Brunn–Minkowski inequality for mixed intersection bodies

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Abstract

Dual of the Brunn–Minkowski inequality for mixed projection bodies are established for mixed intersection bodies.

The intersection operator and the class of intersection bodies were defined by Lutwak [6]. The closure of the class of intersection bodies was studied by Goody et al. [5]. The intersection operator and the class of intersection bodies played a critical role in Zhang [12] and Gardner [2] solution of the famous Busemann–Petty problem. (See also Gardner et al. [4].)

As Lutwak [6] shows (and as is further elaborated in Gardner’s book [3]), there is a duality between projection and intersection bodies (that at present is not yet understood). Consider the following illustrative example: It is well known that the projections

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(onto lower-dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the “dual”: When intersection bodies are intersected with lower-dimensional subspaces, the results are intersection bodies (within the lower-dimensional subspaces). This was proven by Fallert et al. [1].

In [9] (see also [7] and [8]), Lutwak introduced mixed projection bodies and give the Brunn–Minkowski inequality for mixed projection bodies as follows:

\[-\text{I} f \quad K, L \in \mathcal{K}^n, \text{ then for } 0 \leq i < n,\]
\[W_i\left(\Pi(K + L)\right)^{1/(n-i)(n-1)} \geq W_i(\Pi K)^{1/(n-i)(n-1)} + W_i(\Pi L)^{1/(n-i)(n-1)}, \tag{0.1}\]

with equality if and only if $K$ and $L$ are homothetic.

An important generalization of the above Brunn–Minkowski inequality also was established as follows:

\[-\text{I} f \quad K, L \in \mathcal{K}^n, \text{ and } 0 \leq i < n, \text{ while } 0 \leq j < n - 2 \text{ then} \]
\[W_i\left(\Pi_j(K + L)\right)^{1/(n-i)(n-j-1)} \geq W_i(\Pi_j K)^{1/(n-i)(n-j-1)} + W_i(\Pi_j L)^{1/(n-i)(n-j-1)}, \tag{0.2}\]

with equality if and only if $K$ and $L$ are homothetic.

In this paper, by using the same way of [7], we shall prove the dual forms of inequalities (0.1) and (0.2) for mixed intersection body. In this work new contributions that illustrate this mysterious duality will be presented. Our main results can be stated as follows:

\[-\text{I} f \quad K, L \in \mathcal{K}^n, \text{ then for } 0 \leq i < n,\]
\[\tilde{W}_i\left(I(K + L)\right)^{1/(n-i)(n-1)} \leq \tilde{W}_i(I K)^{1/(n-i)(n-1)} + \tilde{W}_i(I L)^{1/(n-i)(n-1)}, \tag{0.3}\]

with equality if and only if $K$ and $L$ are dilates.

A generalization of inequality (0.3) will be established as follows:

\[-\text{I} f \quad K, L \in \mathcal{K}^n, \text{ and } 0 \leq i < n, \text{ while } 0 \leq j < n - 2 \text{ then} \]
\[\tilde{W}_i\left(I_j(K + L)\right)^{1/(n-i)(n-j-1)} \leq \tilde{W}_i(I_j K)^{1/(n-i)(n-j-1)} + \tilde{W}_i(I_j L)^{1/(n-i)(n-j-1)}, \tag{0.4}\]

with equality if and only if $K$ and $L$ are dilates.

Thus, this work may be seen as presenting additional evidence of the natural (but poorly understood) duality between intersection and projection bodies.
1. Notation and preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n > 2$). Let $\mathbb{C}^n$ denote the set of nonempty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors, and the letter $B$ is reserved for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. For $u \in S^{n-1}$, let $E_u$ denote the hyperplane, through the origin, that is orthogonal to $u$. We will use $K^u$ to denote the image of $K$ under an orthogonal projection onto the hyperplane $E_u$. We use $V(K)$ for the $n$-dimensional volume of convex body $K$. The support function of $K \in \mathcal{K}^n$, $h(K, \cdot)$, is defined on $\mathbb{R}^n$ by $h(K, \cdot) = \max\{x \cdot y : y \in K\}$. Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot)$, $\rho(K, u)$, the distance from the point $u$ to the set $K$. We use $\rho(K)$ to denote the radial Hausdorff metric, as follows: if $K, L \in \phi^n$, then $\delta(K, L) = |\rho_K - \rho_L|_\infty$.

1.1. Dual mixed volumes

Now introduce a vector addition on $\mathbb{R}^n$, which we call radial addition, as follows. If $x_1, \ldots, x_r \in \mathbb{R}^n$, then $x_1 + \cdots + x_r$ is defined to be the usual vector sum of $x_1, \ldots, x_r$, provided $x_1, \ldots, x_r$ all lie in 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \ldots, K_r \in \phi^n$, and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski combination $\lambda_1 K_1 + \cdots + \lambda_r K_r$, is defined by $\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r : x_i \in K_i\}$. It has the following important property, for $K, L \in \phi^n$ and $\lambda, \mu \geq 0$:

$$\rho(\lambda K + \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot).$$

(1.1)

For $K_1, \ldots, K_r \in \phi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous $r$th-degree polynomial in the $\lambda_i$,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{\lambda_1, \ldots, \lambda_r} \bar{V}(\lambda_1, \ldots, \lambda_r),$$

(1.2)

where the sum is taken over all $r$-tuples $(\lambda_1, \ldots, \lambda_r)$ whose entries are positive integers not exceeding $r$. If we require the coefficients of the polynomial in (1.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\bar{V}(\lambda_1, \ldots, \lambda_r)$ is nonnegative and depends only on the bodies $K_1, \ldots, K_r$. It is written as $\bar{V}(\lambda_1, \ldots, \lambda_r)$ and is called the dual mixed volume of $K_1, \ldots, K_r$, and if $K_1 = \cdots = K_{n-1} = K$, and $K_{n-1} = \cdots = K_n = L$, the dual mixed volumes is written as $\bar{V}(K, L)$. The dual mixed volume $\bar{V}(K, B)$ is written as $\bar{W}_j(K)$ and $\bar{V}(\lambda_1 K_1 + \cdots + \lambda_r K_r, B)$ is written as $\bar{W}_j(K, B)$. The mixed volume of $K_1 \cap E_{u_1}, \ldots, K_n \cap E_{u_n}$ in $(n - 1)$-dimensional space will be denoted by $\nu(K_1 \cap E_{u_1}, \ldots, K_n \cap E_{u_n})$. If $K_1 = \cdots = K_{n-1} = K$ and $K_{n-1} = \cdots = K_n = L$, then $\nu(K_1 \cap E_{u_1}, \ldots, K_n \cap E_{u_n})$ is written as $\nu(K \cap E_{u_1}, L \cap E_{u_n})$. If $L = B$, then $\nu(K \cap E_{u_1}, B \cap E_{u_2})$ is written as $\nu(K \cap E_{u_1}, B \cap E_{u_2})$. 


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For $K_1, \ldots, K_n \in \varphi^n$, the dual mixed volume of $K_1, \ldots, K_n$, $\tilde{V}(K_1, \ldots, K_n)$, is defined by Lutwak [10],

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \ldots \rho(K_n, u) dS(u). \quad (1.3)$$

From above identity, if $K \in \varphi^n$, $i \in \mathbb{R}$, then

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \quad (1.4)$$

### 1.2. Intersection bodies

For $K \in \varphi^n$, there is a unique star body $IK$ whose radial function satisfies for $u \in S^{n-1}$,

$$\rho(IK, u) = v(K \cap E_u), \quad (1.5)$$

It is called the intersection bodies of $K$. From a result of Busemann, it follows that $IK$ is a convex if $K$ is convex and centrally symmetric with respect to the origin. Clearly any intersection body is centered.

The volume of intersection bodies is given by

$$V(IK) = \frac{1}{n} \int_{S^{n-1}} v(K \cap E_u)^n dS(u).$$

The mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$, $I(K_1, \ldots, K_{n-1})$, whose radial function is defined by

$$\rho(I(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \quad (1.6)$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

The radial function of mixed intersection bodies is also written as

$$\rho(I(K_1, \ldots, K_{n-1}), u) = \frac{1}{n-1} \int_{S^{n-1} \cap E_u} \rho(K_1, v) \ldots \rho(K_{n-1}, v) dS^{n-1}(v). \quad (1.7)$$

If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $I(K_1, \ldots, K_{n-1})$ is written as $I_i(K, L)$. If $L = B$, then $I_i(K, L)$ is written as $I_i K$ and is called the $i$th intersection body of $K$. For $I_0 K$ simply write $IK$. The term is introduced by Zhang [11].

### 2. Lemmas

**Lemma 2.1.** If $K, L \in \varphi^n$, $i < n - 1$, then

$$\tilde{W}_i(K + L)^{1/(n-i)} \leq \tilde{W}_i(K)^{1/(n-i)} + \tilde{W}_i(L)^{1/(n-i)}, \quad (2.1)$$

with equality if and only if $K$ is a dilation of $L$. The inequality is reversed for $i > n$ or $n - 1 < i < n$. 
Proof. From (1.1), (1.4) and in view of the Minkowski inequality for integral, we obtain that for \(i < n - 1\),
\[
\tilde{W}_i(K + L)^{1/(n-i)} = n^{-1/(n-i)} \rho(K + L, u)_{n-i} = n^{-1/(n-i)} \left( \rho(K, u) + \rho(L, u) \right)_{n-i} \\
\leq n^{-1/(n-i)} \left( \| \rho(K, u) \|_{n-i} + \| \rho(L, u) \|_{n-i} \right) \\
= \tilde{W}_i(K)^{1/(n-i)} + \tilde{W}_i(L)^{1/(n-i)}
\]
with equality if and only if \(K\) is a dilation of \(L\).

In view of the inverse Minkowski inequality for integral, similar above the proof, the cases of \(i > n\) or \(n - 1 < i < n\) can also be proved. Here we omit the details. \(\Box\)

A generalization of inequality (2.1) will be established as follows.

Lemma 2.2. If \(K, L, K_1, \ldots, K_i \in \Phi^n\), then for \(0 \leq i < n - 1\),
\[
\tilde{V}(K + L, K_1, \ldots, K_i)^{n-i} \leq \tilde{V}(K, K_1, \ldots, K_i)^{1/(n-i)} + \tilde{V}(L, K_1, \ldots, K_i)^{1/(n-i)}.
\]
(2.2)
The inequality is reversed for \(i > n\) or \(n - 1 < i < n\).

Similar the above way, in view of (1.1), (1.3) and Minkowski inequality for integral, we also prove easy Lemma 2.2. Here we omit the detail.

Remark 2.1. Taking \(i = 0\) to Lemma 2.1, inequality (2.1) changes to the following dual Brunn–Minkowski inequality which was established by Lutwak [10]. If \(K, L \in \Phi^n\), then
\[
V(K + L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n},
\]
with equality if and only if \(K\) is a dilation of \(L\).

Moreover, taking for \(K_1 = \cdots = K_i = B\) in (2.2), (2.2) changes to (2.1).

Lemma 2.3. If \(K, L \in \Phi^n, i < n - 1\), then
\[
\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i} - \tilde{W}_i(L),
\]
(2.3)
with equality if and only if \(K\) is a dilation of \(L\). The inequality is reversed for \(i > n\) or \(n - 1 < i < n\).

Proof. From (1.2), we obtain that
\[
\tilde{W}_i(K + \varepsilon L) = \sum_{j=0}^{n-i} \binom{n-i}{j} \varepsilon^j \tilde{V}(K, \ldots, K, B, \ldots, B, L, \ldots, L).
\]
Hence
\[
\lim_{\varepsilon \to 0} \frac{\tilde{W}_i(K + \varepsilon L) - \tilde{W}_i(K)}{\varepsilon} = (n - i) \tilde{W}_i(K, L).
\]
By using (2.1) and in view of L’Hôpital’s rule, we obtain that for \( i < n - 1 \),
\[
\lim_{\varepsilon \to 0} (\varepsilon \tilde{W_i}(K,L)) = \lim_{\varepsilon \to 0} \left( (n-i) \tilde{W_i}(K,L) + \varepsilon \tilde{W_i}(L) \right)_{n-i}^{1/(n-i)} - \tilde{W_i}(K)
\]
with equality if and only if \( K \) is a dilation of \( L \).

In view of above inverse of general dual Brunn–Minkowski inequality (2.1), similar above the proof, the cases of \( i > n \) or \( n - 1 < i < n \) can also be proved.

**Remark 2.2.** Taking \( i = 0 \) to inequality (2.3), inequality (2.3) changes to the following dual Minkowski inequality which was established by Lutwak [10]. If \( K, L \in \mathcal{K}^n \), then
\[
\tilde{V_1}(K,L) \leq V(K)^{n-1} V(L),
\]
with equality if and only if \( K \) is a dilation of \( L \).

**Lemma 2.4.** If \( K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1} \in \varphi^n \), then
\[
\tilde{V}(K_1, \ldots, K_{n-1}, I(L_1, \ldots, L_{n-1})) = \tilde{V}(L_1, \ldots, L_{n-1}, I(K_1, \ldots, K_{n-1})). \tag{2.4}
\]

**Proof.** From (1.3) and (1.7), it follows that
\[
\tilde{V}(K_1, \ldots, K_{n-1}, I(L_1, \ldots, L_{n-1})) = \frac{1}{n} \int \rho(K_1, u) \rho(K_{n-1}, u) \rho(I(L_1, \ldots, L_{n-1}), u) dS(u) = \frac{1}{n(n-1)} \int \rho(K_1, u) \rho(K_{n-1}, u)
\]
\[
\times \int_{S^{n-1} \cap E_u} \rho(L_1, v) \rho(L_{n-1}, v) dS(v) dS(u) = \tilde{V}(L_1, \ldots, L_{n-1}, I(K_1, \ldots, K_{n-1})).
\]

A special cases of (2.4) will be used twice of proof given later: If \( K_1 = \cdots = K_{n-i-1} = K \), while \( K_{n-i} = \cdots = K_{n-1} = B \), and \( L_1 = \cdots = L_{n-j-1} = L \), while \( L_{n-j} = \cdots = L_{n-1} = B \), then Lemma 2.4 becomes

**Lemma 2.5.** If \( K, L \in \varphi^n \) and \( 0 \leq i, j < n - 1 \) then
\[
\tilde{W_i}(K, I_{i} L) = \tilde{W_j}(L, I_{j} K). \tag{2.5}
\]

Taking \( i = j = 0 \) to (2.5), (2.5) becomes \( \tilde{V}(K, IL) = \tilde{V}(L, IK) \) which was given by Lutwak [6].
3. The Brunn–Minkowski inequality for mixed intersection bodies

The following Brunn–Minkowski inequality for mixed intersection bodies stated in the introduction, which will be established: If \( K, L \in \varphi^n \), then for \( 0 \leq i < n \),

\[
\tilde{W}_i \left( I(K^+L) \right)^{1/(n-i)(n-1)} \leq \tilde{W}_i(IK)^{1/(n-i)(n-1)} + \tilde{W}_i(IL)^{1/(n-i)(n-1)},
\]

with equality if and only if \( K \) and \( L \) are dilates.

In fact a considerably more general inequality will be established the following result.

**Theorem.** If \( K, L \in \varphi^n \) and \( 0 \leq i < n \), while \( 0 \leq j < n-2 \), then

\[
\tilde{W}_j \left( I_j(K^+L) \right)^{1/(n-j-1)} \leq \tilde{W}_j(I_jK)^{1/(n-j-1)}
\]

\[
+ \tilde{W}_j(I_jL)^{1/(n-j-1)},
\]

(3.1)

with equality if and only if \( K \) and \( L \) are dilates.

**Proof.** From (2.5), we have

\[
\tilde{W}_i(M, I_j(K^+L)) = \tilde{W}_j(K^+L, I_iM).
\]

(3.2)

From inequality (2.2), it follows that

\[
\tilde{W}_j(K^+L, I_iM)^{1/(n-j-1)} \leq \tilde{W}_j(K, I_iM)^{1/(n-j-1)} + \tilde{W}_j(L, I_iM)^{1/(n-j-1)},
\]

(3.3)

On the other hand, from (2.5), one has

\[
\tilde{W}_j(K, I_iM)^{1/(n-j-1)} = \tilde{W}_j(M, I_iK)^{1/(n-j-1)},
\]

(3.4)

and hence, inequality (2.3) gives

\[
\tilde{W}_j(K, I_iM)^{1/(n-j-1)} \leq \tilde{W}_j(M)^{(n-i-1)/(n-i)(n-j-1)} \tilde{W}_i(I_jK)^{1/(n-i)(n-j-1)},
\]

(3.5)

with equality if and only if \( M \) and \( I_jK \) are dilates.

In exactly the same way, one obtains

\[
\tilde{W}_j(L, I_iM)^{1/(n-j-1)} \leq \tilde{W}_j(M)^{(n-i-1)/(n-i)(n-j-1)} \tilde{W}_i(I_jL)^{1/(n-i)(n-j-1)},
\]

(3.6)

with equality if and only if \( M \) and \( I_jL \) are dilates.

Combine (3.2)–(3.6), and then the result is

\[
\tilde{W}_i(M, I_j(K^+L))^{1/(n-j-1)} \leq \tilde{W}_i(M)^{(n-i-1)/(n-i)(n-j-1)} \left( \tilde{W}_i(I_jK)^{1/(n-i)(n-j-1)} + \tilde{W}_i(I_jL)^{1/(n-i)(n-j-1)} \right),
\]

(3.7)

with equality if and only if \( I_jK, I_jL \) and \( M \) are dilates.

Taking for \( I_j(K^+L) = M \) in (3.7), inequality (3.7) changes to the inequality of the theorem.

In the following, we will discuss the equality condition of the inequality of the theorem. Suppose there is equality in the inequality (3.1),
\[
\tilde{W}_i(I_j(K+L))^{1/(n-i)(n-j-1)} = \tilde{W}_i(I_jK)^{1/(n-i)(n-j-1)} + \tilde{W}_i(I_jL)^{1/(n-i)(n-j-1)}.
\] (3.8)

From the equality conditions for inequality (3.7), conclude that \(I_jK, I_jL\) and \(I_j(K+L)\) are dilates. In view of intersection bodies are centered, there exist \(\lambda, \mu > 0\), such that
\[
I_jK = \lambda I_j(K+L) \quad \text{and} \quad I_jL = \mu I_j(K+L).
\] (3.9)

But (3.8), combined with (3.9), gives
\[
\lambda^{1/(n-j-1)} + \mu^{1/(n-j-1)} = 1.
\] (3.10)

Suppose \(u \in S^{n-1}\). In view of the fact \((K+L) \cap E_u = K \cap E_u + L \cap E_u\), it follows from (1.6) and (3.10) that
\[
\tilde{w}_j(K \cap E_u) = \lambda \tilde{w}_j(K \cap E_u + L \cap E_u) \quad \text{and} \quad 
\tilde{w}_j(L \cap E_u) = \mu \tilde{w}_j(K \cap E_u + L \cap E_u).
\] (3.11)

Hence, (3.10), combined with (3.11), yields
\[
\tilde{w}_j(K \cap E_u + L \cap E_u)^{1/(n-j-1)} = \tilde{w}_j(K \cap E_u)^{1/(n-j-1)} + \tilde{w}_j(L \cap E_u)^{1/(n-j-1)}.
\]

From the equality conditions of inequality (2.1), it shows that this implies that \(K \cap E_u\) and \(L \cap E_u\) must be dilates. This follows \(K\) and \(L\) are dilates.

This proof is complete. \(\square\)

**Remark.** Taking for \(i = 0, j = 0\) in inequality (3.1), (3.1) changes to
\[
V(I(K+L))^{1/n(n-1)} \leq V(I(K))^{1/n(n-1)} + V(I(L))^{1/n(n-1)},
\]
with equality if and only if \(K\) and \(L\) are dilates.

This is just a dual form of the following Brunn–Minkowski inequality of mixed projection bodies for general volume which was given by Lutwak [7]:
\[
V(\Pi(K+L))^{1/n(n-1)} \geq V(\Pi(K))^{1/n(n-1)} + V(\Pi(L))^{1/n(n-1)},
\]
with equality if and only if \(K\) and \(L\) are homothetic.

**References**