Approximate controllability of semilinear functional equations in Hilbert spaces

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Abstract

In this paper approximate and complete controllability for semilinear functional differential systems is studied in Hilbert spaces. Sufficient conditions are established for each of these types of controllability. The results address the limitation that linear systems in infinite-dimensional spaces with compact semigroup cannot be completely controllable. The conditions are obtained by using the Schauder fixed point theorem when the semigroup is compact and the Banach fixed point theorem when the semigroup is not compact.

Keywords: Approximate controllability; Weak approximate controllability; Complete controllability; Semilinear functional equations; Schauder fixed point theorem; Banach fixed point theorem

1. Introduction

Controllability theory for abstract linear control systems in infinite-dimensional spaces is well-developed, and the details can be found in various papers and monographs (see [1–4] and references therein). Several authors have extended...
these concepts to infinite-dimensional systems represented by nonlinear evolution equations (see [5–20]). Most of the controllability results for nonlinear infinite-dimensional control systems concern the so-called semilinear control system that consists of a linear part and a nonlinear part.

Zhou [5] studied approximate controllability of an abstract semilinear control system by assuming certain inequality conditions that are dependent on the properties of the system components. Naito [6,7] studied the approximate controllability of the same system. He showed that under a range condition on the control action operator, the semilinear control system is approximately controllable. Yamamoto and Park [8] discussed the same problem for parabolic equations with uniformly bounded linear part. Do [9] discussed approximate controllability for a class of semilinear abstract equations. Approximate controllability of semilinear systems was studied by Joshi and Sukavanam [10] and George [11]. Mahmudov showed that under appropriate conditions approximate controllability of semilinear systems is implied by the approximate controllability of its linear part [12]. Constrained controllability of nonlinear systems in abstract spaces has been studied by Chuckwu and Lenhart [13], Klamka [14], Bian [15], and Papageorgiou [16].

Let $C$ be the Banach space of all continuous functions from an interval $[-h, 0]$ to $X$ with the supremum norm. In this paper is studied the controllability of dynamical systems governed by the semilinear evolution equation

$$\begin{align*}
x_t(0) &= x(t) = S(t)\phi(0) + \int_0^t S(t-s)[Bu(s) + f(s, x_s, u(s))] \, ds, \\
x_0(\theta) &= \phi(\theta), \quad -h \leq \theta \leq 0, \quad 0 < t \leq T, \quad I = [0, T],
\end{align*}$$

where the state $x(\cdot)$ takes values in a Hilbert space $X$, the control $u(\cdot) \in L^2(I, U)$ takes values in a Hilbert space $U$, $S(t)$ is a linear semigroup on $X$, $B : U \to X$ is a bounded linear operator, and $\phi \in C$. If $x : [-h, 0] \cup I \to X$ is a continuous function, then $x_t$ is an element in $C$ which has point-wise definition

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in [-h, 0].$$

The purpose of this paper is to show the controllability (approximate and complete) of semilinear differential systems of form (1) in Hilbert space under simple and fundamental assumptions on the system operators, in particular, that the corresponding linear system is appropriately controllable. This is consistent with classical finite-dimensional theory (see [17,18] and references therein).

Section 2 gives the preliminaries for the paper.

In Section 3 approximate controllability of system (1) is studied by using the Schauder fixed point theorem. The corresponding semigroup $S(t), t > 0,$ is assumed to be compact. It is shown that under certain conditions on the nonlinear term approximate controllability of the linear system implies approximate controllability of the semilinear system.
Notice that when the semigroup $S(t)$, $t > 0$, is compact an infinite-dimensional linear system cannot be completely controllable [21,22]. So, the analogue for complete controllability of the results of Section 3 cannot hold in infinite-dimensional space.

In Section 4 this problem is investigated via the Banach fixed point theorem. The compactness of the semigroup $S(t)$, $t > 0$, is not assumed, and results are obtained for complete controllability of the semilinear system (1).

Examples 1 and 2 presented in Section 5 demonstrate the approximate controllability results of Section 3. Example 3 shows the analogous result for complete controllability.

2. Preliminaries

**Definition 1.** System (1) is said to be approximately controllable (completely controllable) on the interval $I$ if

$$\mathcal{R}(T, \phi) = X \quad (\mathcal{R}(T, \phi) = X),$$

where

$$\mathcal{R}(T, \phi) = \{ x_T(\phi; u)(0) : u(\cdot) \in L_2(I, U) \}.$$

The following notations are introduced for convenience:

$$K = \max \{ \| S(t) \| : 0 \leq t \leq T \}, \quad \| \lambda_i \|_1 = \int_0^T |\lambda_i(s)| \, ds,$$

$$k = \max \{ 1, MK, MKTt \}, \quad M = \| B \|,$$

$$a_i = 3kMK^2 \| \lambda_i \|_1, \quad b_i = 3K \| \lambda_i \|_1, \quad c_i = \max \{ a_i, b_i \},$$

$$d_1 = 3MK (\| x_T \| + K \| \phi(0) \|), \quad d_2 = 3K \| \phi(0) \|,$$

$$d = \max \{ d_1, d_2 \}.$$

Concerning the operators $B$ and $f$, assume the following hypothesis:

(H0) The semigroup $S(t)$, $t > 0$, is compact.

(H1) The function $f : I \times C \times U \to X$ is continuous and there exist functions $\lambda_i \in L_1(I, \mathbb{R}^+)$ and $g_i \in L_1(C \times U, \mathbb{R}^+)$, $i = 1, 2, \ldots, q$, such that

$$\| f(t, \phi, u) \| \leq \sum_{i=1}^{q} \lambda_i(t) g_i(\phi, u) \quad \text{for all } (t, \phi, u) \in I \times C \times U.$$

(H2) For each $\alpha > 0$

$$\lim_{r \to \infty} \left( r - \sum_{i=1}^{q} \frac{c_i}{\alpha} \sup \left\{ g_i(\phi, u) : \| (\phi, u) \| \leq r \right\} \right) = \infty.$$
(H3) The function \( f : I \times C \times U \to X \) is continuous and uniformly bounded, i.e., there exists \( L > 0 \) such that
\[
\|f(t, \phi, u)\| \leq L \quad \text{for all } (t, \phi, u) \in I \times C \times U.
\]

(H3') The function \( f : I \times C \times U \to X \) is continuous and there exists \( L > 0 \) such that
\[
\|f(t, \phi, u)\| \leq L \left( 1 + \|\phi\|_C + \|u\| \right) \quad \text{for all } (t, \phi, u) \in I \times C \times U.
\]

(H4) The function \( f : I \times C \times U \to X \) satisfies the Lipschitz condition
\[
\|f(t, \phi_1, u_1) - f(t, \phi_2, u_2)\| \leq L \left( \|\phi_1 - \phi_2\|_C + \|u_1 - u_2\| \right).
\]

It is convenient at this point to introduce two relevant operators and the basic assumptions on these operators:

\[
\Gamma_0^T = \int_0^T S(T - s)BB^*S^*(T - s) \, ds,
\]
\[
R(\alpha, \Gamma_0^T) = (\alpha I + \Gamma_0^T)^{-1}.
\]

(HBA) \( \alpha R(\alpha, \Gamma_0^T) \to 0 \) as \( \alpha \to 0^+ \) in the strong operator topology.

(HBC) \( \alpha R(\alpha, \Gamma_0^T) \to 0 \) as \( \alpha \to 0^+ \) in the uniform operator topology.

It is known that the assumption (HBA) holds if and only if the system
\[
\begin{align*}
x'(t) &= Ax(t) + Bu(t), \\
x(0) &= \phi(0)
\end{align*}
\]
(2)
is approximately controllable on \( I \). Further, it is known that the assumption (HBC) holds if and only if system (2) is completely controllable on \( I \); see [23].

It will be shown that the system (1) is approximately controllable if there exists a continuous function \((x, u) \in C(I, C) \times C(I, U)\) such that
\[
\begin{align*}
u(t) &= B^*S^*(T - t)R(\alpha, \Gamma_0^T)p(x, u), \\
x(t) &= S(t)\phi(0) + \int_0^t S(t - s) \left( Bu(s) + f(s, x_s, u(s)) \right) ds, \\
x_0(\theta) &= \phi(\theta),
\end{align*}
\]
(3)\hspace{1cm}(4)
where
\[
p(x, u) = x_T - S(T)\phi(0) - \int_0^T S(T - s) f(s, x_s, u(s)) \, ds.
\]
Having noticed this fact, the goal in the next section is to find conditions for solvability of system (3), (4) for \( \alpha > 0 \).

### 3. Approximate controllability

Triggiani [21,22] showed that if \( X \) is an infinite-dimensional space, then system (2) cannot be completely controllable if the corresponding semigroup \( S(t), t > 0 \), is compact. Thus, in infinite-dimensional spaces the concept of complete controllability is usually too strong and, indeed, has limited applicability. Approximately controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. For example, the class of systems that lack complete controllability includes the parabolic differential equations [2,3].

It is now shown, using the Schauder fixed point theorem, that under certain conditions approximate controllability of the linear system (2) implies the approximate controllability of the semilinear system (1). This result, Theorem 3, assumes that the linear system has a compact semigroup and consequently is not completely controllable. Therefore, this result cannot have an analogue for the concept of complete controllability.

For \( \alpha > 0 \), define the operator \( \mathbb{F}^\alpha \) on \( C(I,C) \times C(I,U) \) as

\[
\mathbb{F}^\alpha(x, u) = (z, v),
\]

where

\[
v(t) = B^* S^* (T - t) R(\alpha, \Gamma_0^T) p(x, u),
\]

\[
z(t) = S(t) \phi(0) + \int_0^t S(t - s) (Bv(s) + f(s, x, u)) \, ds,
\]

\[
z_0(\theta) = \phi(\theta), \quad \theta \in [-h, 0],
\]

\[
p(x, u) = x_T - S(T) \phi(0) - \int_0^T S(T - s) f(s, x, u) \, ds.
\]

It will be shown that the operator \( \mathbb{F}^\alpha \) from \( C(I,C) \times C(I,U) \) into itself has a fixed point.

In the Banach space \( C(I,C) \times C(I,U) \) introduce the set

\[
Y_r = \{(x, u) \in C(I,C) \times C(I,U): \|x_t\|_C + \|u(t)\| \leq r \text{ for all } t \in I\},
\]

where \( r \) is a positive constant.

**Theorem 2.** Assume assumptions (H0), (H1) and (H2) are satisfied. Then for each \( 0 < \alpha \leq 1 \), the operator \( \mathbb{F}^\alpha \) has a fixed point in \( C(I,C) \times C(I,U) \).
Proof. The proof of the theorem is long and technical. It is therefore split into several steps.

Step 1. For an arbitrary $0 < \alpha \leq 1$ there is a positive constant $r_0 = r_0(\alpha)$ such that $\mathbb{F}^\alpha : Y_{r_0} \to Y_{r_0}$. Let

$$
\mu_i(r) = \sup \{ g_i(\phi, v) : \| (\phi, v) \| \leq r, (\phi, v) \in C \times U \}.
$$

By the assumption (H2), there exists $r_0 > 0$ such that

$$
d + \frac{1}{3k} \sum_{i=1}^q c_i \mu_i(r_0) \leq r_0.
$$

If $(x, u) \in Y_{r_0}$, then from (6) and (7),

$$
\| v(t) \| \leq \frac{1}{\alpha} MK \left( \| x_T \| + K \| \phi(0) \| + K \int_0^T \sum_{i=1}^q \lambda_i(s) g_i(x_s, u(s)) \, ds \right)
$$

$$
\leq \frac{1}{\alpha} MK \left( \| x_T \| + K \| \phi(0) \| \right) + \frac{1}{\alpha} MK^2 \sum_{i=1}^q \| \lambda_i \|_1 \mu_i(r_0)
$$

$$
\leq \frac{1}{\alpha} \left( d + \sum_{i=1}^q c_i \mu_i(r_0) \right) = \frac{1}{3k} \left( d + \sum_{i=1}^q c_i \mu_i(r_0) \right)
$$

$$
\leq \frac{r_0}{3k},
$$

$$
\| z_t(\theta) \| \leq \frac{d}{3} + KMT \| v \| + K \int_0^{t+\theta} \sum_{i=1}^q \lambda_i(s) g_i(x_s, u(s)) \, ds
$$

$$
\leq \frac{d}{3} + k \| v \| + \frac{1}{3} \sum_{i=1}^q c_i \mu_i(r_0) \leq \frac{1}{3} \left( d + \sum_{i=1}^q c_i \mu_i(r_0) \right) + k \| v \|
$$

$$
\leq \alpha r_0 / 3 + r_0 / 3 \leq 2r_0 / 3.
$$

So,

$$
\| (\mathbb{F}^\alpha(x, u))(t) \| = \| v(t) \| + \| z_t(\theta) \| \leq r_0.
$$

Hence, $\mathbb{F}^\alpha$ maps $Y_{r_0}$ into itself.

Step 2. For each $0 < \alpha \leq 1$ (recall $r_0$ depends on $\alpha$), the operator $\mathbb{F}^\alpha$ maps $Y_{r_0}$ into a relatively compact subset of $Y_{r_0}$. According to the infinite-dimensional version of the Ascoli–Arzela theorem, the following two things remain to be shown:

(i) For arbitrary $t \in [0, T]$, the set

$$
V(t) = \left\{ (\mathbb{F}^\alpha(x, u))(t) : (x, u) \in Y_{r_0} \right\}
$$

is relatively compact.
(ii) For arbitrary \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that
\[
\left\| \left( F^\alpha(x,u) \right)(t_1) - \left( F^\alpha(x,u) \right)(t_2) \right\| < \varepsilon,
\]
if \((x,u) \in Y_{r_0}, |t_1 - t_2| \leq \delta, \) and \( t_1, t_2 \in [-h, T] \).

The proof of (i). In fact, the case where \( t = 0 \) is trivial, since \( V(0) = \{ \phi(0) \} \). So let \( t, 0 < t \leq T \), be a fixed and let \( \eta \) be a given real number satisfying \( 0 < \eta < t \). Define
\[
\left( F^\alpha_\eta(x,u) \right)(t) = \left[ S(\eta)z(t - \eta), B^*S^*(T - t)R(\alpha, \Gamma_{T_0}^T)p(x,u) \right].
\]
Since \( S(t) \) is compact and \( z(t - \eta) \) and \( p(x,u) \) are bounded on \( Y_{r_0} \), the set
\[
V_\eta(t) = \left\{ \left( F^\alpha_\eta(x,u) \right)(t): (x,u) \in Y_{r_0} \right\}
\]
is relatively compact set in \( C \times U \). That is, a finite set \( \{ y_i, 1 \leq i \leq m \} \) in \( C \times U \) exists such that
\[
V_\eta(t) \subset \bigcup_{i=1}^{m} N(y_i, \varepsilon/2),
\]
where \( N(y_i, \varepsilon/2) \) is an open ball in \( C \times U \) with the center at \( y_i \) and radius \( \varepsilon/2 \).

On the other hand,
\[
\left\| \left( F^\alpha(x,u) \right)(t) - \left( F^\alpha_\eta(x,u) \right)(t) \right\|
\]
\[
= \left\| \int_{t-\eta}^{t} S(t-s)\left[Bv(s) + f(s,x(s),u(s))\right] ds \right\|
\]
\[
\leq \frac{1}{\alpha} K^2 M^2 \left( \|x_T\| + K \|\phi(0)\| + K \int_{0}^{T} \sum_{i=1}^{q} \lambda_i(s) \mu_i(r_0) ds \right) \eta
\]
\[
+ K \sum_{i=1}^{q} \int_{t-\eta}^{t} \lambda_i(s) \mu_i(r_0) ds \leq \frac{\varepsilon}{2}.
\]
Consequently,
\[
V(t) \subset \bigcup_{i=1}^{m} N(y_i, \varepsilon).
\]

Hence, for each \( t \in [0, T] \), \( V(t) \) is relatively compact in \( C \times U \).

Next the proof of (ii). It must be shown that
\[
V = \left\{ \left( F^\alpha(x,u) \right)(\cdot): (x,u) \in Y_{r_0} \right\}
\]
is equicontinuous on $[0, T]$. In fact, for $0 < t_1 + \theta < t_2 + \theta \leq T$,

$$
\|v(t_1) - v(t_2)\| \\
\leq \left\| B^* S^*(T - t_1) - B^* S^*(T - t_2) \right\| \\
\times \frac{1}{\alpha} \left[ \|x_T\| + K \|\phi(0)\| + K \int_0^T \sum_{i=1}^q \lambda_i(s) g_i(x_s, u(s)) \, ds \right] \\
\leq \left\| B^* S^*(T - t_1) - B^* S^*(T - t_2) \right\| \\
\times \frac{1}{\alpha} \left[ \|x_T\| + K \|\phi(0)\| + K \sum_{i=1}^q \|\lambda_i\|_1 \mu_i(r_0) \right],
$$

$$
z_{t_1}(\theta) - z_{t_2}(\theta) = \left[ S(t_1 + \theta) - S(t_2 + \theta) \right] \phi(0) - \int_{t_1+\theta}^{t_2+\theta} S(t_2 + \theta - s) B v(s) \, ds \\
+ \int_0^{t_1+\theta} \left[ S(t_1 + \theta - s) - S(t_2 + \theta - s) \right] B v(s) \, ds \\
- \int_{t_2+\theta}^{t_1+\theta} S(t_2 + \theta - s) f(s, x_s, u(s)) \, ds \\
+ \int_0^{t_1+\theta} \left[ S(t_1 + \theta - s) - S(t_2 + \theta - s) \right] f(s, x_s, u(s)) \, ds,
$$

$$
\|z(t_1 + \theta) - z(t_2 + \theta)\| \\
\leq \left\| S(t_1 + \theta) - S(t_2 + \theta) \right\| \|\phi(0)\| + K M \int_{t_1+\theta}^{t_2+\theta} \|v(s)\| \, ds \\
+ M \int_0^{t_1+\theta} \left\| S(t_1 + \theta - s) - S(t_2 + \theta - s) \right\| \|v(s)\| \, ds \\
+ K \int_{t_1+\theta}^{t_2+\theta} \sum_{i=1}^q \lambda_i(s) g_i(x_s, u(s)) \, ds \\
+ \int_0^{t_1+\theta} \left\| S(t_1 + \theta - s) - S(t_2 + \theta - s) \right\| \sum_{i=1}^q \lambda_i(s) g_i(x_s, u(s)) \, ds
\[ \leq \left\| S(t_1 + \theta) - S(t_2 + \theta) \right\| \| \phi(0) \| + K M \int_{t_1 + \theta}^{t_2 + \theta} \| v(s) \| \, ds \]

\[ + M \int_{0}^{t_1 + \theta} \left\| S(t_1 + \theta - s) - S(t_2 + \theta - s) \right\| \| v(s) \| \, ds \]

\[ + K \sum_{i=1}^{q} \int_{t_1 + \theta}^{t_2 + \theta} \lambda_i(s) \, ds \mu_i(r_0) \]

\[ + \sum_{i=1}^{q} \int_{0}^{t_1 + \theta} \left\| S(t_1 + \theta - s) - S(t_2 + \theta - s) \right\| \lambda_i(s) \, ds \mu_i(r_0) \]

\[ = I_1 + I_2 + I_3 + I_4 + I_5. \quad (8) \]

Moreover, for all \((x, u) \in Y_{r_0}\),

\[ \| v \| \leq \frac{1}{\alpha} M K \left( \| x_T \| + K \| \phi(0) \| + K \int_{0}^{T} \sum_{i=1}^{q} \lambda_i(s) g_i(x_s, u(s)) \, ds \right) \]

\[ \leq \frac{1}{\alpha} M K \left( \| x_T \| + K \| \phi(0) \| + K \sum_{i=1}^{q} \| \lambda_i \|_1 \mu_i(r_0) \right). \]

Thus, the right-hand side of (8) does not depend on particular choices of \((x, u)\).

It is clear that \(I_2 \to 0\) and \(I_4 \to 0\) as \(t_1 - t_2 \to 0\). Since the semigroup \(S(\cdot)\) is compact,

\[ \| S(t_2 + \theta - s) - S(t_1 + \theta - s) \| \to 0 \]

as \(t_1 - t_2 \to 0\) for arbitrary \(t, s, \theta\) such that \(t + \theta - s > 0\). Then \(I_3 \to 0\) and, by Lebesgue’s dominated convergence theorem, \(I_5 \to 0\) as \(t_1 - t_2 \to 0\). So, the equicontinuity of \(V\) is shown.

Notice that the only case considered is \(0 < t_1 + \theta < t_2 + \theta\), since the other cases, \(t_1 + \theta < t_2 + \theta < 0\) or \(t_1 + \theta < 0 < t_2 + \theta\), are very simple. Thus, \(F_\alpha[Y_{r_0}]\) is equicontinuous and also bounded. By the Ascoli–Arzela theorem, \(F_\alpha[Y_{r_0}]\) is relatively compact in \(C(I, C) \times C(I, U)\).

To apply the Schauder fixed point theorem it remains to show that \(F_\alpha\) is continuous on \(C(I, C) \times C(I, U)\). Let \(\{(y^n, u^n)\} \in C(I, C) \times C(I, U)\) with \((y^n, u^n) \to (y, u)\) in \(C(I, C) \times C(I, U)\). Since \(f(s, y^n_s, u^n(s)) \to f(s, y_s, u(s))\) for each \(s \in I\) and since

\[ \left\| f(s, y^n_s, u^n(s)) - f(s, y_s, u(s)) \right\| \leq 2 \sum_{i=1}^{q} \lambda_i(s) \mu_i(r_0), \]
the Lebesgue dominated convergence theorem implies
\[
\| (\mathcal{F}^\alpha (y^n, u^n))(t) - (\mathcal{F}^\alpha (y, u))(t) \| = \| v^n(t) - v(t) \| + \| z^n_t(\theta) - z_t(\theta) \| \\
+ \left\| \int_0^t S(t-s) \left[ B(v^n(s) - v(s)) \\
+ (f(s, y^n_s, u^n(s)) - f(s, y_s, u(s))) \right] ds \right\| \\
\leq d \int_0^T \| f(s, y^n_s, u^n(s)) - f(s, y_s, u(s)) \| ds \to 0,
\]
where \( d \) is an appropriate constant. Thus, \( \mathcal{F}^\alpha \) is continuous.

Hence, \( \mathcal{F}^\alpha \) is a compact continuous operator on \( C(I, C) \times C(I, U) \) and from the Schauder fixed point theorem, \( \mathcal{F}^\alpha \) has a fixed-point.

**Theorem 3.** Assume assumptions (H0), (H3) and (HBA) are satisfied. Then system (1) is approximately controllable on \( I \).

**Proof.** Let \((x^\alpha, \overline{u}^\alpha)\) be a fixed point of \( \mathcal{F}^\alpha \) in \( Y_{r_0} \). Any fixed point of \( \mathcal{F}^\alpha \) is a mild solution of (1) on \([0, T]\) under the control
\[
\overline{u}^\alpha(t) = B^* S^* (T-t) R(\alpha, \Gamma_{0}^T) p(x^\alpha, \overline{u}^\alpha)
\]
and satisfies
\[
x^\alpha_T(0) = x_T + \alpha R(\alpha, \Gamma_{0}^T) p(x^\alpha, \overline{u}^\alpha).
\]  
By (H3)
\[
\int_0^T \left\| f(s, x^\alpha_s, \overline{u}^\alpha(s)) \right\|^2 ds \leq L^2 T,
\]
and, consequently, the sequence \( \{ f(s, x^\alpha_s, \overline{u}^\alpha(s)) \} \) is bounded in \( L_2(I, X) \). Then there is a subsequence, still denoted by \( \{ f(s, x^\alpha_s, \overline{u}^\alpha(s)) \} \), that weakly converges to, say, \( f(s) \) in \( L_2(I, X) \). Then for
\[
h = S(T)x_0 + \int_0^T S(T-s) f(s) ds - x_T
\]
it follows that
\[ \| p(\bar{x}^\alpha, \bar{u}^\alpha) - h \| = \left\| \int_0^T S(T - s) \left[ f(s, \bar{x}^\alpha_s, \bar{u}^\alpha(s)) - f(s) \right] ds \right\| \]
\[ \leq \sup_{0 \leq t \leq T} \left\| \int_0^t S(t - s) \left[ f(s, \bar{x}^\alpha_s, \bar{u}^\alpha(s)) - f(s) \right] ds \right\| \to 0 \]
as \( \alpha \to 0^+ \) because of compactness of an operator \( g(\cdot) \to \int_0^\cdot S(\cdot - s) g(s) ds : L^2(I, X) \to C(I, X) \). Then by (9)
\[ \| \bar{x}^\alpha_T(0) - x_T \| = \| \alpha R(\alpha, \Gamma^T_0) p(\bar{x}^\alpha, \bar{u}^\alpha) \| \]
\[ \leq \| \alpha R(\alpha, \Gamma^T_0)(h) \| + \| \alpha R(\alpha, \Gamma^T_0)(p(\bar{x}^\alpha, \bar{u}^\alpha) - h) \| \]
\[ \leq \| \alpha R(\alpha, \Gamma^T_0)(h) \| + \| p(\bar{x}^\alpha, \bar{u}^\alpha) - h \| \to 0 \]
as \( \alpha \to 0^+ \). This proves the approximate controllability of system (1). \( \Box \)

4. Complete controllability

Note that compactness of the operator \( g(\cdot) \to \int_0^\cdot S(\cdot - s) g(s) ds : L^2(I, X) \to C(I, X) \) is essential in the proof of Theorem 4. Approximate controllability of system (1) cannot be proved using this method without a compactness assumption. So, in this section the concept of complete controllability is used to explore the controllability of system (1) without the compactness assumption.

In particular, conditions are formulated under which complete controllability of the semilinear system (1) is implied by the complete controllability of its linear part. No assumption of compactness of the linear system is made. This investigation is carried using the Banach fixed point theorem.

Define the operator \( \mathbb{F}^0 \) on \( C(I, C) \times C(I, U) \) as
\[ \mathbb{F}^0(x, u) = (z, v), \]
where
\[ v(t) = B^* S^*(T - t) (\Gamma^T_0)^{-1} p(x, u), \]
\[ z(t) = S(t) \phi(\theta) + \int_0^t S(t - s) \left( Bv(s) + f(s, x_s, u(s)) \right) ds, \]
\[ z_0(\theta) = \phi(\theta), \quad \theta \in [-h, 0], \]
\[ p(x, u) = x_T - S(T) \phi - \int_0^T S(T - s) f(s, x_s, u(s)) ds. \]
It will be shown that the operator $\mathbb{F}^0$ from $C(I, C) \times C(I, U)$ into itself has a fixed point.

Notice that the linear system (2) is completely controllable if and only if there exists a $\gamma > 0$ such that

$$\langle \Gamma_0^T x, x \rangle \geq \gamma \|x\|^2$$

for all $x$ in $X$.

Then $\Gamma_0^T$ is invertible and

$$\| (\Gamma_0^T)^{-1} \| \leq 1/\gamma.$$

**Theorem 4.** Assume that the assumptions (H3’), (H4) and (HBC) hold. The operator $\mathbb{F}^0$ has a unique fixed point in $C(I, C) \times C(I, U)$ if

$$\left( \frac{1}{\gamma} K^2 M + \frac{1}{\gamma} K^3 M^2 T + K \right) T L < 1.$$  \hfill (10)

**Proof.** The proof is based on the classical fixed point theorem for contractions. First it is shown that $\mathbb{F}^0$ maps $C(I, C) \times C(I, U)$ into itself. To do this the following operators are introduced:

$$\mathbb{F}_1(x, u)(t) = \int_0^t S(t-s) B v(s) \, ds,$$

$$\mathbb{F}_2(x, u)(t) = \int_0^t S(t-s) f(s, x_s, u(s)) \, ds.$$

By using the assumption (H1) and notations it can be shown that there exists $C_1 > 0$ such that

$$\| v(t) \| \leq \frac{1}{\gamma} K M \left( \| x_T \| + K \| \phi(0) \| + K \int_0^t \| f(s, x_s, u(s)) \| \, ds \right) \leq C_1 \left( 1 + \sup_{t \in I} \| x_t \| + \sup_{t \in I} \| u(t) \| \right).$$  \hfill (11)

So,

$$\| \mathbb{F}_1(x, u)(t) \| \leq K M C_1 \left( 1 + \sup_{t \in I} \| x_t \| + \sup_{t \in I} \| u(t) \| \right).$$  \hfill (12)

$\mathbb{F}_1$ maps $C(I, C) \times C(I, U)$ into $C(I, C) \times C(I, U)$. The same technique is used to show that this property is also valid for $\mathbb{F}_2$:

$$\| \mathbb{F}_2(x, u)(t) \| \leq K \int_0^t \| f(s, x_s, u(s)) \| \, ds$$
\[ LK \int_0^t (1 + \|x_s\|_C + \|u(s)\|) \, ds \leq C_2 \left( 1 + \sup_{t \in I} \|x_t\|_C + \sup_{t \in I} \|u(t)\| \right). \]  

Therefore, from (11)–(13) there exists \( C_3 > 0 \) such that
\[ \|F_0(x, u)\| = \|v\| + \|z\| \leq C_3 (1 + \|x\| + \|u\|). \]  

So, \( F_0 \) maps \( C(I, C) \times C(I, U) \) into itself.

Secondly, it is shown that \( F_0 \) is a contraction mapping. Let now \((x, u)\) and \((y, w)\) be arbitrary functions from \( C(I, C) \times C(I, U) \). Then
\[ \|F_0(x, u) - F_0(y, w)\| \leq \|v_1 - v_2\| + \|z_1 - z_2\| \]
\[ \leq \|v_1 - v_2\| + \|F_1(x, u) - F_1(y, w)\| + \|F_2(x, u) - F_2(y, w)\| \]
\[ = \|v_1 - v_2\| + I_1 + I_2 \]

and
\[ \|v_1 - v_2\| = \sup_{t \in I} \left\| B^* S(T - t) (\alpha + t_0^T) \right\| \]
\[ \times \left\| \int_0^T S(T - s) \left( f(s, x_s, u(s)) - f(s, y_s, w(s)) \right) \, ds \right\| \]
\[ \leq \frac{1}{\gamma} K^2 M \int_0^T \left\| f(s, x_s, u(s)) - f(s, y_s, w(s)) \right\| \, ds \]
\[ \leq \frac{1}{\gamma} K^2 ML \int_0^T (\|x_s - y_s\|_C + \|u(s) - w(s)\|) \, ds \]
\[ \leq \frac{1}{\gamma} K^2 MTL (\|x - y\| + \|u - w\|). \]  

In a similar way,
\[ I_1 = \sup_{t \in I} \left\| \int_0^t S(t - s) B(v_1(s) - v_2(s)) \, ds \right\| \]
\[ \leq KMT \|v_1 - v_2\| \leq \frac{1}{\gamma} K^3 M^2 T^2 L (\|x - y\| + \|u - w\|), \]  

\[ I_2 = \sup_{t \in I} \left\| \int_0^t S(t - s) \left( f(s, x_s, u(s)) - f(s, y_s, w(s)) \right) \, ds \right\| \]
\[
\leq K \int_0^T \| f(s, x_s, u(s)) - f(s, y_s, w(s)) \| \, ds \\
\leq KTL (\| x - y \| + \| u - w \|).
\] (16)

Summing the obtained estimates (14)–(16) yields

\[
\| F^0(x, u) - F^0(y, w) \|
\leq \left( \frac{1}{\gamma} K^2 MTL + \frac{1}{\gamma} K^3 M^2 T^2 L + KTL \right) (\| x - y \| + \| u - w \|)
\]

for all \((x, u), (y, w) \in C(I, C) \times C(I, U)\). Consequently, if

\[
\left( \frac{1}{\gamma} K^2 M + \frac{1}{\gamma} K^3 M^2 T^2 + KT \right) L < 1,
\]

then the mapping \( F^0 \) has a fixed point \((x, u) \) in \( C(I, C) \times C(I, U) \) which, as it is easy to see, is the solution of Eqs. (6), (7). The theorem is proved. \( \square \)

**Theorem 5.** Under the assumptions \((H3')\), \((H4)\) and \((HBC)\), system (1) is completely controllable on \([0, T]\).

**Proof.** If \((x_0, u_0)\) is a fixed point of an operator \( F^0 \), then (9) holds with \( \alpha = 0 \). In other words in this case \( x_0^0(0) = x_T \) for arbitrary \( x_T \) in \( X \). Thus, system (1) is completely controllable. \( \square \)

**Remark 6.** In Theorem 3, approximate controllability was proved making a compactness assumption on the semigroup \( S(t), t > 0 \). In Theorem 5, a complete controllability result is derived with no compactness assumption on the semigroup. It is obvious that the inequality (10) is fulfilled if the constant \( L \) is sufficiently small.

5. Examples

**Example 1.** Consider the partial differential system of the form

\[
x_t(t, \theta) = x_{\theta\theta}(t, \theta) + b(\theta)u(t) + f(t, x(t - h, \theta)),
\]

\[
x(t, 0) = x(t, \pi) = 0, \quad t > 0,
\]

\[
x(t, \theta) = \phi(t, \theta), \quad -h \leq t \leq 0,
\] (17)

where \( \phi \) is continuous and \( u \in L_2[0, T], X = L_2[0, \pi], b \in X \) and where \( f: R \times R \to R \) is continuous and uniformly bounded.

Let \( B \in \mathcal{L}(R, X) \) be defined as

\[
(Bu)(\theta) = b(\theta)u, \quad 0 \leq \theta \leq \pi, \; u \in R, \; b(\theta) \in L_2[0, \pi],
\]
and let \( A : X \to X \) be operator defined by

\[
A z = z''
\]

with domain

\[
D(A) = \{ z \in X \mid z, z' \text{ are absolutely continuous}, \ z'' \in X, \ z(0) = z(\pi) = 0 \}.
\]

Then

\[
Az = \sum_{n=1}^{\infty} (-n^2)(z, e_n)e_n, \quad z \in D(A),
\]

where \( e_n(\theta) = \sqrt{2/\pi} \sin n\theta, \ 0 \leq x \leq \pi, \ n = 1, 2, \ldots \)

It is known that \( A \) generates a compact semigroup \( S(t), \ t > 0, \) in \( X \) and is given by

\[
S(t)z = \sum_{n=1}^{\infty} e^{-n^2t}(z, e_n)e_n, \quad z \in X.
\]

Therefore, the associated linear system is not completely controllable \([21,22]\), but it is approximately controllable \([3]\) provided that

\[
\int_{0}^{\pi} b(\theta)e_n(\theta) \, d\theta \neq 0 \quad \text{for} \ n = 1, 2, 3, \ldots
\]

Under the above conditions imposed on \( f \) and \( b \), system (17) is approximately controllable on \([0, T]\) by Theorem 3.

**Example 2.** Consider a control system governed by the semilinear heat equation

\[
x_t(t, \theta) = x_{\theta\theta}(t, \theta) + Bu(t, \theta) + f(t, x(t-h, \theta)),
\]

\[
x(t, 0) = x(t, \pi) = 0, \quad t > 0,
\]

\[
x(t, \theta) = \phi(t, \theta), \quad -h \leq t \leq 0,
\]

where \( \phi(t, \theta) \) is continuous, \( f : R \times R \to R \) is continuous and uniformly bounded.

Define an infinite-dimensional space \( U \) by

\[
U = \left\{ u = \sum_{n=2}^{\infty} u_n e_n(\theta) \mid \sum_{n=2}^{\infty} u^2_n < \infty \right\},
\]

where \( e_n(\theta) = \sqrt{2/\pi} \sin n\theta \) and let \( X = L^2[0, \pi] \). The norm in \( U \) is defined by \( \|u\| = \left( \sum_{n=2}^{\infty} u^2_n \right)^{1/2} \). Now define a linear continuous mapping \( B \) from \( U \) to \( X \) as

\[
Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n e_n(\theta), \quad u = \sum_{n=2}^{\infty} u_n e_n \in U.
\]
Because of the compactness of the semigroup $S(t), t > 0$, generated by $A$ and defined as in Example 1, the associated linear system is (again) not completely controllable.

On the other hand, it is easy to see that

$$B^* v = (2v_1 + v_2)e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta),$$

$$B^* S^*(t)x = (2x_1 e^{-t} + x_2 e^{-4t})e_2(\theta) + \sum_{n=3}^{\infty} x_n e^{-n^2 t} e_n(\theta),$$

with $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $x = \sum_{n=1}^{\infty} x_n e_n(\theta)$. Let

$$\|B^* S^*(t)x\| = 0, \quad t \in [0, T].$$

It follows that

$$\|2x_1 e^{-t} + x_2 e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|x_n e^{-n^2 t}\|^2 = 0, \quad t \in [0, T],$$

so

$$x_n = 0, \quad n = 1, 2, \ldots,$$

which implies $x = 0$.

Thus, by Theorem 4.1.7 in [3], the linear system corresponding to (18) is approximately controllable (but not completely controllable). Then, by Theorem 3, system (18) is approximately controllable on $[0, T]$.

**Example 3.** Consider the controlled wave equation with a distributed control $u(t, \cdot) \in L^2[0, 1]$

$$y_{tt}(t, \theta) = y_{\theta\theta}(t, \theta) + u(t, \theta) + f(t, y(t, \theta)), \quad 0 \leq t \leq T,$$

$$y(t, 0) = y(t, 1) = 0, \quad t > 0,$$

$$y(0, \theta) = \alpha(\theta), \quad y_t(0, \theta) = \beta(\theta), \quad 0 \leq \theta \leq 1,$$

(19)

where $\alpha, \beta \in L^2[0, 1]$ and where $f : R \times R \to R$ is continuous, Lipschitz continuous in its second variable and satisfies the linear growth condition.

Proceeding in a similar way to that in [3], introduce the Hilbert space $X = D(A_0^{1/2}) \oplus L^2[0, 1]$, endowed with the inner product

$$\langle w, v \rangle = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \sum_{n=1}^{\infty} \{n^2 \pi^2 \langle w_1, e_n \rangle \langle e_n, v_1 \rangle + \langle w_2, e_n \rangle \langle e_n, v_2 \rangle \},$$

where $e_n(\theta) = \sqrt{2} \sin(n\pi \theta)$. 
Setting

\[ x = \begin{bmatrix} y \\ y_t \end{bmatrix}, \quad x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ I \end{bmatrix}. \]

formulate system (1) as

\[ x'(t) = Ax(t) + Bu(t) + f(t, x(t)), \]
\[ x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \]

where

\[ A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad A_0 h = -(d^2/d\theta^2) h, \]

with domain

\[ D(A_0) = \{ h \in L_2(0, 1): h, (d/d\theta)h \text{ are absolutely continuous}, \]
\[ (d^2/d\theta^2)h \in L_2(0, 1) \text{ and } h(0) = 0 = h(1) \}, \]

and where \( A \) is the infinitesimal generator of a contraction group \( S(t) \) on \( X \) given by

\[ S(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \cos(n\pi t) & (n\pi)^{-1} \sin(n\pi t) \\ -n\pi \sin(n\pi t) & \cos(n\pi t) \end{bmatrix} \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} e_n. \]

Therefore \( S(t) \) is not compact. On the other hand, it is known that the linear system corresponding to (19) is completely controllable [3]. Thus, by Theorem 5, system (19) is completely controllable on \([0, T] \) provided that the Lipschitz constant is sufficiently small.

References