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Tauberian Theorems for Power Series Methods Applied to Double Sequences

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We discuss the relations between power series methods, weighted mean methods, and ordinary convergence for double sequences. In particular, we study Tauberian theorems for methods being products of the related one-dimensional summability methods. © 1997 Academic Press

1. INTRODUCTION

Let (p_{mn}) be a double sequence of nonnegative numbers (where, if not indicated otherwise, the indices run through $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) with $p_{00} > 0$ such that

$$p(x, y) := \sum_{m,n=0}^{\infty} p_{mn} x^m y^n < \infty \qquad \text{for } x, y \in (0, 1) \tag{1.1}$$

and

$$p(x, y) \to \infty \qquad \text{as } x, y \to 1-, \tag{1.2}$$

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where within this paper a limit in two variables is meant in the sense of Pringsheim, i.e., the two variables tend to their limit independently. Conditions (1.1) and (1.2) will be assumed throughout the paper without further mentioning them. Since

$$p(x,y) \geq \sum_{k,l=0}^{m,n} p_{kl} x^k y^l \qquad \text{for } x, y \in (0,1),$$

we have that $p(x,y) \to \infty$ as $x, y \to 1-$, iff

$$P_{mn} := \sum_{k,l=0}^{m,n} p_{kl} := \sum_{k=0}^{m} \sum_{l=0}^{n} p_{kl} \to \infty \qquad \text{as } m, n \to \infty.$$

We consider complex double sequences $\mathsf{S} = (s_{mn})$ with increments (a_{kl}), i.e.,

$$s_{mn} = \sum_{k,l=0}^{m,n} a_{kl}.$$

Let

$$\begin{aligned} \sigma_{mn} &:= \frac{1}{P_{mn}} \sum_{k,l=0}^{m,n} p_{kl} s_{kl}, \\ p_{\mathsf{S}}(x,y) &:= \sum_{m,n=0}^{\infty} s_{mn} p_{mn} x^m y^n, \end{aligned} \tag{1.3}$$

and

$$\sigma_{\mathsf{S}}(x,y) := p_{\mathsf{S}}(x,y)/p(x,y). \tag{1.4}$$

We say that:

(i) S is boundedly convergent to s and write b-lim $s_{mn} = s$ if

$$\lim s_{mn} := \lim_{m,n\to\infty} s_{mn} = s \qquad \text{and} \qquad \sup_{m,n} |s_{mn}| < \infty;$$

(ii) S is summable to s by the power series mean J_p and write J_p-lim $s_{mn} = s$ if the double power series $p_{\mathsf{S}}(x,y)$ converges for all $(x,y) \in (0,1)^2$ and

$$\sigma_{\mathsf{S}}(x,y) \to s \qquad \text{as } x, y \to 1-;$$

(iii) S is boundedly summable to s by the power series mean J_p and write bJ_p-lim $s_{mn} = s$ if

$$J_p\text{-lim } s_{mn} = s \qquad \text{and} \qquad \sup_{x,y\in(0,1)} |\sigma_{\mathsf{S}}(x,y)| < \infty;$$

(iv) S is summable to s by the weighted mean method M_p and write M_p-lim $s_{mn} = s$ if $\sigma_{mn} \to s$ as $m, n \to \infty$;

(v) S is boundedly summable to s by the weighted mean method M_p and write bM_p-lim $s_{mn} = s$ if

$$M_p\text{-lim } s_{mn} = s \qquad \text{and} \qquad \sup_{m,n} |\sigma_{mn}| < \infty.$$

Under the conditions

$$P_{ml}/P_{mn} \to 0 \quad \text{and} \quad P_{kn}/P_{mn} \to 0 \qquad \text{as } m, n \to \infty \text{ for any fixed } k, l, \tag{1.5}$$

we have by the theorem of Kojima and Robinson (see, e.g., [11, Theorem 20] that for bounded sequences the corresponding weighted mean method is regular (shortly is b-regular), i.e.,

$$s_{mn} \to s \text{ as } m, n \to \infty \text{ and } \sup_{m,n} |s_{mn}| < \infty \qquad \text{implies} \qquad \text{b}M_p\text{-lim } s_{mn} = s.$$

Furthermore, under the condition that, for any fixed μ, ν,

$$\sum_{k=0}^{\infty} p_{k\nu} x^k / p(x, y) \to 0, \qquad \sum_{l=0}^{\infty} p_{\mu l} y^l / p(x, y) \to 0 \qquad \text{as } x, y \to 1- \tag{1.6}$$

holds, we have that the corresponding power series method J_p is b-regular (see, e.g., [5, p. 84]). It is the aim of this paper to derive converse conclusions, i.e., Tauberian results. However, this can only be true under additional assumptions on the sequence S, the so-called Tauberian conditions.

So far for power series methods J_p our main Tauberian result is restricted to weights (p_{kl}) which factorize, i.e., we have $p_{kl} = p_k q_l$ with nonnegative sequences $p = (p_n)$ and $q = (q_n)$ satisfying

$$\begin{gathered} 0 < P_n := \sum_{k=0}^{n} p_k \to \infty, \qquad 0 < Q_n := \sum_{k=0}^{n} q_k \to \infty \qquad \text{as } n \to \infty, \\ p(x) := \sum_{k=0}^{\infty} p_k x^k < \infty \qquad \text{for } x \in (0,1), \\ q(y) := \sum_{l=0}^{\infty} q_l y^l < \infty \qquad \text{for } y \in (0,1). \end{gathered} \tag{1.7}$$

We denote the associated power series method based on weights $(p_k q_l)$ for double sequences by J_{pq} and the associated arithmetic mean method by M_{pq}. Both methods are b-regular under (1.7). In the literature on Tauberian results for power series methods applied to double sequences there exist (to our knowledge) only results on the Abel method for double sequences, where $p_{kl} = p_q q_l$ with $p_k = q_l \equiv 1$ (see [25] for references and also [1, 2]). For weighted mean methods in particular the Cesàro method $C_{1,1}$ was studied (see, e.g., the book [25] for references or [9, 20] where $C_{\alpha,\beta}$ methods are discussed). For more general weighted mean methods there is also a recent result in [4].

2. MAIN RESULTS

First we have to introduce the following quantities which were used in a series of papers [8, 12–14, 16–19, 23] to study power series methods in the one-dimensional case:

$$\Delta_m^p = \inf_{0<x<1} p(x) x^{-m} \qquad \text{and} \qquad \Delta_n^q = \inf_{0<y<1} q(y) y^{-n}. \tag{2.1}$$

The infima for, e.g., Δ_m^p are attained at points $x_m \in (0,1)$. For details and the most important properties of these quantities Δ_m, see Lemma 1 below or consult the papers [7, 16–18]. They are in many cases of the same order as the quantities P_m resp. Q_n but not always.

EXAMPLE. (i) In case $p_n = 1/(n+1)$ we find

$$1 - x_n \sim \frac{1}{n \log n}, \qquad \Delta_n^p \sim \log n, \qquad P_n \sim \log n$$

so we have $\Delta_n^p \sim P_n$.

(ii) In case $p_n = (n+1)^\gamma$ or $p_n = \Gamma(\gamma+1)\binom{n+\gamma}{n}$ with $\gamma > -1$ we obtain that

$$1 - x_n \sim (\gamma+1)/(n+\gamma+1),$$

$$\Delta_n^p \sim \left(\frac{e}{\gamma+1}\right)^{\gamma+1} n^{\gamma+1}, \qquad P_n \sim n^{\gamma+1}/(\gamma+1).$$

So we obtain again that Δ_n^p and P_n are of the same order.

(iii) In case $p_n \sim \exp(n^\gamma)$ with $0 < \gamma < 1$ we have (see, e.g., [7, 8, 18])

$$1 - x_n \sim -\gamma n^{\gamma-1}, \qquad \Delta_n^p \sim \left(\frac{2\pi}{\gamma(1-\gamma)}\right)^{1/2} n^{1-\gamma/2} \exp(n^\gamma),$$

$$P_n \sim n^{1-\gamma} \exp(n^\gamma)/\gamma.$$

Now P_n is of smaller order than Δ_n^p. Similar calculations can be done for a much larger class of weights; see, e.g., [18].

For a general discussion of the relation between Δ_n^p and P_n, see Lemma 1 below.

The following condition will be used as our basic Tauberian condition:

$$\begin{aligned} \sup_{m \in \mathbb{N}_0} \left| \sum_{\mu=0}^{m} a_{\mu n} \right| &\leq \beta_n \frac{q_n}{\Delta_n^q} \qquad \text{for } n = 1, 2, \ldots, \\ \sup_{n \in \mathbb{N}_0} \left| \sum_{\nu=0}^{n} a_{m \nu} \right| &\leq \alpha_m \frac{p_m}{\Delta_m^p} \qquad \text{for } m = 1, 2, \ldots, \end{aligned} \tag{2.2}$$

with suitable nonnegative sequences (α_m) and (β_n) to be specialized below. In the case of the Abel method in two dimensions we have (see example (ii) above)

$$\frac{q_n}{\Delta_n^q} = \frac{p_n}{\Delta_n^p} \asymp \frac{1}{n+1}.$$

Our first result is very general for product-power series methods.

THEOREM 1. *Assume that the double sequence* S *satisfies* (2.2) *with null sequences* $(\alpha_m), (\beta_n)$*. Then we have*

$$J_{pq}\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \lim s_{mn} = s$$

and

$$\mathrm{b}J_{pq}\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \text{b-lim } s_{mn} = s.$$

Remark 1. (i) In the case of the Abel method or the generalized Abel methods $A_{\gamma,\delta}$ with $p_m = \binom{m+\gamma}{m}$, $q_n = \binom{n+\delta}{n}$ where $\gamma, \delta > -1$, a sufficient condition for (2.2) is given by

$$a_{mn} = o(1) \frac{1}{m^2 + n^2} \qquad \text{as } m \text{ or } n \to \infty$$

(see, e.g., [15] for the ordinary Abel method). This condition can be replaced (see [9]) by

$$a_{mn} = o(1) \frac{1}{m^p + n^q} \qquad \text{as } m \text{ or } n \to \infty,$$

with conjugate indices $p, q > 1$ with $1/p + 1/q = 1$.

Considering the logarithmic method where $p_n = q_n = 1/(n+1)$, we obtain the somewhat stronger assumption

$$a_{mn} = o(1)\frac{1}{(m\log(m+1))^2 + (n\log(n+1))^2} \qquad \text{as } m \text{ or } n \to \infty.$$

However, for methods using weights $p_m = \exp(m^\gamma)$, $q_n = \exp(n^\delta)$ where $\gamma, \delta \in (0,1)$ a sufficient condition for (2.2) is given by

$$a_{mn} = o(1)\frac{1}{m^{2-(1-\gamma/\delta+\gamma/2)} + n^{2-(1-\delta/\gamma+\delta/2)}} \qquad \text{as } m \text{ or } n \to \infty.$$

(ii) The Tauberian condition (2.2) is called o-type for obvious reasons.

The question arises now whether the o-Tauberian condition in Theorem 1 can be replaced by the so-called O-type condition. This is indeed the case if we assume that the weight sequences (p_{mn}) behave nicely. We assume in the following that the partial sums (P_m) and (Q_n) are unbounded regularly varying sequences (see, e.g., [6] for the notation and basic properties). That means we have representations

$$P_m = (m+1)^\alpha L_1(m+1) \qquad \text{and} \qquad Q_n = (n+1)^\beta L_2(n+1), \quad (2.3)$$

with constants $\alpha, \beta \geq 0$ and slowly varying functions L_1 and L_2 on $(0,\infty)$, i.e., they are positive, measurable, and satisfy

$$\frac{L_1(\lambda t)}{L_1(t)}, \frac{L_2(\lambda t)}{L_2(t)} \to 1 \qquad \text{as } t \to \infty \text{ for all } \lambda > 0.$$

Now we can give an O-type Tauberian theorem.

THEOREM 2. *Assume that for the partial sums of the sequences* $(p_m), (q_n)$ *property* (2.3) *holds and that* S *satisfies* (2.2) *with bounded sequences* $(\alpha_m), (\beta_n)$. *Then we have*

$$\text{b}J_{pq}\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \text{b-lim } s_{mn} = s.$$

Remark 2. (i) In the case of the two dimensional Abel methods a sufficient condition for (2.2) is given by

$$|a_{mn}| \leq \frac{c}{(m+1)^2 + (n+1)^2} \qquad \text{for } m, n \in \mathbb{N}_0 \text{ with some } c > 0,$$

a condition used by various authors. Actually this condition works for any power series method based on weight sequences $(p_m), (q_n)$ satisfying (2.3) (see Remark 5). So in particular this condition applies to the generalized Abel methods $A_{\gamma,\delta}$ where $\gamma, \delta > -1$.

For the logarithmic method we obtain as a Tauberian condition

$$|a_{mn}| \le \frac{c}{((m+1)\log(m+1))^2 + ((n+1)\log(n+1))^2} \quad \text{for } m, n \in \mathbb{N},$$

with some $c > 0$.

(ii) Condition (2.3) can be relaxed somewhat. Assume that a positive unbounded sequence $(\tilde{P}_n)$ satisfies

$$\limsup_{n\to\infty} \frac{\tilde{P}_{2n}}{\tilde{P}_n} < \infty \qquad \text{and} \qquad M_{\tilde{P}}(t) = \limsup_{n\to\infty} \frac{\tilde{P}_{[tn]}}{\tilde{P}_n} \to 1 \qquad \text{as } t \to 1+, \tag{2.4}$$

i.e., $M_{\tilde{P}}(t)$ exists for $t > 1$ and is continuous at $t = 1$. Following [10], we could call such sequences intermediate regularly varying. Instead of (2.3) we may assume that (2.4) holds for both (P_n) and (Q_n).

The second result will be proven in two steps, which are of interest themselves. In a first step J_p- and M_p-summability are related. First we show that the J_p-method is stronger than the M_p-method and then we discuss an inverse Tauberian result.

PROPOSITION 1. *Assume that (p_{mn}) is a weight sequence satisfying* (1.6). *Then we have*

$$\mathrm{b}M_p\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \mathrm{b}J_p\text{-lim } s_{mn} = s.$$

Proof. If (1.6) holds then the power series method J_P with weights (P_{mn}) is b-regular as well. Hence we find from (1.3) with the notation $P(x, y) := \sum_{m,n=0}^{\infty} P_{mn} x^m y^n$ the relations

$$s \leftarrow \frac{1}{P(x,y)} \sum_{m,n=0}^{\infty} P_{mn} \sigma_{mn} x^m y^n = \frac{1}{p(x,y)} \sum_{m,n=0}^{\infty} p_{mn} s_{mn} x^m y^n$$

since $p(x, y) = (1 - x)(1 - y)P(x, y)$ and obviously the J_p-mean is bounded as well. ∎

Now we return to the Tauberian aspect. Assume in the following that (P_{mn}) satisfies (1.6) and we define

$$\pi(u,v) := \sum_{\substack{0\le k<u\\ 0\le l<v}} p_{kl} \qquad \text{for } u, v \ge 0 \tag{2.5}$$

and put, for any $\varepsilon \in (0,1)$,

$$B(1,\varepsilon) := \big\{(e_1,e_2) \in \mathbb{R}^2 \,|\, e_1,e_2 > 0 \\ \text{such that } \varepsilon \le e_1/e_2 \le 1/\varepsilon \text{ and } e_1^2 + e_2^2 = 1\big\}.$$

Suppose now that, with some $\varepsilon \in (0,1)$ and any fixed $(e_1,e_2) \in B(1,\varepsilon)$,

$$\lim_{m,n\to\infty} \pi\frac{\pi(um,vn)}{(e_1m,e_2n)} \qquad \begin{array}{l}\text{exists for all } u,v > 0 \\ \text{and is continuous at } (u,v) = (e_1,e_2).\end{array} \tag{2.6}$$

Furthermore, we suppose that for any $\delta > 0$ there exists some $M_\delta > 0$ such that, for all $m,n \in \mathbb{N}$,

$$\frac{\pi(um,vn)}{\pi(e_1m,e_2n)} \le M_\delta e^{\delta(u+v)} \qquad \text{for } u,v \ge 1 \text{ and } (e_1,e_2) \in B(1,\varepsilon) \tag{2.7}$$

holds. In the unit square the quotient is bounded then by monotonicity arguments. Conditions (2.6) and (2.7) are satisfied, e.g., in case $P_{mn} = P_m Q_n$ where $(P_m),(Q_n)$ satisfy (2.3) (or also (2.4)) with arbitrary $\varepsilon \in (0,1)$. Observe that in this case

$$H_{m,n}(u,v;e_1,e_2) := \frac{\pi(um,vn)}{\pi(e_1m,e_2n)} \to \left(\frac{u}{e_1}\right)^\alpha \left(\frac{v}{e_2}\right)^\beta \\ \text{for all } u,v > 0 \text{ and } e_1,e_2 \in B(1,\varepsilon)$$

and

$$H_{m,n}(u,v;e_1,e_2) \le C_{\delta,\varepsilon} \max\{1, u^{\alpha+\delta} v^{\beta+\delta}\} \qquad \text{for all } u,v > 0$$

for any $\varepsilon,\delta \in (0,1)$ with a suitable constant $C_{\delta,\varepsilon}$. Observe that slowly varying functions can be bounded by arbitrarily small powers (consult, e.g., the book [6, Theorem 1.5.6]) and that $(P_n),(Q_n)$ are nondecreasing. Then we have the following Tauberian theorem.

THEOREM 3. *Assume that (P_{mn}) is a weight sequence satisfying* (2.6) *and* (2.7). *Then $s_{mn} = O(1)$ is a Tauberian condition for the conclusion*

$$J_p\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \mathrm{b}M_p\text{-lim } s_{mn} = s.$$

Remark 3. For bounded sequences S the methods J_p and M_p are equivalent and consistent provided the weights satisfy (2.6) and (2.7).

The second step in the proof of Theorem 2 uses a Tauberian result from M_{pq}-summability to convergence for regularly varying weights.

THEOREM 4. *Assume that the weights* $(P_m), (Q_n)$ *satisfy* (2.3). *Then*

$$M_{pq}\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \lim s_{mn} = s$$

and

$$\mathrm{b}M_{pq}\text{-lim } s_{mn} = s \qquad \textit{implies} \qquad \text{b-lim } s_{mn} = s,$$

provided the following Tauberian condition holds:

$$\begin{aligned} \sup_{m \in \mathbb{N}_0} \left| \sum_{\mu=0}^{m} a_{\mu n} \right| &\le C \frac{q_n}{Q_n} \qquad \text{for } n = 1, 2, \ldots, \\ \sup_{n \in \mathbb{N}_0} \left| \sum_{\nu=0}^{n} a_{m \nu} \right| &\le C \frac{p_m}{P_m} \qquad \text{for } m = 1, 2, \ldots . \end{aligned} \tag{2.8}$$

Remark 4. (i) For the class of weights considered in Theorem 4, conditions (2.2) with bounded sequences $(\alpha_n), (\beta_n)$ and conditions (2.8) are equivalent, because of Lemma 1(iii). Furthermore, if, e.g., q is already regularly varying then $q_m/Q_m \asymp 1/m$ as $m \to \infty$.

(ii) Conditions (2.8) are also sufficient if $(P_n), (Q_m)$ satisfy (2.4) instead of (2.3) (see again Lemma 1 and the proof of Theorem 4 below).

3. AUXILIARY RESULTS

For dealing with the quantities Δ_n^p the following lemma is important.

LEMMA 1 (cf. [7, 16, 17]). *Let there be given a nonnegative sequence* $p = (p_n)$ *satisfying* (1.7). *For the associated sequence* (Δ_n^p) *defined in* (2.1) *the following hold*:

(i) *There exists a sequence* $(x_n) \nearrow 1$ *such that* $\Delta_n^p = p(x_n) x_n^{-n}$ *for* $n \in \mathbb{N}_0$;

(ii) $\sum_{k=1}^{\infty} p_k/\Delta_k^p = \infty$;

(iii) $P_n \le \Delta_n^p$ *and* ($\Delta_n^p = O(P_n)$ *iff* $P_{2n} = O(P_n)$).

Remark 5. So in particular we have $\Delta_n^p = O(P_n)$ in case that (P_n) is regularly varying. Furthermore, if (p_n) is already regularly varying, then we have $p_n/\Delta_n^p \sim c_1 p_n/P_n \sim c_2/n$ with suitable constants $c_1, c_2 > 0$.

The following lemma is the key to our results.

LEMMA 2. *Assume that the double sequence* S *satisfies* (2.2) *and let* $\sigma_{\mathsf{S}}(x, y)$ *denote the* J_{pq}*-means as defined in* (1.4). *Then we have, for any* $\mu, \nu, m, n \in \mathbb{N}_0$,

(i) $$|s_{\mu\nu} - s_{mn}| \le \sum_{k=\rho_u+1}^{\rho_o} \alpha_k \frac{p_k}{\Delta_k^p} + \sum_{l=\theta_u+1}^{\theta_o} \beta_l \frac{q_l}{\Delta_l^q},$$

where $\rho_u = \min\{\mu, m\}$, $\rho_o = \max\{\mu, m\}$ *and* $\theta_u = \min\{\nu, n\}$, $\theta_o = \max\{\nu, n\}$.

(ii)
$$\left|\sigma_{\mathsf{S}}(x_m, y_n) - s_{mn}\right| \le \sum_{k=1}^{\infty} \alpha_k p_k x_m^k / p(x_m) + \sum_{l=1}^{\infty} \beta_l q_l y_n^l / q(y_n).$$

COROLLARY. *If in Lemma* 2 *the sequences* $(\alpha_m), (\beta_n)$ *are bounded then we have*

$$\sup_{m,n} \left|\sigma_{\mathsf{S}}(x_m, y_n) - s_{mn}\right| < \infty;$$

if they are null sequences then we have

$$\lim_{m,n\to\infty} \left|\sigma_{\mathsf{S}}(x_m, y_n) - s_{mn}\right| = 0.$$

4. PROOFS

Proof of Lemma 2. (i) For $\mu \ge m$ and $\nu \ge n$ we have

$$\begin{aligned} |s_{\mu\nu} - s_{mn}| &= \left| \sum_{k,l=0}^{\mu,\nu} a_{kl} - \sum_{k,l=0}^{m,n} a_{kl} \right| \le \left| \sum_{k=m+1}^{\mu} \sum_{l=0}^{\nu} a_{kl} \right| + \left| \sum_{k=0}^{m} \sum_{l=n+1}^{\nu} a_{kl} \right| \\ &\le \sum_{k=m+1}^{\mu} \left| \sum_{l=0}^{\nu} a_{kl} \right| + \sum_{l=n+1}^{\nu} \left| \sum_{k=0}^{m} a_{kl} \right| \\ &\le \sum_{k=m+1}^{\mu} \alpha_k \frac{p_k}{\Delta_k^p} + \sum_{l=n+1}^{\nu} \beta_l \frac{q_l}{\Delta_l^q}, \end{aligned}$$

where we used (2.2).

For $\mu < m$ and $\nu \geq n$ we find

$$|s_{\mu\nu} - s_{mn}| \leq \sum_{k=\mu+1}^{m} \left| \sum_{l=0}^{n} a_{kl} \right| + \sum_{l=n+1}^{\nu} \left| \sum_{k=0}^{\mu} a_{kl} \right|$$
$$\leq \sum_{k=\mu+1}^{m} \alpha_k \frac{p_k}{\Delta_k^p} + \sum_{l=n+1}^{\nu} \beta_l \frac{q_l}{\Delta_l^q},$$

where we used (2.2) again. The remaining cases are to be dealt with similarly.

(ii) using the sequences $(x_m), (y_n)$ defined in Lemma 1(i) for Δ_m^p and Δ_n^q respectively, we obtain by Lemma 2(i) that

$$\left| \sigma_S(x_m, y_n) - s_{mn} \right|$$
$$\leq \frac{1}{p(x_m) q(y_n)} \sum_{\mu,\nu=0}^{\infty} p_\mu q_\nu |s_{\mu\nu} - s_{mn}| x_m^\mu y_n^\nu$$
$$\leq \frac{1}{p(x_m) q(y_n)} \sum_{\mu,\nu=0}^{\infty} p_\mu q_\nu \left(\sum_{k=\rho_u+1}^{\rho_o} \alpha_k \frac{p_k}{\Delta_k^p} + \sum_{\theta_u+1}^{\theta_o} \beta_l \frac{q_l}{\Delta_l^q} \right) x_m^\mu y_n^\nu$$
$$\leq \frac{1}{p(x_m)} \left(\sum_{\mu=0}^{m-1} p_\mu x_m^\mu \sum_{k=\mu+1}^{m} \alpha_k \frac{p_k}{\Delta_k^p} + \sum_{\mu=m+1}^{\infty} p_\mu x_m^\mu \sum_{k=m+1}^{\mu} \alpha_k \frac{p_k}{\Delta_k^p} \right)$$
$$+ \frac{1}{q(y_n)} \left(\sum_{\nu=0}^{n-1} q_\nu y_n^\nu \sum_{l=\nu+1}^{n} \beta_l \frac{q_l}{\Delta_l^q} + \sum_{\nu=n+1}^{\infty} q_\nu y_n^\nu \sum_{l=n+1}^{\nu} \alpha_k \frac{q_l}{\Delta_l^q} \right)$$
$$= \frac{1}{p(x_m)} \left(\sum_{k=1}^{m} \alpha_k \frac{p_k}{\Delta_k^p} \frac{x_m^k}{x_k^k} \sum_{\mu=0}^{k-1} p_\mu x_k^\mu \left(\frac{x_m}{x_k} \right)^{\mu-k} \right.$$
$$\left. + \sum_{k=m+1}^{\infty} \alpha_k \frac{p_k}{\Delta_k^p} \frac{x_m^k}{x_k^k} \sum_{\mu=k}^{\infty} p_\mu x_k^\mu \left(\frac{x_m}{x_k} \right)^{\mu-k} \right)$$
$$+ \frac{1}{q(y_n)} \left(\sum_{l=1}^{n} \beta_l \frac{q_l}{\Delta_l^q} \frac{y_n^l}{y_l^l} \sum_{\nu=0}^{l-1} q_\nu y_l^\nu \left(\frac{y_n}{y_l} \right)^{\nu-l} \right.$$
$$\left. + \sum_{l=n+1}^{\infty} \beta_l \frac{q_l}{\Delta_l^q} \frac{y_n^l}{y_l^l} \sum_{\nu=l}^{\infty} q_\nu y_l^\nu \left(\frac{y_n}{y_l} \right)^{\nu-l} \right).$$

Now observe that $(x_m), (y_n)$ are nondecreasing. Hence $(x_m/x_k)^{\mu-k}$, $(y_n/y_l)^{\nu-l} \leq 1$ in all occurring cases and that by Lemma 1(i) $\Delta_l^q y_l^l = q(y_l)$, $\Delta_k^p x_k^k = p(x_k)$ and we end with

$$\left| \sigma_S(x_m, y_n) - s_{mn} \right| \leq \frac{1}{p(x_m)} \sum_{k=1}^{\infty} \alpha_k p_k x_m^k + \frac{1}{q(y_n)} \sum_{l=1}^{\infty} \beta_l q_l y_n^l.$$

The corollary follows from the regularity and positivity of the one-dimensional power series methods. ∎

Proof of Theorem 1. The proof follows directly from the corollary. ∎

Proof of Theorem 3. The proof of Theorem 3 is based on a Tauberian theorem for multivariate Laplace transforms in [22]. For the moment we assume that S is real. Now we use the following notation:

$$S(u,v) := \sum_{\substack{0\le\mu<u\\0\le\nu<v}} s_{\mu\nu} p_{\mu\nu}.$$

Then

$$\begin{aligned} p_S(x,y) &= \sum_{\mu,\nu=0}^{\infty} p_{\mu\nu} s_{\mu\nu} x^{\mu} y^{\nu} \\ &= \sum_{\mu,\nu=0}^{\infty} p_{\mu\nu} s_{\mu\nu} e^{-\xi\mu-\eta\nu} \qquad \text{with } x=e^{-\xi},\ y=e^{-\eta} \\ &= \int_0^{\infty}\int_0^{\infty} e^{-\xi u-\eta v}\, dS(u,v), \end{aligned}$$

and analogously we write

$$p(x,y) = \int_0^{\infty}\int_0^{\infty} e^{-\xi u-\eta v}\, d\pi(u,v).$$

where π is defined by (2.5). If the sequence S is bounded we are led by partial integration to the formula

$$\sigma_S(\xi,\eta) := \frac{p_S(e^{-\xi},e^{-\eta})}{p(e^{-\xi},e^{-\xi})} = \frac{\int_0^\infty \int_0^\infty e^{-\xi u-\eta v} S(u,v)\,du\,dv}{\int_0^\infty \int_0^\infty e^{-\xi u-\eta v} \pi(u,v)\,du\,dv},$$

which can be checked directly, partitioning the integrals in rectangles where S,π are constant. Hence bJ_p-lim $s_{mn}=s$ implies $\sigma_S(e^{-\xi},e^{-\eta})\to s$ as $\xi,\eta\to 0+$. Since S is bounded we can w.l.o.g. assume that $s_{mn}\ge 0$ so that $S(u,v)$ is nondecreasing. Then Theorem 1 in [22] applies and it yields

$$\frac{S(u,v)}{\pi(u,v)} \to s \qquad \text{as } u,v\to\infty$$

and the quotient is obviously bounded since S is bounded. Observe that in Theorem 1 in [22] the vector-valued function p can be written as $p(m) =$

$(m, n(m))$ with $n(m) \nearrow \infty$ arbitrarily fast or slow. This yields the convergence in the Pringsheim sense as stated.

In case of a complex sequence S we split it into its real and imaginary parts and proceed as before for both parts separately. The conclusion above translates to

$$\mathrm{b}M_p\text{-lim } s_{mn} = s. \qquad \blacksquare$$

Proof of Theorem 4. We proceed as in the one-dimensional case thereby using similar ideas as in [15, p. 579] for the $C_{1,1}$-mean applied to double sequences. We have with

$$\sigma_{mn} = \frac{1}{P_m Q_n} \sum_{k,l=0}^{m,n} s_{kl} p_k q_l,$$

for $\mu > m$, $\nu > n$,

$$\begin{aligned}
\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} p_k q_l s_{kl} \\
&= P_\mu Q_\nu \sigma_{\mu\nu} - P_\mu Q_n \sigma_{\mu n} - P_m Q_\nu \sigma_{m\nu} + P_m Q_n \sigma_{mn} \\
&\quad + \{P_\mu Q_\nu - P_\mu Q_n - P_m Q_\nu\} \sigma_{mn} \\
&\quad - \{P_\mu Q_\nu - P_\mu Q_n - P_m Q_\nu\} \sigma_{mn} \\
&= P_\mu Q_\nu (\sigma_{\mu\nu} - \sigma_{mn}) - P_\mu Q_n (\sigma_{\mu n} - \sigma_{mn}) \\
&\quad - P_m Q_\nu (\sigma_{m\nu} - \sigma_{mn}) + \sigma_{mn}\big((P_\mu - P_m)(Q_\nu - Q_n)\big).
\end{aligned}$$

Consequently, we find, with the abbreviation $D := (P_\mu - P_m)(Q_\nu - Q_n)$,

$$\begin{aligned}
\frac{1}{D} \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} p_k q_l s_{kl} = \sigma_{mn} + \frac{P_\mu Q_\nu}{D}(\sigma_{\nu\mu} - \sigma_{mn}) \\
- \frac{P_\mu Q_n}{D}(\sigma_{\mu n} - \sigma_{mn}) - \frac{P_m Q_\nu}{D}(\sigma_{m\nu} - \sigma_{mn}).
\end{aligned}$$

Thus we obtain

$$|\sigma_{mn} - s_{mn}| \leq \frac{P_\mu Q_\nu}{D}|\sigma_{\nu\mu} - \sigma_{mn}| + \frac{P_\mu Q_n}{D}|\sigma_{\mu n} - \sigma_{mn}| + \frac{P_m Q_\nu}{D}|\sigma_{m\nu} - \sigma_{mn}| + \frac{1}{D}\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu} p_k q_l |s_{kl} - s_{mn}|. \tag{4.1}$$

Now we put $\mu = \operatorname{argmin}\{P_k \geq \lambda P_m\}$ and $\nu = \operatorname{argmin}\{Q_k \geq \rho Q_n\}$ with $\lambda, \rho > 1$. By (2.8), Lemma 1(iii), and Lemma 2 there exist constants $c_i > 0$ $(i = 1, 2, 3)$ such that

$$|s_{kl} - s_{mn}| \leq c_1 \sum_{r=m+1}^{\mu} \frac{p_r}{P_r} + c_2 \sum_{s=n+1}^{\nu} \frac{q_s}{Q_s} \leq c_3((\lambda - 1) + (\rho - 1)) \tag{4.2}$$

holds for all k, l such that $m \leq k \leq \mu$ and $n \leq l \leq \nu$.

Now observe that by (2.3) $P_{\mu+1}/P_\mu \to 1$ and $Q_{\nu+1}/Q_\nu \to 1$ and hence

$$\frac{P_\mu}{P_m} \to \lambda \qquad \text{as } m \to \infty \qquad \text{and} \qquad \frac{Q_\nu}{Q_n} \to \rho \qquad \text{as } n \to \infty. \tag{4.3}$$

Using in addition that by assumption $\sigma_{mn} \to s$ and hence $\sigma_{kl} - \sigma_{mn} \to 0$ for $k = \mu$ or m and $l = \nu$ or n as $m, n \to \infty$, we obtain from (4.1), using (4.2) and (4.3), that

$$|\sigma_{mn} - s_{mn}| \leq \frac{\lambda\rho + \lambda + \rho}{(\lambda - 1)(\rho - 1)}(1 + o(1)) \cdot o(1) + c_3 \cdot ((\lambda - 1) + (\rho - 1))(1 + o(1)).$$

Hence, for any $\lambda, \rho > 1$ we obtain

$$\limsup_{m,n\to\infty} |\sigma_{mn} - s_{mn}| \leq c_3 \cdot ((\lambda - 1) + (\rho - 1)),$$

which yields the desired result as $\lambda, \rho \to 1+$. ∎

Remark 6. Actually the same proof shows that instead of (2.8) the following Tauberian condition would work here as well:

$$\limsup_{m,n\to\infty} \max_{\substack{P_m \leq k \leq \lambda P_m \\ Q_n \leq l \leq \rho Q_n}} |s_{kl} - s_{mn}| \to 0 \qquad \text{as } \lambda, \rho \to 1+.$$

5. FINAL REMARKS

There are several open questions:

(1) Do there exist O-Tauberian results with the weights from Example 1(iii) as in the one-dimensional case (see [18])?

(2) What is the situation in the case of nonmultiplicative weights? The only answer we can give is: The results above apply if there exist constants $0 < c_1 < c_2$ and sequences $(p_m), (q_n)$ satisfying (1.7) such that

$$c_1 p_m q_n \leq p_{mn} \leq c_2 p_m q_n.$$

(3) What is the proper version of regular variation in dimension 2 to be applied in this context?

(4) What about one-sided Tauberian conditions?

(5) Is the order of the Tauberian conditions optimal?

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