $\{x, y, t, b\}$, $\{x, z, t, e\}$, $\{t, c, d, e\}$, $\{x, y, d, e\}$ and $\{y, a, c, d\}$, it turns out that $\{y, b, d\}$, $\{t, b, e\}$, $\{x, c, e\}$, $\{a, d, e\}$ and $\{a, b, c\}$ must be lines of $S$. Hence $Q$ is contained in a linear space $S'$ consisting of the 9 points $x, y, z, t, a, b, c, d, e$ and isomorphic to $AG(2, 3)$. If $S$ contains a point $u\notin S'$, the quadrangle $\{x, y, z, u\}$ must also have a unique diagonal point; this forces one of the sets $\{x, u, e\}$, $\{y, u, b\}$, $\{z, u, a\}$ to be a line of $S$, contradicting the fact that $S$ is a linear space. Therefore $S$ is isomorphic to $AG(2, 3)$ and it is easy to check that $S$ is homogeneous. However, $AG(2, 3)$ is not ultrahomogeneous: for example, the linear structure induced on the union of two disjoint lines $L_1$ and $L_2$ admits an automorphism

$\alpha'$ interchanging two points of $L_1$ and fixing the other 4 points of $L_1 \cup L_2$, but $\alpha'$ does not extend to an automorphism of $AG(2, 3)$.

This ends the proof of the theorem.

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