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A NOTE ON SHORT CYCLES IN DIGRAPHS

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In 1977, Caccetta and Haggkvist conjectured that if G is a directed graph with n vertices and minimal outdegree k, then G contains a directed cycle of length at most $\lceil n/k \rceil$. This conjecture is known to be true for $k \leqslant 3$. In this paper, we present a proof of the conjecture for the cases $k=4$ and $k=5$. We also present a new conjecture which implies that of Caccetta and Haggkvist.

In 1977, Caccetta and Haggkvist [1] conjectured that if G is a directed graph with n vertices and if each vertex of G has outdegree at least k, then G contains a directed cycle of length at most $\lceil n/k \rceil$. We shall refer to this conjecture as the C–H conjecture. Trivially, this conjecture is true for $k=1$; and it has been proved for $k=2$ (Caccetta and Haggkvist [1]) and $k=3$ (Hamildoune [3]). Chvátal and Szemerédi [2] proved that the C–H conjecture holds if we replace the bound on the length of the cycle by $\lceil n/k \rceil + 2500$. In this paper, we prove that the C–H conjecture holds for $k \leqslant 5$. Our main results are obtained by extending the arguments of Chvátal and Szemerédi.

In order to prove the C–H conjecture for $k \leqslant 5$, we show first that if the conjecture fails for a small value of k, then it must fail on a reasonably small graph.

Theorem 1. *Suppose that the C–H conjecture is not true. Let k_1 be the smallest k for which the C–H conjecture does not hold. Then the conjecture fails on some graph G, with minimal outdegree k_1, such that G has at most $3k_1^2$ vertices.*

Proof. Let k_1 be the smallest k for which the C–H conjecture fails. Let G be the smallest graph on which the C–H conjecture fails for $k=k_1$. By removing edges (if necessary), we can ensure that the outdegree of each vertex in G is k_1. Let n be the number of vertices of G and write $t=\lceil n/k_1 \rceil$. We define $\operatorname{dist}(x, y)$ to be the number of edges in a shortest path from x to y. For any vertex in G, we set $S_i^x = \{y \mid \operatorname{dist}(x,y)=i\}$ and $T_i^x = \bigcup_{j=1}^{i} S_j^x$.

First we claim that

$$\text{There exists a vertex } x \text{ such that } |T_i^x| < ik \text{ for some integer } i \leqslant \lceil \tfrac{1}{2}t \rceil. \tag{1}$$

Suppose that (1) is false, then for every x we have $|T_{\lceil \frac{1}{2}t \rceil}^x| \geqslant \lceil \frac{1}{2}t \rceil k_1$. Now, by an averaging argument, we know that there exists an x such that the set $U^x_{\lceil\frac{1}{2}t\rceil} = \{y \mid \text{dist}(y,x) \leqslant \lceil\frac{1}{2}t\rceil\}$ contains at least $\lceil\frac{1}{2}t\rceil k_1$ vertices. We also know that $|T^x_{\lfloor\frac{1}{2}t\rfloor}| \geqslant \lceil\frac{1}{2}t\rceil k_1$. Now, $|U^x_{\lceil\frac{1}{2}t\rceil}| + |T^x_{\lfloor\frac{1}{2}t\rfloor}| \geqslant \lceil\frac{1}{2}t\rceil k_1 + \lfloor\frac{1}{2}t\rfloor k_1 > |G-x|$, and so at least one vertex y is in $T^x_{\lfloor\frac{1}{2}t\rfloor} \cap U^x_{\lceil\frac{1}{2}t\rceil}$. This implies that $\text{dist}(x,y) + \text{dist}(y,x) \leqslant t$, and it follows that G has a cycle of length at most t, a contradiction. Thus, (1) holds.

In the remainder of the proof, x will be a vertex which satisfies (1) and i will be the smallest integer for which (1) is satisfied with this x. We claim that

$$\tfrac{1}{2}k_1 < |S^x_i| < k_1. \tag{2}$$

By the minimality of i, we have $|S^x_i| < k_1$. Write $r = |S^x_i|$ and let F be the subgraph of G induced by T^x_{i-1}. Since each vertex in F has outdegree at least $k_1 - r$, the graph F contains a cycle of length at most t', where $t' = |F|/(k_1 - r)$. Since this cycle is contained in G, it follows that $t' > t$. Now, $|F| = |T^x_i| - r < \lceil\frac{1}{2}t\rceil k_1 - r$. Thus,

$$\frac{\lceil\frac{1}{2}t\rceil k_1 - r}{k_1 - r} > t. \tag{3}$$

It is easy to see that (3) is satisfied only if $r > \frac{1}{2}k_1$. (To see this; note that $t \geqslant 2\lceil\frac{1}{2}t\rceil - 1$, so

$$\begin{aligned}(3) \;&\Rightarrow\; \lceil\tfrac{1}{2}t\rceil k_1 - r > 2\lceil\tfrac{1}{2}t\rceil k_1 - k_1 + r - 2\lceil\tfrac{1}{2}t\rceil r\\ &\Rightarrow\; 2\lceil\tfrac{1}{2}t\rceil r + k_1 - 2r > \lceil\tfrac{1}{2}t\rceil k_1.\end{aligned}$$

Substituting $a = r - \frac{1}{2}k_1$ we get $2(\lceil\frac{1}{2}t\rceil - 1)a > 0$.
Clearly, $\lceil\frac{1}{2}t\rceil - 1 \geqslant 0$, it follows that $a > 0$ and $r > \frac{1}{2}k_1$.) We have shown that $r < k_1$ and $r > \frac{1}{2}k_1$, which is precisely (2).

From (2), it is easy to see that

$$|S^x_{i-1}| < 2k_1 - |S^x_i| \leqslant \lfloor\tfrac{3}{2}k_1\rfloor. \tag{4}$$

(By the minimality of i, we have $|S^x_{i-1}| + |S^x_i| < 2k_1$. So (2) implies that $|S^x_{i-1}| < \lfloor\frac{3}{2}k_1\rfloor$. In fact, $|S^x_i| \geqslant \lfloor\frac{1}{2}k_1\rfloor + 1$, so $|S^x_{i-1}| \leqslant 2k_1 - \lfloor\frac{1}{2}k_1\rfloor - 1$.)

Finally, we claim that

$$G \text{ contains a cycle of length at most } \left\lceil\frac{T^x_{i-1}}{k_1}\right\rceil + |S^x_{i-1}| \leqslant \lceil\tfrac{1}{2}t\rceil + \lfloor\tfrac{3}{2}k_1\rfloor. \tag{5}$$

To prove (5), we may assume that S^x_{i-1} is acyclic for otherwise it contains a cycle of length at most $|S^x_{i-1}|$, and we are done. It follows that from every vertex y in S^x_{i-1}, there exists a path in S^x_{i-1} to a vertex $y' \in S^x_{i-1}$ such that the outdegree of y' in S^x_{i-1} is zero. Now, since $|S^x_i| < k_1$, there must be a vertex $y'' \in T^x_{i-2}$ such that $y'y''$ is an edge of G.

Now consider the graph H obtained from the vertices of T^x_{i-1} in the following manner:

(i) Keeping all edges uv (of G) with $u, v \in T^x_{i-1}$;

(ii) For each vertex y in S^x_{i-1}, we find y'', and add 'new' edge yz whenever $y''z$ is an edge of G.

Clearly, the minimal outdegree of H is k_1. Thus, by the minimality of G, H contains a cycle C of length at most $\lceil |H|/k_1 \rceil = \lceil T^x_{i-1}/k_1 \rceil \leqslant \lceil \frac{1}{2}t \rceil$. Now, each 'new' edge yz of C with $y \in S^x_{i-1}$ can be replaced by a path, consisting of only 'old' edges of G, from y to z going through y' and y''.

Thus from C, we create a subgraph C' of T^x_{i-1} which contains a cycle. Now C' has at most $\lceil |T^x_{i-1}|/k_1 \rceil$ vertices in T^x_{i-2} (if there are m vertices y in $C \cap S^x_{i-1}$, then we add at most m vertices y'' to C', so $|C' \cap T^x_{i-1}| \leqslant \lceil |T^x_{i-1}|/k_1 \rceil$). Clearly, C' contains a cycle of length at most $\lceil |T^x_{i-1}|/k_1 \rceil + |S^x_{i-1}|$. This establishes (5).

Now, (5) implies that G contains a cycle of length at most $\lceil \frac{1}{2}t \rceil + |S^x_{i-1}| \leqslant \lceil \frac{1}{2}t \rceil + \lfloor \frac{3}{2}k_1 \rfloor$. Since every cycle of G has length greater than

$$t = \left\lceil \frac{n}{k_1} \right\rceil < \lceil \tfrac{1}{2}t \rceil + |S^x_{i-1}| \leqslant \lceil \tfrac{1}{2}t \rceil + \lfloor \tfrac{3}{2}k_1 \rfloor, \tag{6}$$

it follows trivially that $n < 3k_1^2$. □

Using Theorem 1, we can prove that the C–H conjecture holds for $k \leqslant 5$. In particular, for $k = 2$ and $k = 3$, we only need verify that the conjecture holds for graphs with at most 12 vertices, and at most 27 vertices, respectively. In fact, when the value of k is small, a much sharper bound on the size of a minimal counter example can be obtained. For example, if $k = 2$, then (2) implies that $1 < |S^x_i| < 2$; thus the C–H conjecture must hold for $k = 2$. If $k = 3$, then (2) implies that $|S^x_i| = 2$, (4) implies that $|S^x_{i-1}| \leqslant 3$, (5) and (6) imply that $n \leqslant 15$; thus it remains to show that, for $k = 3$, the C–H conjecture holds for all graphs with at most 15 vertices. We omit these details as they are analogous to, and much simpler than, arguments presented in the proof of the following theorem.

Theorem 2. *The C–H conjecture holds for $k \leqslant 5$.*

Proof. We are going to present a detailed proof for the case $k = 4$. For the case $k = 5$, we shall only show that a minimal counter example can contain at most 55 vertices. An easy (and long) argument showing that the C–H conjecture holds for all graphs with at most 55 vertices is omitted.

Case 1. $k = 4$

We know that the C–H conjecture holds for $k \leqslant 3$. Assume that the conjecture fails for $k = 4$. Let G be the smallest graph (with minimal outdegree four) for which the conjecture fails. Define n and t as usual. First note that the conjecture holds trivially for $t \leqslant 2$. When $t \geqslant 3$, our arguments shall rely on the arguments used in the proof of Theorem 1. Choose a vertex x and a smallest integer i such that x and i satisfy (1). With $k_1 = 4$, (2) implies that $|S_i^x| = 3$. Now, we have $|T_{i-1}^x| = |T_i^x| - |S_i^x|$. Since i is chosen so that $|T_i^x| < 4i$, we have $|T_{i-1}^x| < 4i - 3$, or $|T_{i-1}^x| < 4(i-1) < 4(\lceil \frac{1}{2}t \rceil - 1)$. But (5) implies that G contains a cycle of length at most $t' = \frac{1}{4} \cdot 4(\lceil \frac{1}{2}t \rceil - 1) + 4 = \lceil \frac{1}{2}t \rceil + 3$. By our assumption, we must have $t' > t$ (recall that $t = \frac{1}{4}|G|$). If $t > 5$, then we have $t' = \lceil \frac{1}{2}t \rceil + 3 \leqslant t$, a contradiction. It remains to show that the conjecture holds for $t = 3$, 4 and 5.

If $t = 3$, then an averaging argument establishes the existence of a vertex x with $|S_i^x| = 4$, $|S_2^x| = 3$, $|U_1^x| = 4$ (recall that U_1^x consists of all vertices z such that zx is an edge). Now S_1^x contains a cycle and S_2^x must be acyclic (since $|S_2^x| = 3$). Note that the cycle in S_1^x must be of length 4 and so each vertex in S_1^x has outdegree 1 in S_1^x. It follows that yz is an edge whenever $y \in S_1^x$, $z \in S_2^x$. Also, since S_2^x is acyclic and $|U_1^x| = 4$, it follows that there is a vertex $u \in S_2^x$ such that uv is an edge whenever $v \in U_1^x$. If there is a vertex $a \in U_1^x$ such that ay is an edge for some y in S_1^x, then $\{a, y, u\}$ is a (directed) triangle. Now, the subgraph induced by $S_2^x \cup U_1^x$ has minimal outdegree 3, and it has 7 vertices. So it must contain a directed triangle, and we are done.

If $t = 4$, then $i = 2$, and so S_1^x contains a cycle of length at most $|S_1^x| = 4$.

If $t = 5$, then $i = 3$, (if $i = 2$ then S_1^x contains a cycle of length at most 4), it follows that $|S_1^x| = |S_2^x| = 4$, and that $|S_3^x| = 3$. Now the subgraph F of G induced by T_2^x has 8 vertices, it must contain a cycle, and it has at least 20 edges. Using these facts, it is easy to see that F must contain a cycle of length at most 5.

The above results show that G cannot exist.

Case 2. $k = 5$

Choosing x and i as usual, we note that $|S_i^x| \geqslant 3$, so $|S_{i-1}^x| \leqslant 6$. It follows from (6) that $t < \lceil \frac{1}{2}t \rceil + 6$, and so $t \leqslant 11$, or $n \leqslant 55$. Thus a minimal counter example contains at most 55 vertices. The case analysis for graphs with at most 55 vertices is too involved to be presented here. □

It is obvious that the same technique could also be used for larger values, of k. However, as k increases, the reasonably small graph gets larger and larger. Thus, new approaches may be needed to resolve the C–H conjecture, and in the remaining paragraphs we would like to discuss one possible approach.

We think that the C–H conjecture could be proved by considering how the cycles in a graph interrelate. In particular, the following conjecture implies the C–H conjecture.

Conjecture. *If G is a directed graph with minimal outdegree k, then there exists a sequence of directed cycles $C_1, C_2, \ldots, C_k$ in G such that $|(\bigcup_{i<j} C_i) \cap C_j| \leqslant 1$ for each $j \leqslant k$.*

It is easy to see that our conjecture implies the C–H conjecture. We only need consider the number of vertices in $\bigcup_{i \leqslant k} C_i$. If the shortest cycle in G has size t, then each of the cycles C_i has size at least t, and so our conjecture implies that $|\bigcup_{i\leqslant k} C_i| \geqslant k(t-1)+1$. It follows that $n > k(t-1)$ and $n/k > (t-1)$.

Thomassen (4) proved our conjecture for the case $k = 2$. It would be interesting to consider our conjecture for other small values of k.

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