Journal of Algebra 343 (2011) 143–159

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Non-characteristic approximate roots of polynomials

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ARTICLE INFO

Article history:
Received 5 May 2010
Available online 20 July 2011
Communicated by Steven Dale Cutkosky

Keywords:
Approximate roots of polynomials
Characteristic sequences

ABSTRACT

We generalize Abhyankar–Moh's theory of approximate roots of polynomials to the case of approximate roots of non-characteristic degrees of an irreducible element of $\mathbb{K}((X))[Y]$.

1. Introduction

Let $\mathbb{K}[[X]]$ be the power series ring in one variable X with coefficients in an algebraically closed field $\mathbb{K}$, $\mathbb{K}((X))$ be its field of fractions (the field of Laurent series in the variable X) and $f \in \mathbb{K}((X))[Y]$ – a monic and irreducible polynomial. For any $l \mid \deg_Y f$ the approximate l-th root of f is a monic polynomial $g \in \mathbb{K}((X))[Y]$ such that

$$\deg_Y\left(f - g^l\right) < \deg_Y f - \frac{\deg_Y f}{l}.$$

In [1,2] Abhyankar and Moh proved many properties (see Theorem 2 for a compilation of their results) of l-th approximate roots for so-called characteristic divisors l of $\deg_Y f$, and applied them in affine algebraic geometry (embedding of the line in the plane [3], the Jacobian conjecture [4], analytic irreducibility at ∞ [5]).

In the paper we show that almost all of the nice properties of approximate roots found by Abhyankar and Moh have their 'non-characteristic' analogues (Theorem 5), at least in the case when $\operatorname{char}\mathbb{K} = 0$. The results are a continuation of the investigations started in [6], cf. also [7].

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.06.030

2. Basic notions and known results

In what follows we concentrate on [4] as our main source of references. For the convenience of the Reader we recall some basic notions from the Abhyankar–Moh's theory. We start with the fundamental definition (cf. [4, Definition (4.3)]).

Definition 1. Let R be a commutative ring with unity, let $f \in R[Y]$ be a *monic* polynomial of degree k and let $l|k$ be a positive divisor of k such that $1/l \in R$ (i.e. l is invertible in R). A *monic* polynomial $g \in R[Y]$ satisfying the relation

$$\deg_Y\left(f - g^l\right) < k - \frac{k}{l}$$

is called *an approximate l-th root of f*.

The following theorem is well known.

Theorem 1. *(See [4, Theorem (4.4)].) Under the above assumptions an l-th approximate root of f exists and is uniquely determined.*

Notation 1. In what follows, the unique element of Theorem 1 will be denoted by $\sqrt[l]{f}$.

Remark 1. There exists an easy-to-implement algorithm for computing approximate roots. It is based on the so called Tschirnhausen transformation, which in turn reduces to the division with remainder (cf. [4, §3]).

In the sequel we will make use of a more precise version of Theorem 1. We state it as a lemma (cf. [7]).

Lemma 1. *Under the assumptions of the theorem, let additionally $\mathbb{Q} \subset R$. Put $\hat{f} := Y^k f(Y^{-1}) \in R[Y]$. There exists $\hat{g} \in R[[Y]]$, such that $\hat{g}(0) = 1$ and $\hat{g}^l = \hat{f}$. What is more, if such a $\hat{g}$ is of the form*

$$\hat{g} = \underbrace{\sum_{j=0}^{\frac{k}{l}} a_j Y^j}_{g:=} + \textit{terms of order greater than } \frac{k}{l}, \tag{2.1}$$

then

$$\sqrt[l]{f} = Y^{\frac{k}{l}} g\left(Y^{-1}\right) = \sum_{j=0}^{\frac{k}{l}} a_j Y^{\frac{k}{l}-j}. \tag{2.2}$$

Proof. Consider $h \in R[[Y]]$ of the form $h := \sum_{j=0}^{+\infty} \binom{\frac{1}{l}}{j} Y^j$. It is clear that $h(0) = 1$ and $h^l = 1 + Y$. Composing h with the series $\hat{f} - 1$, which has no free term since f is monic, we find the required series $\hat{g}$. Suppose $\hat{g}$ and g are of the form (2.1). Then $(Y^{\frac{k}{l}} \hat{g}(Y^{-1}))^l = Y^k \hat{f}(Y^{-1}) = f$. Notice also that $\deg Y^{\frac{k}{l}} g(Y^{-1}) = \frac{k}{l}$ and that

$$Y^{\frac{k}{l}} \hat{g}\left(Y^{-1}\right) = Y^{\frac{k}{l}} g\left(Y^{-1}\right) + \text{terms of order less than } 0.$$

Raising both sides of this equality to the l-th power we thus get

$$f = \left(Y^{\frac{k}{l}} g\left(Y^{-1}\right)\right)^l + \underbrace{\text{terms of order less than } (l-1)\frac{k}{l}}_{r:=}.$$

Since $f, Y^{\frac{k}{l}} g(Y^{-1}) \in R[Y]$, then also $r \in R[Y]$. Since by (2.1) $g(0) = 1$, $Y^{\frac{k}{l}} g(Y^{-1})$ is monic and $\deg_Y r < k - \frac{k}{l}$. From the definition of an approximate root it follows that $\sqrt[l]{f} = Y^{\frac{k}{l}} g(Y^{-1})$. $\square$

Remark 2. Lemma 1 and its proof are also valid in the general case (i.e. without the assumption $\mathbb{Q} \subset R$). One must only prove that $\binom{\frac{1}{l}}{j} \in \mathbb{Z}[l^{-1}]$ for any $j \in \mathbb{N}_0$ and use the canonical homomorphism $\mathbb{Z}[l^{-1}] \to R$.

2.1. *Characteristic sequences of a parametrization (cf. [4, §6])*

Let there be given: a positive integer $k \in \mathbb{N}$ and a Laurent series $y(t) \in \mathbb{K}((t))$ with coefficients in a field $\mathbb{K}$ of characteristic $p \in \mathbb{N}_0$. The couple $(t^k, y(t))$ will be called a *parametrization*. The *support* of $y(t)$ (the set of those exponents of the powers of t that occur with a non-zero coefficient in the Laurent expansion of $y(t)$) will be denoted by $\operatorname{Supp}_t y(t)$.

From the expansion of $y(t)$ in the powers of t we read off so-called *characteristic sequences of the parametrization* $(t^k, y(t))$. Namely, if $y(t) = 0$ we define $m_0 := k$,[1] $m_1 := +\infty$, $d_1 := k$ and $h := 0$. If $y(t) \neq 0$, we put $m_0 := k$, $d_1 := k$, $m_1 := \operatorname{ord}_t y(t)$, $d_2 := \gcd(m_0, m_1)$ and, inductively, if $m_0, \ldots, m_i$ and $d_1, \ldots, d_{i+1}$ are already defined for some $i \geqslant 1$, we put

$$m_{i+1} := \inf\left\{j \in \operatorname{Supp}_t y(t)\colon\ j \not\equiv 0 \pmod{d_{i+1}}\right\}.$$

If, now, $m_{i+1} < +\infty$, we also define

$$d_{i+2} := \gcd(m_0, \ldots, m_{i+1}),$$

and in the case when $m_{i+1} = +\infty$, we put $h := i$ and finish the inductive definition.

Since in the above construction, there is always $0 < d_{j+1} < d_j$ for $j \geqslant 2$, the process ends after finitely many steps. Thus we end up with two sequences:

$$m := (m_0, m_1, \ldots, m_{h+1})$$

and

$$d := (d_1, \ldots, d_{h+1}).$$

We call them, respectively: *the characteristic* (*of the parametrization* $(t^k, y(t))$) and *the sequence of characteristic divisors* (*of the parametrization* $(t^k, y(t))$).

Immediately from the above definition, we get:

Property 1.

1. $h \geqslant 1$ *if* $y(t) \neq 0$,

[1] We could also take $m_0 = -k$; the value of m_0 isn't important for the results of this work.

2. $m_1 < m_2 < \cdots < m_{h+1} = +\infty$,
3. $d_{i+1} = \gcd(m_0, \ldots, m_i)$ *for* $0 \leqslant i \leqslant h$,
4. $d_{h+1} | d_h | \ldots | d_1 = k$ *and* $d_{h+1} < d_h < \cdots < d_2$,
5. *if* $M \in \mathbb{Z} \cup \{+\infty\}$ *and* $m_{i-1} < M \leqslant m_i$ *for some* $i \in \{2, \ldots, h+1\}$ (*or only* $M \leqslant m_i$, *if* $i = 1$),

then

$$\gcd\bigl(\{k\} \cup \bigl(\operatorname{Supp}_t y(t) \cap (-\infty, M)\bigr)\bigr) = \gcd(m_0, \ldots, m_{i-1}) = d_i.$$

On the basis of the sequences m and d we also define the following *derived characteristic sequences*:

$$s = (s_0, \ldots, s_{h+1}),$$

putting $s_0 := m_0$, $s_i := m_1 d_1 + \sum_{2 \leqslant j \leqslant i} (m_j - m_{j-1}) d_j$ for $1 \leqslant i \leqslant h$, and $s_{h+1} := +\infty$;

$$r = (r_0, \ldots, r_{h+1}),$$

putting $r_0 := m_0$, $r_i := \frac{s_i}{d_i}$ for $1 \leqslant i \leqslant h$, and $r_{h+1} := +\infty$;

$$n = (n_1, \ldots, n_h),$$

putting $n_i = \frac{d_i}{d_{i+1}}$ for $1 \leqslant i \leqslant h$.

The following property is self-evident.

Property 2. *The sequences m, d, s, r, n are integer-valued (or $+\infty$). What is more*

$$d_{i+1} = \gcd(r_0, \ldots, r_i) \quad \textit{for } 0 \leqslant i \leqslant h,$$
$$s_i = s_{i-1} + (m_i - m_{i-1}) d_i \quad \textit{for } 2 \leqslant i \leqslant h,$$
$$s_1 < s_2 < \cdots < s_{h+1} = +\infty.$$

Remark 3. Although all the sequences defined above depend on the parametrization $(t^k, y(t))$, we will omit this dependency, since it will always be clear from the context which parametrization they belong to. If we face the necessity of distinguishing characteristic sequences of two parametrizations, we will use decorations, e.g. $\overline{m}$, $\bar{d}$, etc.

2.2. The Basic Assumptions and the results of Abhyankar and Moh

The following assumptions will be made in our main results. We will call them the *Basic Assumptions.*

Let $U_k(\mathbb{K}) := \{\varepsilon \in \mathbb{K}\colon\ \varepsilon^k = 1\}$. Let f be an irreducible and monic element of $\mathbb{K}((X))[Y]$, $\mathbb{K} = \overline{\mathbb{K}}$, $\operatorname{char} \mathbb{K} = 0$, $\deg_Y f = k$. Then, by Newton–Puiseux Theorem,

$$f\bigl(t^k, Y\bigr) = \prod_{\varepsilon \in U_k(\mathbb{K})} \bigl(Y - y(\varepsilon t)\bigr)$$

for some $y(t) \in \mathbb{K}((t))$ of the form

$$y(t) = \sum_{j \in \mathbb{Z}} y_j t^j, \quad \text{where } y_j = 0 \text{ for } j \ll 0.$$

Using the parametrization $(t^k, y(t))$ (or any of its ε-conjugates) we define the *characteristic sequences of* f as the characteristic sequences m, d, s, r and n of $(t^k, y(t))$. Note that since $\gcd(\{k\} \cup \operatorname{Supp}_t y(t)) = 1$, by Property 1 item 5 it follows that $d_{h+1} = 1$.

To formulate the Abhyankar–Moh theorem we need a few definitions; all of them can be found in [4].

Notation 2. Let $\mathbb{K}$ be a field. The symbol θ is to denote any (unspecified) non-zero element of this field.

Notation 3. Let $\mathbb{K}$ be a field. By $\mathbb{K}((t^*))$ we denote the field of Puiseux series in the variable t with coefficients in $\mathbb{K}$. In the sequel, if $z(t) \in \mathbb{K}((t^*))$ then the notation $z(t) = \sum_{q\in\mathbb{Q}} z_q t^q$ (a formal sum) means that there exists $k \in \mathbb{N}$ such that $z_q = 0$ for $kq \notin \mathbb{Z}$ and $z(t^k) \in \mathbb{K}((t))$; so in fact $z(t)$ can be written as $z(t) = \sum_{i\in\mathbb{Z}} z_{i/k} t^{i/k}$ with $z_{i/k} = 0$ for $i \ll 0$.

Definition 2. Let $\mathbb{K}$ be a field, $z(t) \in \mathbb{K}((t^*))$ and $\operatorname{ord}_t z(t) = q$, $q \in \mathbb{Q}$. Then $z(t) = \alpha t^q + \bar{z}(t)$ for some $\alpha = \theta$ and $\bar{z}(t) \in \mathbb{K}((t^*))$, $\operatorname{ord}_t \bar{z}(t) > q$. We define the *leading form* $\operatorname{info}_t z(t)$ *of* $z(t)$ as the term αt^q and the *leading coefficient* $\operatorname{inco}_t z(t)$ *of* $z(t)$ as α. If $z(t) = 0$, we put $\operatorname{info}_t z(t) := 0$ and $\operatorname{inco}_t z(t) := 0$.

Definition 3. Let $\mathbb{K}$ be a field, U – an indeterminate, $Q \in \mathbb{Q}$ and $z(t) \in \mathbb{K}((t^*))$ be of the form $z(t) = \sum_{q\in\mathbb{Q}} z_q t^q$. We say that $z^*(t)$ is a (Q, U)*-deformation of* $z(t)$ if $z^*(t) \in \mathbb{L}^*((t))$, where $\mathbb{L}$ is an extension field of $\mathbb{K}(U)$, and

$$\operatorname{info}_t\left(z^*(t) - \sum_{q<Q} z_q t^q\right) = U \cdot t^Q.$$

Remark 4. The definition of a deformation in [4] is slightly different (cf. [4, Definition (7.14)]), because it allows only deformations on 'characteristic places'.

The most important properties of approximate roots of characteristic degrees, found by Abhyankar and Moh, can be summarized as follows.

Theorem 2. *Under the Basic Assumptions, let* $l = d_i$ *for some* $1 \leqslant i \leqslant h+1$. *Then*:

1. $\sqrt[l]{f}$ *is an irreducible element of* $\mathbb{K}((X))[Y]$,
2. *if* $2 \leqslant i$, *then for every Puiseux root* $z(t) \in \mathbb{K}((t^*))$ *of the polynomial* $\sqrt[l]{f}$ *and every* $\sigma \in U_k(\mathbb{K})$,
$$\operatorname{ord}_t\big(y(\sigma t) - z\big(t^k\big)\big) \leqslant m_i,$$
3. *if* $2 \leqslant i$, *then for every* $\varepsilon \in U_k(\mathbb{K})$ *there exists a Puiseux root* $z(t)$ *of the polynomial* $\sqrt[l]{f}$ *such that*
$$\operatorname{ord}_t\big(y(\varepsilon t) - z\big(t^k\big)\big) = m_i,$$
4. *if* $2 \leqslant i$, *then for every Puiseux root* $z(t)$ *of the polynomial* $\sqrt[l]{f}$ *there exists* $\varepsilon \in U_k(\mathbb{K})$ *such that*
$$\operatorname{ord}_t\big(y(\varepsilon t) - z\big(t^k\big)\big) = m_i,$$
5. *if* $2 \leqslant i$, *then*
$$\operatorname{ord}_t\big(\sqrt[l]{f}\big(t^k, y(t)\big)\big) = r_i,$$

6. *if* $2 \leqslant i \leqslant h$, *then for every* (m_i, U)*-deformation* $y^*(t)$ *of* $y(t)$ *it is*

$$\operatorname{info}_t\left(\sqrt[l]{f}\left(t^k, y^*(t)\right)\right) = \cancel{o}Ut^{r_i}.$$

Proof. The items 1 and 3 are the content of [4, Theorem (13.2)]. The item 5 is proved in [4, Theorem (8.2)], and the item 6 – in [4, Theorem (7.19)]. It remains to prove the items 4 and 2. Let, then, $i \geqslant 2$ and let us consider any Puiseux root $z(t)$ of $\sqrt[l]{f}$. Define $w(t) := z(t^{\frac{k}{l}})$. Then $\sqrt[l]{f}(t^{\frac{k}{l}}, w(t)) = 0$. Since $\sqrt[l]{f}$ is irreducible and $\deg_Y \sqrt[l]{f} = \frac{k}{l}$, then from the Newton–Puiseux theorem it follows that $w(t) \in \mathbb{K}((t))$. Let, according to item 3, $z_0(t)$ be such a Puiseux root of $\sqrt[l]{f}$ that

$$\operatorname{ord}_t\left(y(t) - z_0\left(t^k\right)\right) = m_i.$$

Like above, $z_0(t^{\frac{k}{l}}) \in \mathbb{K}((t))$. Again by Newton–Puiseux theorem, there exists $\varepsilon_0 \in U_{\frac{k}{l}}(\mathbb{K})$ such that $w(\varepsilon_0 t) = z_0(t^{\frac{k}{l}})$. Hence

$$\operatorname{ord}_t\left(y(t) - w\left(\varepsilon_0 t^l\right)\right) = m_i$$

that is

$$\operatorname{ord}_t\left(y\left(\varepsilon_0^{-\frac{1}{l}} t\right) - z\left(t^k\right)\right) = m_i.$$

Thus, the item 4 is proved.

As for item 2, note that the relation $\operatorname{ord}_t(y(\sigma t) - z(t^k)) > m_i$, for some Puiseux root $z(t) \in \mathbb{K}((t^*))$ of $\sqrt[l]{f}$ and some $\sigma \in U_k(\mathbb{K})$, implies that $z(t^{\frac{k}{l}})$ has a non-zero (equal to $y_{m_i}\sigma^{m_i}$) coefficient by $t^{\frac{m_i}{l}}$. Since on the one hand $z(t^{\frac{k}{l}}) \in \mathbb{K}((t))$ and on the other – $\frac{m_i}{l} = \frac{m_i}{d_i} \notin \mathbb{Z}$ by the definition of m_i, this is absurd. $\square$

Definition 4. Under the Basic Assumptions, a positive divisor l of k such that $l \in \{d_1, \ldots, d_{h+1}\}$ will be called a *characteristic divisor of* k (*with regard to* f). A positive divisor l of k that is not characteristic will be called a *non-characteristic divisor of* k (*with regard to* f).

In [6] we've examined non-characteristic approximate roots and we've proved that, in the above theorem: in general Property 1 is not true while Properties 2 and 4 partly are – in the form of greater-or-equal-inequalities (Theorems 1 and 3 of [6]), that happen to be equalities in some special case (Theorems 2 and 3 of [6]). In the present work we improve those results, obtaining almost full analogue of Theorem 2 (in the case of $\operatorname{char}\mathbb{K} = 0$; see Remark 7 for directions for the general case).

For the current purpose, we cite only the following theorem, which is a combination of Theorems 1 and 3 of [6].

Theorem 3. *Under the Basic Assumptions, let* l *be a non-characteristic divisor of* k *and let* $i := \max\{1 \leqslant j \leqslant h+1\colon l | d_j\}$. *Then*:

1. *for every Puiseux root* $z(t)$ *of the polynomial* $\sqrt[l]{f}$ *there exists* $\varepsilon \in U_k(\mathbb{K})$ *such that*

$$\operatorname{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) \geqslant m_i,$$

2.
$$\operatorname{ord}_t\left(\sqrt[l]{f}\left(t^k, y(t)\right)\right) \geqslant r_i \frac{d_i}{l}.$$

3. Auxiliary results

Some of the facts stated here are simple and well known. Nevertheless they are crucial for the prove of the main theorem, so we decided to include them in the work.

We start with the following theorem, which is stated (without a proof) in Section 2 of [5].

Theorem 4 *(Newton's polygon method). Let $\mathbb{K}$ be a field, $\operatorname{char}\mathbb{K} = 0$, and let g be a monic element of $\mathbb{K}((X))[Y]$ that splits into linear factors $Y - z^j(X)$, where $z^j(X) \in \mathbb{K}((X^*))$ for $1 \leqslant j \leqslant \deg_Y g$. In another words, let*

$$g(X,Y) = \prod_{1 \leqslant j \leqslant \deg_Y g} \big(Y - z^j(X)\big). \tag{3.1}$$

Let us consider any $u(t) := \sum_{q \leqslant Q} u_q t^q \in \mathbb{K}((t^*))$, where $Q \in \mathbb{Q}$. Then the following two conditions are equivalent:

(i) *there exists $1 \leqslant j_0 \leqslant \deg_Y g$ such that $\operatorname{ord}_t(u(t) - z^{j_0}(t)) > Q$,*
(ii) *the polynomial $h := \operatorname{inco}_t g(t, u(t) + Ut^Q) \in \mathbb{K}[U]$ is not constant and one of its roots is $U = 0$.*

What is more, if $U = 0$ has multiplicity $l > 0$ as a root of h, then there exist exactly l different indices $j_1, \dots, j_l \in \{1, \dots, \deg_Y g\}$ for which $\operatorname{ord}_t(u(t) - z^{j_i}(t)) > Q$, for $i = 1, \dots, l$.

Proof. (i) $\Rightarrow$ (ii). If there exists $1 \leqslant j_0 \leqslant \deg_Y g$ such that $\operatorname{ord}_t(u(t) - z^{j_0}(t)) > Q$, then $\operatorname{inco}_t(u(t) + Ut^Q - z^{j_0}(t)) = U$. Hence and from (3.1), $h = Uh_1$, for some non-zero $h_1 \in \mathbb{K}[U]$. This gives (ii).

$\sim$ (i) $\Rightarrow$ $\sim$ (ii). If for every $1 \leqslant j \leqslant \deg_Y g$ it is $q^j := \operatorname{ord}_t(u(t) - z^j(t)) \leqslant Q$, then

$$\operatorname{inco}_t\big(u(t) + Ut^Q - z^j(t)\big) = \begin{cases} \theta, & \text{if } q^j < Q \\ U + \theta, & \text{if } q^j = Q \end{cases} = \delta_Q^{q^j} U + \theta,$$

where δ is the Kronecker delta. Hence

$$h = \prod_{1 \leqslant j \leqslant \deg_Y g} \big(\delta_Q^{q^j} U + \theta\big),$$

which means that the polynomial h has no roots equal to zero, that is $\sim$ (ii).

The last assertion follows by a careful examination of the above reasoning. For, let $A := \{1 \leqslant j \leqslant \deg_Y g\colon \operatorname{ord}_t(u(t) - z^j(t)) > Q\}$ and $B := \{1 \leqslant j \leqslant \deg_Y g\colon \operatorname{ord}_t(u(t) - z^j(t)) \leqslant Q\}$. Then, like before,

$$\operatorname{inco}_t\left(\prod_{j \in A}\big(u(t) + Ut^Q - z^j(t)\big)\right) = \prod_{j \in A} U = U^{\operatorname{card} A}$$

and

$$\operatorname{inco}_t\left(\prod_{j \in B}\big(u(t) + Ut^Q - z^j(t)\big)\right) = \prod_{j \in B}\big(\delta_Q^{q^j} U + \theta\big),$$

where again $q^j := \operatorname{ord}_t(u(t) - z^j(t))$ for $j \in B$. Therefore, together we get

$$h = U^{\operatorname{card} A} \prod_{j \in B}\big(\delta_Q^{q^j} U + \theta\big),$$

which means that the polynomial h has a zero of exactly $\operatorname{card} A$ multiplicity, at zero. $\square$

Remark 5. The above theorem and the next two results are also valid (along with their proofs) in the case $\operatorname{char}\mathbb{K} =: p \neq 0$ if one replaces the Puiseux series with Kedlaya's generalized series (cf. [8]) and assumes that $k \not\equiv 0 \pmod p$ (see below).

As an easy consequence of Theorem 4, we prove the following.

Property 3. *Let $\mathbb{K}$ be a field, $\mathbb{K} = \overline{\mathbb{K}}$, $\operatorname{char}\mathbb{K} = 0$ and let there be given a parametrization $(t^k, w(t))$ of the form $w(t) := \sum_{j<a} w_j t^j$, where $a \in \mathbb{Z}$. Put $k_1 := \gcd(\{k\} \cup \operatorname{Supp} w(t))$. Then:*

1. $$\operatorname{card}\bigl\{w(\varepsilon t)\colon\ \varepsilon \in U_k(\mathbb{K})\bigr\} = \frac{k}{k_1},$$
2. *if U is an indeterminate and for some $g \in \mathbb{K}((X))[Y]$ it is*
$$\operatorname{inco}_t g\bigl(t^k, w(t) + Ut^a\bigr) = \theta P(U),$$
where $\theta \in \mathbb{K}$, $P \in \mathbb{K}[U]$, and $\deg_U P =: l > 0$, then to every $\varepsilon \in U_k(\mathbb{K})$ there exist exactly l (counting multiplicities) roots of $g(t^k, Y)$ of the form
$$w(\varepsilon t) + \text{terms of order } \geqslant a. \tag{3.2}$$

Therefore
$$\deg_Y g \geqslant \frac{k}{k_1} l.$$

What is more, if every root of the polynomial $g(t^k, Y)$ is of the form (3.2) for a suitable $\varepsilon \in U_k(\mathbb{K})$, then $\deg_Y g = \frac{k}{k_1} l$.

Proof. Concerning item 1. Notice that for any $\varepsilon_1, \varepsilon_2 \in U_k(\mathbb{K})$ the following equivalences take place:

$$\bigl(w(\varepsilon_1 t) = w(\varepsilon_2 t)\bigr) \quad\Leftrightarrow\quad \forall_{j \in \operatorname{Supp}_t w(t) \cup \{k\}} \bigl(\varepsilon_1^j = \varepsilon_2^j\bigr) \quad\Leftrightarrow\quad \forall_{j \in \operatorname{Supp}_t w(t) \cup \{k\}} \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^j = 1$$
$$\Leftrightarrow\quad \left(\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{\gcd(\{k\} \cup \operatorname{Supp} w(t))} = 1\right) \quad\Leftrightarrow\quad \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{k_1} = 1 \quad\Leftrightarrow\quad \bigl(\varepsilon_1^{k_1} = \varepsilon_2^{k_1}\bigr).$$

This means that

$$\operatorname{card}\bigl\{w(\varepsilon t)\colon\ \varepsilon \in U_k(\mathbb{K})\bigr\} = \operatorname{card}\bigl\{\varepsilon^{k_1}\colon\ \varepsilon \in U_k(\mathbb{K})\bigr\} = \frac{k}{k_1}. \tag{3.3}$$

Concerning item 2. We can assume that g is monic. Fix $\varepsilon_0 \in U_k(\mathbb{K})$. From the assumption,

$$\operatorname{inco}_t g\bigl(t^k, w(\varepsilon_0 t) + U(\varepsilon_0 t)^a\bigr) = \theta P(U).$$

Let $x \in \mathbb{K}$ be any root of P and let $i(x)$ be its multiplicity. Substituting $\varepsilon_0^{-a} U + x$ for U in the above equality, we get

$$\operatorname{inco}_t g\bigl(t^k, w(\varepsilon_0 t) + \bigl(U + \varepsilon_0^a x\bigr)t^a\bigr) = \theta P\bigl(\varepsilon_0^{-a} U + x\bigr) = \theta U^{i(x)} H_x$$

or

$$\mathrm{inco}_t\, g\big(t, w\big(\varepsilon_0 t^{\frac{1}{k}}\big) + \big(U + \varepsilon_0^a x\big)t^{\frac{a}{k}}\big) = \theta P\big(\varepsilon_0^{-a}U + x\big) = \theta U^{i(x)} H_x,$$

for some polynomial $H_x \in \mathbb{K}[U]$ such that $H_x(0) \neq 0$. Using Theorem 4 we conclude that there exist exactly $i(x)$ roots $z_{x,1}(t), \ldots, z_{x,i(x)}(t) \in \mathbb{K}((t^*))$ of the polynomial $g(t, Y)$ (written according to their multiplicities) such that

$$\mathrm{ord}_t\big(w(\varepsilon_0 t) + \varepsilon_0^a x t^a - z_{x,j}\big(t^k\big)\big) > a \quad \text{for } j = 1, \ldots, i(x).$$

From the last formula it follows that if x_1, x_2 are different roots of P, then $z_{x_1,j_1}(t) \neq z_{x_2,j_2}(t)$ for $j_1 = 1, \ldots, i(x_1)$, $j_2 = 1, \ldots, i(x_2)$. Hence to the given ε_0 there correspond exactly $\deg_U P = l$ roots of the polynomial $g(t^k, Y)$ and all of them are of the form $w(\varepsilon_0 t) +$ terms of order $\geqslant a$.

Since ε_0 was an arbitrarily fixed element of $U_k(\mathbb{K})$, then from (3.3) we get that $g(t^k, Y)$ has at least $\frac{k}{k_1} l$ roots, hence that $\deg_Y g \geqslant \frac{k}{k_1} l$. What is more, if every root of the polynomial $g(t^k, Y)$ is of the form $w(\varepsilon t) +$ terms of order $\geqslant a$ for some $\varepsilon \in U_k(\mathbb{K})$, then – again by Theorem 4 – $\deg_Y g \leqslant \frac{k}{k_1} l$, which gives the required equality. □

Remark 6. Although Property 3 is simple, it can be treated as a generalization of the Main Lemma 1 of [4]. Indeed, using this property, Newton's Polygon Method and Proposition 1 below, one can give a one-liner proof of the Main Lemma 1.

Although for aesthetic reasons we stated Theorem 4 in its most natural form, it is general enough to be used with the deformations of Definition 3, too. Such possibility is a consequence of the following observation.

Proposition 1. *Let $\mathbb{K}$ be a field,* $\mathrm{char}\,\mathbb{K} = 0$*, let $g \in \mathbb{K}((X))[Y]$ and let $u(t) := \sum_{q \leqslant Q} u_q t^q \in \mathbb{K}((t^*))$. If*

$$\mathrm{info}_t\big(g\big(t^k, u(t) + Ut^Q\big)\big) = \theta P \cdot t^M, \tag{3.4}$$

where $k \in \mathbb{N}$, $\theta \in \mathbb{K}_0$, $\mathbb{K}_0$ – a subfield of $\mathbb{K}$, $P \in \mathbb{K}[U]$, $P \neq 0$, $M \in \mathbb{Q}$, and if $u^*(t)$ is any (Q, U)-deformation of $u(t)$, then

$$\mathrm{info}_t\big(g\big(t^k, u^*(t)\big)\big) = \theta P(U - u_q) \cdot t^M \quad \textit{with } \theta \in \mathbb{K}_0.$$

In particular, if $u_Q = 0$ then

$$\mathrm{info}_t\big(g\big(t^k, u^*(t)\big)\big) = \mathrm{info}_t\big(g\big(t^k, u(t) + Ut^Q\big)\big).$$

Proof. From the definition of a deformation it follows that $u^*(t) = \sum_{q<Q} u_q t^q + Ut^Q + \bar{u}(t)$, where $\mathrm{ord}_t\, \bar{u}(t) > Q$. Assume first that $u_Q = 0$, which gives

$$u^*(t) = \sum_{q \leqslant Q} u_q t^q + Ut^Q + \bar{u}(t) = u(t) + \left(U + \frac{\bar{u}(t)}{t^Q}\right)t^Q \tag{3.5}$$

where $\mathrm{ord}_t \frac{\bar{u}(t)}{t^Q} > 0$. It means that for every $j \in \mathbb{N}_0$ and $q \in \mathbb{Q}$ it is $\mathrm{info}_t((U + \frac{\bar{u}(t)}{t^Q})^j t^q) = U^j t^q = \mathrm{info}_t(U^j t^q)$. Hence also for any $H \in \mathbb{K}[U]$ and $q \in \mathbb{Q}$,

$$\operatorname{info}_t\left(H\left(U+\frac{\bar{u}(t)}{t^Q}\right)t^q\right)=\operatorname{info}_t\big(H(U)t^q\big). \tag{3.6}$$

If we write

$$g\big(t^k,u(t)+Ut^Q\big)=\theta P(U)\cdot t^M+\sum_{q>M}P_q(U)\cdot t^q,$$

where $P_q(U)\in\mathbb{K}[U]$, then, substituting $U+\frac{\bar{u}(t)}{t^Q}$ for U in this equality, we get by (3.5)

$$g\big(t^k,u^*(t)\big)=\theta P\left(U+\frac{\bar{u}(t)}{t^Q}\right)\cdot t^M+\sum_{q>M}P_q\left(U+\frac{\bar{u}(t)}{t^Q}\right)\cdot t^q$$

which by (3.6) means that

$$\operatorname{info}_t\big(g\big(t^k,u^*(t)\big)\big)=\operatorname{info}_t\left(\theta P\left(U+\frac{\bar{u}(t)}{t^Q}\right)\cdot t^M\right)=\theta P(U)\cdot t^M=\operatorname{info}_t\big(g\big(t^k,u(t)+Ut^Q\big)\big).$$

For $u_Q\neq 0$ it is $u^*(t)=u(t)+(U-u_Q+\frac{\bar{u}(t)}{t^Q})t^Q$, where $\operatorname{ord}_t\frac{\bar{u}(t)}{t^Q}>0$. Similarly as above, $\operatorname{info}_t(H(U-u_Q+\frac{\bar{u}(t)}{t^Q})t^q)=H(U-u_Q)t^q$, for every $H\in\mathbb{K}[U]$ and $q\in\mathbb{Q}$. In this case, the substitution of $U-u_Q+\frac{\bar{u}(t)}{t^Q}$ for U in (3.4) leads to

$$\operatorname{info}_t\big(g\big(t^k,u^*(t)\big)\big)=\theta P(U-u_Q)\cdot t^M.$$

This ends the proof. □

In the following the symbol $\lfloor\cdot\rfloor$ denotes the integer-part function and the symbol $\{\cdot\}$ – the fractional-part function.

Lemma 2. *Let $q,Q\in\mathbb{Q}$, $q\geqslant 1$ and $P:=\sum_{0\leqslant e\leqslant Q}\binom{q}{e}(-1)^e\cdot U^{\lfloor Q\rfloor-e}\in\mathbb{Q}[U]$. Then*

$$U(U-1)P'(U)=\lfloor Q\rfloor(U-1)P(U)+qP(U)+(-1)^{\lfloor Q\rfloor+1}q\binom{q-1}{\lfloor Q\rfloor}.$$

Proof. A calculation. □

4. Main results

We start with the following corollary from Theorem 3.

Corollary 1. *Under the assumptions of Theorem 3, let $y^*(t)$ be an (m_i,U)-deformation of $y(t)$. Then*

$$\deg_U\big(\operatorname{inco}_t\big(\sqrt[l]{f}\big(t^k,y^*(t)\big)\big)\big)=\frac{d_i}{l}.$$

Proof. Let us consider $\bar{y}(t):=\sum_{j<m_i}y_jt^j$. Then by Property 1 item 5, $\gcd(\{k\}\cup\operatorname{Supp}\bar{y}(t))=d_i$ and by Property 3,

$$\operatorname{card}\{\bar{y}(\varepsilon t)\colon \varepsilon \in U_k(\mathbb{K})\} = \frac{k}{d_i}. \tag{4.1}$$

Let $z_0(t)$ be any of the Puiseux roots of the polynomial $\sqrt[l]{f}$. From Theorem 3 it follows that $\bar{y}(\varepsilon_0 t) + Ut^{m_i}$, for a suitable $\varepsilon_0 \in U_k(\mathbb{K})$, is an (m_i, U)-deformation of $z_0(t^k)$. Put

$$h_0(U) := \operatorname{inco}_t \sqrt[l]{f}\big(t^k, \bar{y}(\varepsilon_0 t) + Ut^{m_i}\big). \tag{4.2}$$

By Theorem 4 it is $\deg_U h_0 > 0$, and since according to Theorem 3 every root of the polynomial $\sqrt[l]{f}(t^k, Y)$ is of the form

$$\bar{y}(\varepsilon t) + \text{terms of order} \geqslant m_i$$

for a suitable $\varepsilon \in U_k(\mathbb{K})$, then by Property 3 and (4.1) we conclude that

$$\frac{k}{d_i} \deg_U h_0 = \deg_Y \sqrt[l]{f} = \frac{k}{l}$$

and so $\deg_U h_0 = \frac{d_i}{l}$. From equality (4.2) we easily deduce that

$$\deg_U\big(\operatorname{inco}_t\big(\sqrt[l]{f}\big(t^k, \bar{y}(t) + Ut^{m_i}\big)\big)\big) = \frac{d_i}{l},$$

and using Proposition 1 we finish the proof. □

Now we prove the main theorem.

Theorem 5. *Under the Basic Assumptions, let l be a non-characteristic divisor of k. Define $i := \max\{1 \leqslant j \leqslant h+1\colon l|d_j\}$, $a := \lfloor \frac{d_{i+1}}{l} \rfloor$ and $b := n_i\{\frac{d_{i+1}}{l}\}$. Then:*

1. *if $d_{i+1} > l$ then $\sqrt[l]{f}$ is reducible in $\mathbb{K}((X))[Y]$ and*

$$\sqrt[l]{f} = f_1 \cdots f_a \cdot g,$$

where $f_1, \ldots, f_a, g \in \mathbb{K}((X))[Y]$ are monic, $f_1, \ldots, f_a$ are irreducible in $\mathbb{K}((X))[Y]$ and pairwise different, $\deg_Y f_j = \frac{k}{d_{i+1}}$ for $j = 1, \ldots, a$, $\deg_Y g = \frac{k}{d_i} b$; what is more, for every (m_i, U)-deformation $y^*(t)$ of $y(t)$,

$$\operatorname{info}_t f_j\big(t^k, y^*(t)\big) = \cancel{o}\big(U^{n_i} - (\alpha_j y_{m_i})^{n_i}\big) \cdot t^{r_i n_i} \quad \text{for } j = 1, \ldots, a \tag{4.3}$$

and

$$\operatorname{info}_t g\big(t^k, y^*(t)\big) = \cancel{o} U^b \cdot t^{r_i b},$$

where $\alpha_1, \ldots, \alpha_a \in \mathbb{K}^* \setminus U_{n_i}(\mathbb{K})$ and $\alpha_1^{n_i}, \ldots, \alpha_a^{n_i}$ are pairwise different,

2. *for every Puiseux root $z(t) \in \mathbb{K}((t^*))$ of the polynomial $\sqrt[l]{f}$ and every $\sigma \in U_k(\mathbb{K})$,*

$$\operatorname{ord}_t\big(y(\sigma t) - z(t^k)\big) \leqslant m_i,$$

3. *for every $\varepsilon \in U_k(\mathbb{K})$ there exists a Puiseux root $z(t)$ of the polynomial $\sqrt[l]{f}$ such that*

$$\operatorname{ord}_t\bigl(y(\varepsilon t) - z\bigl(t^k\bigr)\bigr) = m_i,$$

4. *for every Puiseux root $z(t)$ of the polynomial $\sqrt[l]{f}$ there exists $\varepsilon \in U_k(\mathbb{K})$ such that*

$$\operatorname{ord}_t\bigl(y(\varepsilon t) - z\bigl(t^k\bigr)\bigr) = m_i,$$

5. $$\operatorname{ord}_t\bigl(\sqrt[l]{f}\bigl(t^k, y(t)\bigr)\bigr) = \frac{s_i}{l} = r_i\frac{d_i}{l},$$

6. *for every (m_i, U)-deformation $y^*(t)$ of $y(t)$ it is*

$$\begin{aligned}
\operatorname{info}_t \sqrt[l]{f}\bigl(t^k, y^*(t)\bigr) &= \theta \sqrt[l]{\bigl(U^{n_i} - y_{m_i}^{n_i}\bigr)^{d_{i+1}}} \cdot t^{\frac{s_i}{l}} \\
&= \theta \sum_{0 \leqslant e \leqslant a} \binom{\frac{d_{i+1}}{l}}{e} (-1)^e y_{m_i}^{n_i e} U^{n_i a - n_i e} U^b \cdot t^{\frac{s_i}{l}} \qquad (4.4) \\
&= \theta \bigl(U^{n_i} - (\alpha_1 y_{m_i})^{n_i}\bigr) \cdots \bigl(U^{n_i} - (\alpha_a y_{m_i})^{n_i}\bigr) U^b \cdot t^{\frac{s_i}{l}}, \qquad (4.5)
\end{aligned}$$

where $\theta \in \mathbb{K}$, and $\alpha_1, \ldots, \alpha_a$ are defined as above.

Proof. We will consider the items of theorem in the following order: 6, 2, 5, 4, 3, 1.

Concerning item 6. Using Proposition 1 we can assume that $y^*(t) = \sum_{e<m_i} y_e t^e + U t^{m_i}$. By [4, Lemma (7.16)] it is

$$\operatorname{info}_t f\bigl(t^k, y^*(t)\bigr) = \alpha\bigl(U^{n_i} - y_{m_i}^{n_i}\bigr)^{d_{i+1}} t^{s_i}, \quad \text{for some } \alpha \in \mathbb{K}\setminus\{0\}, \qquad (4.6)$$

and for $P(U) := \operatorname{inco}_t \sqrt[l]{f}(t^k, y^*(t))$ we have, by Corollary 1,

$$\deg_U P(U) = \frac{d_i}{l}. \qquad (4.7)$$

According to the Definition 1,

$$f = \bigl(\sqrt[l]{f}\bigr)^l + H, \qquad (4.8)$$

where $H \in \mathbb{K}((X))[Y]$ and $\deg_Y H < k - \frac{k}{l}$. Let $\bar{k} := \deg_U(\operatorname{inco}_t H(t^k, y^*(t)))$. Applying Property 3 to the parametrization $(t^k, \sum_{e<m_i} y_e t^e)$ we get

$$\frac{k}{d_i}\bar{k} \leqslant \deg_Y H < k - \frac{k}{l},$$

which leads to

$$\bar{k} < d_i - \frac{d_i}{l}. \qquad (4.9)$$

Since by (4.6) it is $\deg_U(\operatorname{inco}_t f(t^k, y^*(t))) = d_i > \bar{k}$ and at the same time $d_i = \deg_U P^l(U)$, it follows from (4.8) that

$$\operatorname{ord}_t\bigl(\sqrt[l]{f}\bigr)^l\bigl(t^k, y^*(t)\bigr) = \operatorname{ord}_t f\bigl(t^k, y^*(t)\bigr) = s_i \tag{4.10}$$

or

$$\operatorname{ord}_t \sqrt[l]{f}\bigl(t^k, y^*(t)\bigr) = \frac{s_i}{l} = r_i\frac{d_i}{l}. \tag{4.11}$$

Since by definition $d_{i+1} \not\equiv 0 \pmod{l}$ and $y_{m_i} \neq 0$, it cannot be $\alpha(U^{n_i} - y_{m_i}^{n_i})^{d_{i+1}} = P(U)^l$, and so it is

$$\operatorname{ord}_t H\bigl(t^k, y^*(t)\bigr) = s_i. \tag{4.12}$$

Combining (4.6)–(4.12) we get

$$\alpha\bigl(U^{n_i} - y_{m_i}^{n_i}\bigr)^{d_{i+1}} = P(U)^l + \left(\text{a poly of degree} < d_i - \frac{d_i}{l}\right). \tag{4.13}$$

Putting $P_1 := \alpha^{-1/l} y_{m_i}^{-d_i/l} \cdot P(U \cdot y_{m_i})$, substituting $U \cdot y_{m_i}$ for U into the above equality and simplifying the coefficients of the highest powers of U, we get

$$\bigl(U^{n_i} - 1\bigr)^{d_{i+1}} = P_1(U)^l + \left(\text{a poly of degree} < d_i - \frac{d_i}{l}\right).$$

In the above equality one can choose P_1 to be monic. It follows, then, that $P_1(U)$ is the approximate l-th root of the polynomial $(U^{n_i} - 1)^{d_{i+1}}$. Making use of the power series expansion of $(1 - U^{n_i})^{d_{i+1}/l}$, Lemma 1 and the uniqueness of an approximate root, we conclude that

$$P_1(U) = \sum_{0 \leqslant n_i e \leqslant \frac{d_i}{l}} \binom{\frac{d_{i+1}}{l}}{e}(-1)^e \cdot U^{\frac{d_i}{l} - n_i e} = \sum_{0 \leqslant e \leqslant \frac{d_{i+1}}{l}} \binom{\frac{d_{i+1}}{l}}{e}(-1)^e \cdot U^{\frac{d_i}{l} - n_i e}.$$

From the above equality we get

$$P_1(U) = \sum_{0 \leqslant e \leqslant a} \binom{\frac{d_{i+1}}{l}}{e}(-1)^e \cdot U^{\frac{d_i}{l} - n_i e} = U^{\frac{d_i}{l} - n_i a} \cdot \sum_{0 \leqslant e \leqslant a} \binom{\frac{d_{i+1}}{l}}{e}(-1)^e \cdot U^{n_i a - n_i e}.$$

But $\frac{d_i}{l} - n_i a = n_i(\frac{d_{i+1}}{l} - \lfloor \frac{d_{i+1}}{l} \rfloor) = n_i\{\frac{d_{i+1}}{l}\} = b$, so, rewriting the above equality in terms of $P(U)$ and using (4.11), (4.13), we arrive at the first two equalities of (4.4).

In the rest of the reasoning we assume that $\frac{d_{i+1}}{l} > 1$ since there is nothing more to prove in the opposite case. Let as consider the polynomial $P_2 \in \mathbb{K}[U]$ such that $P_2(U^{n_i}) = \frac{P_1(U)}{U^b} = \theta \frac{P(U \cdot y_{m_i})}{U^b}$. In other words,

$$P_2 := \sum_{0 \leqslant e \leqslant a} \binom{\frac{d_{i+1}}{l}}{e}(-1)^e \cdot U^{a-e} = \bigl(U - \alpha_1^{n_i}\bigr) \cdots \bigl(U - \alpha_a^{n_i}\bigr),$$

for some $\alpha_1, \ldots, \alpha_a \in \mathbb{K}$. In order to finish the proof of (4.5), we thus need to check that $\alpha_1^{n_i}, \ldots, \alpha_a^{n_i} \notin \{0, 1\}$ and that they are pairwise different, which in turn reduces to checking if P_2 has no root equal to 0 or 1 and if it is square-free. According to Lemma 2 it is

$$U(U-1)P_2'(U)=a(U-1)P_2(U)+\frac{d_{i+1}}{l}P_2(U)+(-1)^{a+1}\frac{d_{i+1}}{l}\binom{\frac{d_{i+1}}{l}-1}{a}.$$

Since $\frac{d_{i+1}}{l}\notin\mathbb{Z}$ by the definition of i, $\binom{\frac{d_{i+1}}{l}-1}{a}\neq 0$ and from the above equality we conclude that $P_2(0)\neq 0$, $P_2(1)\neq 0$ and that P_2 has only simple roots. This ends the proof of the item 6.

Concerning items 2 and 5. By item 6, $\alpha_1,\dots,\alpha_a\in\mathbb{K}\backslash U_{n_i}(\mathbb{K})$ which is equivalent to any of the following conditions:

$$\big(\mathrm{inco}_t\sqrt[l]{f}\big(t^k,y^*(t)\big)\big)\big|_{U=y_{m_i}}\neq 0,\quad \text{for every } (m_i,U)\text{-deformation } y^*(t) \text{ of } y(t),$$

$$⁎\qquad \left(\mathrm{inco}_t\sqrt[l]{f}\left(t^k,\sum_{e\leqslant m_i}y_et^e+Ut^{m_i}\right)\right)\Bigg|_{U=0}\neq 0\quad \text{(Proposition 1)},$$

$$⁎⁎\qquad \big(\mathrm{inco}_t\sqrt[l]{f}\big(t^k,y(t)+Ut^{m_i}\big)\big)\big|_{U=0}\neq 0\quad \text{(Proposition 1)}.$$

By Newton's Polygon Method, the inequality ⁎ means exactly that any Puiseux root $z(t)$ of $\sqrt[l]{f}$ fulfills

$$\mathrm{ord}_t\left(\sum_{e\leqslant m_i}y_et^e-z\big(t^k\big)\right)\leqslant m_i,$$

so equivalently

$$\mathrm{ord}_t\big(y(t)-z\big(t^k\big)\big)\leqslant m_i.$$

From the fact that the series $y(t)$ in Basic Assumptions is an arbitrarily fixed root of $f(t^k,Y)$ and by Newton–Puiseux Theorem, it follows that the item 2 is proved.

The condition ⁎⁎ is equivalent to

$$\mathrm{ord}_t\sqrt[l]{f}\big(t^k,y(t)+Ut^{m_i}\big)=\mathrm{ord}_t\sqrt[l]{f}\big(t^k,y(t)\big).$$

Since $\mathrm{ord}_t\sqrt[l]{f}(t^k,y(t)+Ut^{m_i})=\mathrm{ord}_t\sqrt[l]{f}(t^k,y(t)+(U-y_{m_i})t^{m_i})=r_i\frac{d_i}{l}$, where the last equality follows from (4.4), then we have proved the item 5 of the theorem.

The item 4 follows from the item 2 proved above and the item 1 of Theorem 3.

The item 3 is a consequence of the formula (4.4) applied to $y^*(t)=\sum_{e<m_i}y_et^e+Ut^{m_i}$ and Property 3. Indeed, we conclude that given any $\varepsilon\in U_k(\mathbb{K})$, the polynomial $\sqrt[l]{f}(t^k,Y)$ has a root $z(t^k)$ of the form $\sum_{e<m_i}y_e(\varepsilon t)^e+$terms of order $\geqslant m_i$ and so $\mathrm{ord}_t(y(\varepsilon t)-z(t^k))\geqslant m_i$, what together with the item 2 proved above gives the equality.

Concerning item 1. Assume that $d_{i+1}>l$. Were $\sqrt[l]{f}$ irreducible in $\mathbb{K}((X))[Y]$, then from Newton–Puiseux Theorem it would follow that all the Puiseux roots of $\sqrt[l]{f}$ have simultaneously zero or simultaneously non-zero coefficient beside $t^{\frac{m_i}{k}}$ in their expansions. But $a,b>0$ (since $d_{i+1}>l$) and this by Theorem 4 and the item 6 proved above means that, among the aforementioned roots, there exist both kinds: such with zero and such with non-zero coefficients by $t^{\frac{m_i}{k}}$. Contradiction.

Let, by Theorem 4 and item 6, $z_1(t),\dots,z_a(t)$ be such Puiseux roots of $\sqrt[l]{f}$ that for any $j\in\{1,\dots,a\}$,

$$z_j\big(t^k\big)=\sum_{e<m_i}y_et^e+\alpha_jy_{m_i}t^{m_i}+\text{h.o.t.},\tag{4.14}$$

where $\alpha_j \in \mathbb{K}^*$ are those of (4.5). Let us define $f_j \in \mathbb{K}((X))[Y]$ as the monic minimal polynomial of $z_j(t)$ over $\mathbb{K}((t))$, for $j = 1, \ldots, a$. Then for $j = 1, \ldots, a$, $f_j(t, z_j(t)) = 0$ and $\sqrt[l]{f}(t, z_j(t)) = 0$, which by the minimality of f_j means that $f_j | \sqrt[l]{f}$ in $\mathbb{K}((X))[Y]$. Let $\deg_Y f_j =: k_j$, $j = 1, \ldots, a$. Since by Newton–Puiseux Theorem $z_j(t^{k_j})$ is a Laurent root of $f_j(t^{k_j}, Y)$, then by (4.14) f_j has the characteristic of the form $(m_0 \frac{k_j}{k}, \ldots, m_i \frac{k_j}{k}, \ldots)$ and the divisor sequence of the form $(d_1 \frac{k_j}{k}, \ldots, d_{i+1} \frac{k_j}{k}, \ldots)$. In particular, $d_{i+1} \frac{k_j}{k} \in \mathbb{Z}$. As a consequence, according to [4, Lemma (7.16)], for $y^*(t) := \sum_{e<m_i} y_e t^e + U t^{m_i}$ it is

$$\operatorname{info}_t f_j\big(t^{k_j}, y^*\big(t^{\frac{k_j}{k}}\big)\big) = \theta\big(U^{n_i} - (\alpha_j y_{m_i})^{n_i}\big)^{d_{i+1} \frac{k_j}{k}} \cdot t^{s_i (\frac{k_j}{k})^2}$$

or

$$\operatorname{info}_t f_j\big(t^k, y^*(t)\big) = \theta\big(U^{n_i} - (\alpha_j y_{m_i})^{n_i}\big)^{d_{i+1} \frac{k_j}{k}} \cdot t^{s_i \frac{k_j}{k}}, \quad \text{for } j = 1, \ldots, a. \tag{4.15}$$

Since $(\alpha_1 y_{m_i})^{n_i}, \ldots, (\alpha_a y_{m_i})^{n_i}$ are pairwise different, (4.15) shows that also $f_1, \ldots, f_a$ are pairwise different and since $f_1, \ldots, f_a$ are irreducible, too, they are pairwise coprime in $\mathbb{K}((X))[Y]$. Therefore $(f_1 \cdots f_a) | \sqrt[l]{f}$ in $\mathbb{K}((X))[Y]$ and

$$\big(\operatorname{inco}_t f_1\big(t^k, y^*(t)\big) \cdots \operatorname{inco}_t f_a\big(t^k, y^*(t)\big)\big) \big| \operatorname{inco}_t \sqrt[l]{f}\big(t^k, y^*(t)\big) \quad \text{in } \mathbb{K}[U]. \tag{4.16}$$

However $\mathbb{K}[U]$ is factorial, so using (4.15) and (4.4) in (4.16) we see that for $j = 1, \ldots, a$ it is

$$d_{i+1} \frac{k_j}{k} = 1, \quad \text{that is} \quad k_j = \frac{k}{d_{i+1}}.$$

By (4.15) this gives (4.3). Putting $g := \frac{\sqrt[l]{f}}{f_1 \cdots f_a} \in \mathbb{K}((X))[Y]$, we get

$$\deg_Y g = \frac{k}{l} - a \frac{k}{d_{i+1}} = \frac{k}{l} - n_i a \frac{k}{d_i} = \frac{k}{l} - \left(\frac{d_i}{l} - b\right) \frac{k}{d_i} = \frac{k}{d_i} b$$

and by (4.5),

$$\operatorname{inco}_t g\big(t^k, y^*(t)\big) = \frac{\operatorname{inco}_t \sqrt[l]{f}(t^k, y^*(t))}{(\operatorname{inco}_t f_1(t^k, y^*(t)) \cdots \operatorname{inco}_t f_a(t^k, y^*(t)))} = \theta U^b.$$

Since $\frac{s_i}{l} - a \cdot (r_i n_i) = \frac{s_i}{l} - r_i(\frac{d_i}{l} - b) = r_i b$, from the definition of g it follows that $\operatorname{ord}_t g(t^k, y^*(t)) = r_i b$. Using Proposition 1, we can see that the proof of the theorem is finished. □

Remark 7. One can prove that if $\operatorname{char} \mathbb{K} =: p \neq 0$ and $k \not\equiv 0 \pmod p$, then the contents of items 2 and 5 of Theorem 5 are equivalent to the inequality $\binom{\frac{d_{i+1}}{l} - 1}{a} \cdot \mathbf{1} \neq 0$ in $\mathbb{K}$ (cf. Remark 2 and Remark 5). The reducibility assertion of item 1 of Theorem 5 is also valid in that case. The details can be found in [9].

From Theorem 5 it follows that the characteristics of the f_j's are all the same – equal to $(\frac{m_0}{d_{i+1}}, \ldots, \frac{m_i}{d_{i+1}})$, which is also the characteristic of $\sqrt[d_{i+1}]{f}$. This allows a possibility of giving a geometric interpretation of the connection between $\sqrt[d_{i+1}]{f}$ and the f_j's in terms of blowing-ups (see the work of Spivakovsky [10]). Note however, that it is not in general possible to say anything about the (full) characteristics of g (and as a consequence – also about the full process of resolution of singularities of $\sqrt[l]{f}$). It is the content of the following example.

Example 1. We will show that unlike in the case of the f_j's, the behavior of g *does not depend only on the characteristic part of the parametrization* $(t^k, y(t))$ *of* f. Namely, we will find two irreducible elements f and $\overline{f}$ of $\mathbb{K}[[X]][Y]$ with the same characteristic and totally different g and $\overline{g}$ (for the same non-characteristic divisor l).

First, consider the parametrization $(t^{72}, y(t)) := (t^{72}, t^{48} + t^{88} + t^{91})$ and let $f \in \mathbb{K}[[X]][Y]$ be the irreducible monic polynomial with the above parametrization. It is seen that the characteristic of f is equal to $(72, 48, 88, 91)$ and the divisor sequence – to $(72, 24, 8, 1)$. Consider the non-characteristic divisor $l = 3$. Then $i = 2$. Using any computer algebra system one can compute $\sqrt[3]{f}$ (cf. Remark 1) and check that $\operatorname{inco}_t \sqrt[3]{f}(t^{72}, t^{48} + \sqrt{2}t^{108} + Zt^{132}) = \theta(10Z - 63\sqrt{2})$. By Theorem 4 and Proposition 1 there exists a Puiseux root $z(t) = t^{\frac{2}{3}} + \sqrt{2}t^{\frac{3}{2}} + \theta t^{\frac{11}{6}} + \cdots$ of $\sqrt[3]{f}(t, Y)$ and by Theorem 5 it follows that $z(t)$ is a root of $g(t, Y)$ and is not a root of any of the f_j's. Let h be the irreducible monic polynomial vanishing on $(t, z(t))$. A simple consequence of Theorem 4 and Property 3 is that $\deg_Y h \geqslant 6 \cdot 1 = 6$. Since h is irreducible, $h|g$ and so $\deg_Y g \geqslant 6$. On the other hand Theorem 5 says that $\deg_Y g = 6$. It follows that $h = g$ and g is *irreducible*.

Note that the sequence of characteristic Puiseux exponents of g 'goes further' than that of the f_j's – it is of the form $(\frac{2}{3}, \frac{3}{2})$ while the f_j's have the exponents equal to $(\frac{2}{3}, \frac{11}{9})$. In another words, it takes more steps to desingularize g than the f_j's (or than $\sqrt[d_{i+1}]{f} = \sqrt[8]{f}$).

In order to construct $\overline{f}$, we will now change $y(t)$ a little. Namely, let $(t^{72}, \overline{y}(t)) := (t^{72}, y(t) + t^{92})$. It is clear that the characteristic of this parametrization is the same as the one of $(t^{72}, y(t))$. As before, consider the irreducible and monic polynomial $\overline{f}$ with the parametrization $(t^{72}, \overline{y}(t))$. One can compute $\sqrt[3]{\overline{f}}$ and then

$$\operatorname{inco}_t \sqrt[3]{\overline{f}}\bigl(t^{72}, t^{48} + Zt^{96}\bigr) = \theta(Z + 18 - \sqrt{10\,914})(Z + 18 + \sqrt{10\,914}). \tag{4.17}$$

Like before, by Theorem 4 and Theorem 5 we conclude that in that case it has to be $\operatorname{inco}_t \overline{g}(t^{72}, t^{48} + Zt^{96}) = \theta \operatorname{inco}_t \sqrt[3]{\overline{f}}(t^{72}, t^{48} + Zt^{96})$. Were $\overline{g}$ irreducible, it would have the characteristic of the form $(6, 4, \ldots)$ and since then $t^4 + Zt^8$ would be a deformation at a non-characteristic place of a parametrization of $\overline{g}$, it should be the case that $\operatorname{inco}_t \overline{g}(t^6, t^4 + Zt^8)$ is a power of a linear polynomial (see [4, Theorem (14.2)]). However, since this inco is equal to the right-hand side of (4.17), it cannot be such a power. The contradiction shows, that $\overline{g}$ is *reducible*. Similarly as in the first part of the example, one can show that it is $\overline{g} = \overline{g}_1 \cdot \overline{g}_2$, where the $\overline{g}_j$'s have the same characteristic, namely $(3, 2)$.

In contrast with the behavior of g above, the characteristic Puiseux exponents of $\overline{g}$ are just the sequence $(\frac{2}{3})$ so $\overline{g}$ desingularizes 'faster' than the $\overline{f}_j$'s (or than $\sqrt[d_{i+1}]{\overline{f}} = \sqrt[8]{\overline{f}}$).

Acknowledgment

I would like to thank the anonymous referee whose remarks inspired the search for Example 1 and led to a more complete exposition.

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