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The Strong Converse Inequality for Bernstein-Kantorovich Operators*

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Abstract—We give a solution to a problem posed by Totik at the 1992 Texas conference concerning the strong converse inequality for approximation by Bernstein-Kantorovich operators. The approximation behaviour of these operators is characterized for $1 \le p \le \infty$ by using an appropriate K-functional which, for $1 < p \le \infty$, is equivalent to a second order modulus and an extra term. Crucial in our approach are estimates for the derivatives of iterated Kantorovich operators.

Keywords—Bernstein-Kantorovich operators, Strong converse inequality, Degree of approximation, K-functional.

1. INTRODUCTION AND MAIN RESULTS

Recently two independent proofs of a strong converse inequality (of type A, according to the classification of Ditzian and Ivanov [1]) were given for the classical Bernstein operator B_n by Knoop and Zhou [2][1] and by Totik [3][2]. As is well known, these operators are defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad f \in C[0,1], \ x \in [0,1],$$

where the fundamental functions are given by

$$p_{n,k}(x) = \frac{n!}{k!(n-k)!}\, x^k (1-x)^{n-k}, \qquad 0 \le k \le n.$$

For convenience, we shall suppose $p_{n,k}(x) = 0$ in case $k < 0$ or $k > n$. The three authors mentioned showed that for some constant $C > 0$ independent of f and n one has

$$C^{-1}\omega_\varphi^2\left(f, n^{-1/2}\right)_\infty \le \|f - B_n f\|_\infty \le C\omega_\varphi^2\left(f, n^{-1/2}\right)_\infty, \qquad \forall f \in C[0,1]. \tag{1.1}$$

Here $\omega_\varphi^2(f,t)_\infty$ denotes the second order modulus of smoothness with weight function $\varphi(x) = (x(1-x))^{1/2}$ (see [4] for details). Moreover, all quantities subscribed by ∞ are taken with respect to the uniform norm in $C[0,1]$.

*The main result was presented at the Second International Conference in Functional Analysis and Approximation Theory held in Acquafredda di Maratea, Italy, September 14–19, 1992.

[1]The first part of this paper will be published in *Constr. Approx.* The second part is published in *Results in Mathematics*, **25** (1994), 300–315.

[2]In fact, in [3] only the strong converse inequality for Szász-Mirakjan operator was proved.

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There is one particular modification of the Bernstein operator for the approximation of L_p functions, $1 \le p \le \infty$, (for $p = \infty$, we will always consider $C[0,1]$ instead of $L_\infty[0,1]$) which has been attracting special interest in the past. This is given by the Kantorovich operators K_n which are obtained if one replaces $f(k/n)$ in the definition of Bernstein operators by

$$(n+1)\int_{k/n+1}^{(k+1)/(n+1)} f(t)\,dt.$$

The operators obtained in this way are thus

$$K_n(f,x) = (n+1)\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt p_{n,k}(x).$$

It is a natural question to ask if there is a strong converse inequality (of type A) for Kantorovich operators as well, and what it should look like. In his survey paper [5] for the 1992 Texas conference proceedings, Totik asked if it were possible to have

$$\omega_\varphi^2\left(f, n^{-1/2}\right)_p \le C\|f - K_n f\|_p. \tag{1.2}$$

Obviously, to have an upper estimate like (1.1) is impossible as the simple example $f(x) = x$, with $\omega_\varphi^2(f, n^{-1/2})_p = 0$, but $\|f - K_n f\|_p \sim n^{-1}$, shows. However, the problem is that for some p even the relation (1.2) is not valid. In fact, for $p = 1$ and $f = \ln x$ one has $\omega_\varphi^2(f,t)_1 \ge Ct^2|\ln t|$. But (see [6]) $\|f - K_n f\|_1 \le Cn^{-1}$.

However, it is not the point to focus on some special modulus, but to give a full analogy of (1.1), i.e., to find a functional which is equivalent to $\|f - K_n f\|_p$. It is the aim of our present note to find such an analogy. In order to formulate the main result of this paper, we will need the following conventions.

The symbol $P(D)$ will denote the differential operator given by

$$P(D)f := \left(\varphi^2 f'\right)', \qquad \forall f \in C^2[0,1].$$

We define the functional $K(f,t)_p$ for $f \in L_p[0,1]$, $1 \le p \le \infty$, as below:

$$K(f,t)_p := \inf\left\{\|f-g\|_p + t^2\|P(D)g\|_p : g \in C^2[0,1]\right\}.$$

Using this functional, we shall prove the following theorem.

THEOREM 1.1. *There exists an absolute positive constant C such that for all $f \in L_p[0,1]$, $1 \le p \le \infty$, there holds*

$$C^{-1}K\left(f, n^{-1/2}\right)_p \le \|f - K_n f\|_p \le CK\left(f, n^{-1/2}\right)_p. \tag{1.3}$$

In order to characterize the K-functional used in Theorem 1.1, we also show the following theorem.

THEOREM 1.2. *We have*

$$K(f,t)_p \sim \omega_\varphi^2(f,t)_p + t^2 E_0(f)_p, \qquad 1 < p < \infty,$$

and

$$K(f,t)_\infty \sim \omega_\varphi^2(f,t)_\infty + \omega(f,t^2)_\infty.$$

Here $\omega(f,t)_p$ is the classical modulus and $E_0(f)_p$ denotes the best approximation constant of f defined by

$$E_0(f)_p = \inf_c \|f - c\|_p.$$

REMARK 1.3. We note that one cannot drop the term $t^2E_0(f)_p$ on the right-hand side of the relations of Theorem 1.2 in case $1 < p < \infty$, and a term $\omega(f,t^2)_\infty$ for $p = \infty$. Moreover, one cannot replace $\omega(f,t^2)_\infty$ in the second relation of Theorem 1.2 by $t^2E_0(f)_\infty$. For, otherwise we would get that $\omega_\varphi^2(f,t)_\infty = O(t^2)$ implies $f \in \text{Lip } 1$. However, this is not the case as for $f(x) = x\ln x$, $\omega_\varphi^2(f,t)_\infty = O(t^2)$, but $\omega(f,t^2)_\infty \neq O(t^2)$. Of course, we can replace $t^2E_0(f)_\infty$ in the first relation of Theorem 1.2 by $\omega(f,t^2)_\infty$. Thus, one may reformulate Theorem 1.2 as

$$K(f,t)_p \sim \omega_\varphi^2(f,t)_p + \omega\left(f,t^2\right)_p, \qquad 1 < p \leq \infty.$$

2. UPPER ESTIMATION

Throughout this paper, we shall denote by C absolute positive constants and by $C_{\alpha,\beta}$ constants depending on α and β. These constants may be different on each occurrence. As usual, by Π_m we denote the set of algebraic polynomials of degree $\leq m$. In this section, we will give some upper estimates for the operator K_n and some inequalities concerning polynomials. Our first result is

LEMMA 2.1. *Let $1 \leq p \leq \infty$. Then for $g \in C^6[0,1]$ one has*

$$\left\|K_ng - g - \frac{1}{2(n+1)}P(D)g\right\|_p \leq Cn^{-2}\left\{\|\varphi^4g^{(4)}\|_p + \|g''\|_p + \|g\|_p\right\}, \tag{2.1}$$

$$\left\|\varphi\left(K_ng - g - \frac{1}{2(n+1)}P(D)g\right)'\right\|_p \leq Cn^{-2}\left\{\|\varphi^5g^{(5)}\|_p + \|g\|_p\right\}. \tag{2.2}$$

PROOF. Following (3.7) of [6], we have (2.1) for all $g \in \Pi_{[\sqrt{n}]}$. In general, let $P_i \in \Pi_i$ be the polynomial of best approximation of g. Then (see [4, p. 79])

$$E_i(g)_p = \|g - P_i\|_p \leq \frac{C_j}{i^j}\|\varphi^jg^{(j)}\|_p, \qquad i \geq j. \tag{2.3}$$

Thus, for $j = 4$ writing $g - P_{[\sqrt{n}]}$ as an infinite telescoping sum of terms of the form $P_{2^i[\sqrt{n}]} - P_{2^{i-1}[\sqrt{n}]}$ and using Bernstein's inequality for each term, we obtain by (2.3)

$$\left\|P(D)\left(g - P_{[\sqrt{n}]}\right)\right\|_p \leq \frac{C}{n}\|\varphi^4g^{(4)}\|_p. \tag{2.4}$$

Therefore, if we write $g = (g - P_{[\sqrt{n}]}) + P_{[\sqrt{n}]}$, by (2.3) and (2.4) we see that in order to verify (2.1) it is enough to show

$$\left\|\varphi^4P_{[\sqrt{n}]}^{(4)}\right\|_p \leq C\|\varphi^4g^{(4)}\|_p \tag{2.5}$$

and

$$\left\|P_{[\sqrt{n}]}''\right\|_p \leq C\left(\left\|\varphi^4g^{(4)}\right\|_p + \|g''\|_p + \frac{1}{n}\|g\|_p\right). \tag{2.6}$$

The estimate of (2.5) follows directly from the following inequality (see [4, p. 84]):

$$\left\|\varphi^4P_{[\sqrt{n}]}^{(4)}\right\|_p \leq Cn^2\omega_\varphi^4\left(g,n^{-1/2}\right)_p \leq C\left\|\varphi^4g^{(4)}\right\|_p.$$

To prove (2.6), we notice that

$$\left\|P_{[\sqrt{n}]}'' - n^2\Delta_{1/n}^2P_{[\sqrt{n}]}\right\|_p \leq \frac{C}{n}\left\|P_{[\sqrt{n}]}'''\right\|_p.$$

On the other hand, one may write $P_{[\sqrt{n}]}$ as a sum of terms of the form $P_{2^i} - P_{2^{i-1}}$ with $2^i \leq \sqrt{n}$ and then use (2.3) and Markov's inequality to get

$$\left\|P_{[\sqrt{n}]}'''\right\|_p \leq C\left(n\left\|\varphi^4g^{(4)}\right\|_p + \|g\|_p\right).$$

Combining these two inequalities with (2.3) we deduce (2.6).

To verify (2.2), we first consider $g \in \Pi_m, m \le \sqrt{n}$. Thus, it follows from Taylor's formula that

$$g(t) = \sum_{j=0}^{5} \frac{(t-x)^j}{j!} g^{(j)}(x) + R_6(g,t,x)$$

and

$$K_n(g,x) - g(x) - \sum_{j=1}^{5} \frac{g^{(j)}(x)}{j!} K_n\left((\cdot - x)^j, x\right) = K_n(R_6(g,\cdot,x),x). \tag{2.7}$$

Now, as the left-hand side of (2.7) is a polynomial of degree $\le m$, so is the right-hand side. Hence we can use Bernstein's inequality for the interval $[0,1]$ to obtain the first inequality below and then using the estimates of [4, p. 134] we arrive at

$$\begin{aligned}\|\varphi(K_n(R_6(g,\cdot,x),x))'\|_p &\le Cm\|(K_n(R_6(g,\cdot,x),x)\|_p \\ &\le Cmn^{-3}\left(\|\varphi^6 g^{(6)}\|_p + \|g\|_p\right) \\ &\le Cm^2n^{-3}\left(\|\varphi^5 g^{(5)}\|_p + \|g\|_p\right),\end{aligned}$$

where in the last step we have again used the Bernstein inequality. Using (2.7) and the last estimate, we deduce

$$\begin{aligned}\left\|\varphi\left(K_ng - g - \frac{1}{2(n+1)}P(D)g\right)'\right\|_p &\le \left\|\varphi\left(\sum_{j=3}^{5}\frac{g^{(j)}(x)}{j!}K_n((\cdot - x)^j,x)\right)'\right\|_p + \\ &+ Cn^{-2}\left(\left\|\varphi\left((\varphi^4)''g''\right)'\right\|_p + \|\varphi^5 g^{(5)}\|_p + \|g\|_p\right).\end{aligned} \tag{2.8}$$

To complete the proof, we also need the following estimate, the proof of which can be carried out by using a Hardy-type inequality (see e.g., [4, p. 135]): for $i = 1,2$,

$$\left\|\varphi^{5-2i} f^{(5-i)}\right\|_p \le C\left\|\varphi^5 f^{(5)}\right\|_p + \|f\|_p.$$

Using this inequality and the estimate of $K_n((\cdot - t)^j, x)$ (see [4, p. 139]), one may get (2.2) from (2.8) for $g \in \Pi_m, m \le \sqrt{n}$. Just using the approach in proving (2.1), we get (2.2) for all $g \in C^6[0,1]$. ∎

The following upper estimate is due to Berens and Xu (see [7]).

THEOREM 2.2. *For $g \in C^2[0,1]$*

$$\|g - K_ng\|_p \le \frac{C}{n}\|P(D)g\|_p \tag{2.9}$$

with $1 \le p \le \infty$.

To prove some further inequalities, we need also the so-called Bernstein-Durrmeyer operator M_n, which is defined if one replaces $f(k/n)$ in the definition of B_n by

$$(n+1)\int_0^1 f(t)p_{n,k}(t)\,dt.$$

This operator has many interesting properties. We collect some of them below (for details see, e.g., [7–9]): for $f \in L_p[0,1]$ and $g \in C^2[0,1]$, we have

$$P(D)M_ng = M_nP(D)g, \qquad P(D)M_nf = n(n+1)(M_{n-1}f - M_nf),$$

$$\left\|\varphi^4(M_nf)^{(4)}\right\|_p \le Cn^2\|f\|_p,$$

and

$$\|g - M_ng\|_p \le \frac{C}{n}\|P(D)g\|_p. \tag{2.10}$$

Using these estimates, we deduce

$$\varphi^4\left(M_ng\right)^{(4)} = -\sum_{k=2}^{n} \frac{\varphi^4\left(M_kP(D)g\right)^{(4)}}{k(k+1)}.$$

Thus,

$$\left\|\varphi^4(M_ng)^{(4)}\right\|_p \le Cn\|P(D)g\|_p, \qquad \forall g \in C^2[0,1]. \tag{2.11}$$

The last inequality can be used to prove the following useful assertion.

LEMMA 2.3. *For $g \in C^2[0,1]$, one has*

$$E_n(g)_p \le Cn^{-2}E_n(P(D)g)_p, \qquad 1 \le p \le \infty. \tag{2.12}$$

PROOF. It is enough to prove

$$E_n(g)_p \le Cn^{-2}\|P(D)g\|_p. \tag{2.13}$$

In fact, if this holds, one may replace g by $g - P$ with $P \in \Pi_n$. Thus

$$E_n(g)_p \le Cn^{-2}\inf_{P\in\Pi_n}\|P(D)g - P(D)P\|_p. \tag{2.14}$$

On the other hand, the eigenfunctions of $P(D)$ are the Legendre polynomials defined on $[0,1]$. More clearly, let $P_k \in \Pi_k$ be the Legendre polynomial, then $P(D)P_k = \lambda_kP_k$ with $\lambda_0 = 0$, $\lambda_k \neq 0$ if $k \neq 0$. As every polynomial $P^* \in \Pi_n$ can be written as a linear combination of $\{P_k\}_{k=0}^n$, we have

$$\begin{aligned} P^* &= \sum_{k=0}^{n} a_kP_k = a_0P_0 + P(D)\sum_{k=1}^{n} a_k\lambda_k^{-1}P_k \\ &=: a_0P_0 + P(D)\overline{P}. \end{aligned}$$

Suppose

$$\inf_{P\in\Pi_n}\|P(D)g - P\|_p = \|P(D)g - P^*\|_p.$$

By orthogonality, we have

$$|a_0| = \left|\int_0^1 \left(P(D)g - P^*\right)dx\right| \le \|P(D)g - P^*\|_p.$$

Hence we can replace $P(D)P$ in (2.14) by P and C by $2C$. In this way we get (2.12).

It remains to prove (2.13). Due to (2.3), we get by (2.10) and (2.11),

$$\begin{aligned} E_n(g)_p &\le E_n(g - M_mg)_p + \frac{C}{n^4}\left\|\varphi^4(M_mg)^{(4)}\right\|_p \\ &\le \frac{C}{m}\|P(D)g\|_p + \frac{Cm}{n^4}\|P(D)g\|_p. \end{aligned}$$

Choosing $m = n^2$ we deduce (2.13) from the above. ∎

For future purposes, we also need the Bernstein-type inequalities for Bernstein-Kantorovich polynomials.

LEMMA 2.4. *Let $1 \le p \le \infty$ and f be a polynomial. Then*

$$\left\|\varphi^j (K_n f)^{(j+i)}\right\|_p \le C_{j,i} n^{((v+u)/2)+\tau} \left\|\varphi^{j-v+u} f^{(j-v+i-\tau)}\right\|_p, \tag{2.15}$$

where $0 \le v, u \le j$, $0 \le v-u \le j$, and $0 \le \tau \le i$;

$$\left\|\varphi^4 (K_n f)^{(4)}\right\|_p \le C n^{1/2} \|\varphi (P(D)f)'\|_p, \tag{2.16}$$

$$\left\|\varphi^5 (K_n f)^{(5)}\right\|_p \le C n^{1/2} \|P^2(D) f\|_p, \tag{2.17}$$

$$\left\|\varphi^3 (K_n f)^{(3)}\right\|_p \le C n^{1/2} \|P(D) f\|_p. \tag{2.18}$$

PROOF. (2.15) can be found in [4, p. 125 and 156] if j is an even number and $u = i = 0$. For other cases the proof is analogous.

To prove (2.16) we notice: if $P_i \in \Pi_i$ is the best polynomial approximation of f, then using the linearity of K_n, (2.12) and (2.3) for any m we get

$$\left\|\varphi^4 (K_n f)^{(4)}\right\|_p \le C \left(n^2 m^{-3} \|\varphi (P(D)f)'\|_p + \left\|\varphi^4 P_m^{(4)}\right\|_p \right).$$

Representing P_m as a sum of terms of the form $P_{2^i} - P_{2^{i+1}}$ and using Bernstein's inequality, Lemma 2.3 and (2.3), we get for the last term

$$\left\|\varphi^4 P_m^{(4)}\right\|_p \le C m \|\varphi (P(D)f)'\|_p.$$

Choosing $m = [\sqrt{n}]$, (2.16) follows.

(2.17) and (2.18) can be proved in a similar fashion. ∎

REMARK 2.5. The reader may find out that in Lemmas 2.1 and 2.4 the conditions on g and f are too strong. Of course, we can weaken them in some sense. However, this is not necessary, since later we will replace them by their Bernstein-Kantorovich polynomials. On the other hand, such restriction makes the results neater. In the next section, we will also often use such consideration.

3. ESTIMATES FOR THE ITERATES K_n^N

The results in this section play a centre role in proving the lower estimation of Theorem 1.1. Some lemmas (Lemmas 3.3, 3.4 and 3.6) have the same form as in [2]. But, since now the situation is somewhat more complicated, we have to prove them in this paper again. The proofs of Lemmas 3.3 and 3.4 are given completely. We omit the proof of Lemma 3.6, because it can be carried out almost word for word as in [2].

In the next section, we will prove a theorem for the iterates of K_n (see Theorem 3.1). This result is in fact the key step in order to get the lower estimate for Bernstein-Kantorovich operators. It shows that these operators behave similarly as a semi-group operator in the sense that the N^{th} iteration of a Bernstein-Kantorovich operator of degree n has similar properties as the same operator with degree n/N. This property will be understood better after seeing Lemma 3.5.

As usual, we write $K_n^0 f = f$, $K_n^i f = K_n(K_n^{i-1} f)$, $i = 1, 2, \dots$. For the iterates K_n^N we prove the following theorem.

THEOREM 3.1. *For $1 \le p \le \infty, 2 \le N \le Cn$ and $f \in \Pi_n$, one has*

$$\left\|P^2(D) K_n^N f\right\|_p \le \frac{Cn}{\sqrt{N}} \|P(D) f\|_p + C_N \Phi_{n,p}(f); \tag{3.1}$$

$$\left\|\varphi^2 (K_n^N f)^{(4)}\right\|_p \le C \left(\frac{\ln N}{\sqrt{N}}\right)^3 n^2 \|f'\|_p; \tag{3.2}$$

$$\left\|(K_n^N f)^{(i)}\right\|_p \le C \left(\frac{\ln N}{\sqrt{N}}\right)^{i-1} n^{i-1} \|f'\|_p, \qquad i = 2, 3, 4. \tag{3.3}$$

Here

$$\Phi_{n,p}(f) := \frac{1}{n}\left\{\left\|\varphi^2 f^{(4)}\right\|_p + \frac{1}{n}\left\|f^{(4)}\right\|_p + \frac{1}{n}\|f'\|_p\right\}.$$

The main aim of this section is to prove two iterate inequalities (see Lemmas 3.3 and 3.4), which will be needed in order to verify Theorem 3.1. We begin with the following computational result.

LEMMA 3.2. *The following five inequalities hold:*

$$\int_0^1\int_0^1\int_0^1 e^{\frac{(t_1+t_2-1)^2}{t_3+1}}\,dt_1\,dt_2\,dt_3 \le 1.15; \tag{3.4}$$

$$\int_0^1\int_0^1\int_0^1 \frac{1+t_1+t_2}{(1+t_1+t_2+t_3)^2} e^{\frac{t_3^2}{1+t_1+t_2}}\,dt_1\,dt_2\,dt_3 \le \frac{1}{2}; \tag{3.5}$$

$$\int_0^1\int_0^1 \frac{1+t_1}{(1+t_1+t_2)^2}\left(1-e^{-\frac{(1+t_1+t_2)^2}{1+t_1}}\right) e^{\frac{t_2^2}{1+t_1}}\,dt_1\,dt_2 \le \frac{1}{2}; \tag{3.6}$$

$$\int_0^1\int_0^1 \frac{j+t_1}{(j+t_1+t_2)^2} e^{\frac{t_2^2}{j+t_1}}\,dt_1\,dt_2 \le \frac{1}{j+1}, \qquad j\ge 2; \tag{3.7}$$

$$\int_0^1\int_0^1 \frac{1}{j+t_1} e^{\frac{t_2^2}{j+t_1}}\,dt_1\,dt_2 \le \frac{1}{j}, \qquad j\ge 1. \tag{3.8}$$

Moreover, let $0<\alpha\le 1$ *and* $G_0(t)=t$, $G_k(t)=1-e^{-\alpha G_{k-1}(t)}$; $k=1,2,\dots$ *Then, for* $0\le\mu\le\nu-1$ *and* $b\ge 0$, *we have*

$$\int_0^1 \frac{t^\mu G_N^\nu(t)}{t^\nu} e^{-bG_N(t)}\,dt \le C_\nu \frac{\ln(N+1)}{b^{\mu+1}}, \tag{3.9}$$

where C_ν *depends only on* ν.

PROOF. The first three inequalities can be verified directly. Of course, they can also be proved using the method of proving (3.7) and (3.8). Next we verify (3.7). Using Taylor's formula

$$e^{\frac{t_2^2}{j+t_1}} \le \sum_{i=0}^{2} \frac{t_2^{2i}}{i!(j+t_1)^i} + \frac{t_2^6}{3(j+t_1)^3} \tag{3.10}$$

and

$$\frac{j+t_1}{(j+t_1+t_2)^2} = \frac{1}{j+t_1}\sum_{i=0}^{\infty}(-1)^i(i+1)\left(\frac{t_2}{j+t_1}\right)^i, \tag{3.11}$$

we deduce that the integral on the left side of (3.7) with respect to t_2 is smaller than

$$\frac{1}{1+x}+\frac{1}{3x^2}-\frac{2}{5x^3}+\frac{5}{6x^3(1+x)}$$

with $x=j+t_1$. To see this, we notice that after integrating with respect to t_2 we obtain

$$\sum_{i=0}^{\infty}(-1)^i\frac{(i+1)}{(j+t_1)^{i+1}}\left\{\frac{1}{i+1}+\frac{1}{(i+3)(j+t_1)}+\frac{1}{2(i+5)(j+t_1)^2}+\frac{1}{3(i+7)(j+t_1)^3}\right\}.$$

We consider this as four sums. Then the first one is $1/(1+x)$, the first term of the second sum is $1/3x^2$. Adding the rest of the second sum to the third one, then the first term of the obtained sum is $-2/5x^3$. Again adding the rest to the fourth sum, this is smaller than $5/6x^3(1+x)$. Therefore, to prove (3.7), it is enough to show, with $x=j+t$, that

$$\int_0^1\left(\frac{1}{1+x}+\frac{1}{3x^2}-\frac{2}{5x^3}+\frac{5}{6x^3(1+x)}\right)dt \le \frac{1}{j+1}.$$

Indeed, for $j = 2, 3$ the integral is $0.332549\ldots$, $0.245791\ldots$, respectively. If $j \geq 4$, then, since

$$\int_0^1 \frac{dt}{1+x} = -\ln\left(1 - \frac{1}{j+2}\right)$$
$$\leq \frac{1}{j+2} + \frac{1}{2(j+2)^2} + \frac{1}{3}\sum_{i=2}^{\infty} \frac{1}{(j+2)^{i+1}},$$
$$\int_0^1 \left(\frac{1}{3x^2} - \frac{2}{5x^3}\right) dt \leq \frac{1}{2(j+2)^2} \quad \text{and} \quad \int_0^1 \frac{5}{6x^4}\, dt \leq \frac{2}{3(j+2)^3},$$

the integral is less than

$$\sum_{i=0}^{\infty} \frac{1}{(j+2)^{i+1}} \qquad \left(\equiv \frac{1}{j+1}\right).$$

To prove (3.8), we use (3.10) to obtain for the case $j \geq 2$

$$\int_0^1 \int_0^1 \frac{1}{j+t_1} e^{\frac{t_2^2}{j+t_1}}\, dt_1\, dt_2 \leq \ln\frac{j+1}{j} + \frac{1}{3j(j+1)} + \frac{2j+1}{20j^2(j+1)^2} + \frac{3j^2+3j+1}{63j^3(j+1)^3}.$$

Thus direct calculation shows that the right-hand side of the above is not larger than $1/j$. For $j = 1$ we use (3.10) with $e/6$ in place of $1/3$.

It remains to verify (3.9). We consider the function G_N. It is clear that $G_N(t) \geq 0$ if $0 \leq t \leq 1$ and by induction one gets $G_N(t) \leq \alpha^N t$, $0 \leq t \leq 1$. On the other hand, by the definition,

$$G_N'(t) = \alpha^N \prod_{k=1}^{N} e^{-\alpha G_{k-1}(t)}.$$

Thus, $G_N'(0) = \alpha^N$, $|G_N''(0)| \leq N\alpha^N$ and $G_N'''(t) \geq 0$, which implies

$$\alpha^N\left(t - \frac{N}{2}t^2\right) \leq G_N(t) \leq \alpha^N t, \qquad 0 \leq t \leq 1. \tag{3.12}$$

Next we divide $[0, 1]$ into $[0, 2/3N]$ and $[2/3N, 1]$. The integral in (3.9) can be written as

$$\left\{\int_0^{2/3N} + \int_{2/3N}^1\right\} \frac{t^\mu G_N^\nu(t)}{t^\nu} e^{-bG_N(t)}\, dt := I_1 + I_2.$$

Using (3.12), we deduce

$$I_1 \leq \alpha^{N\nu} \int_0^{2/3N} t^\mu e^{-\frac{2b}{3}\alpha^N t}\, dt \leq \frac{C}{b^{\mu+1}}.$$

To estimate I_2, we notice that, as $\mu \leq \nu - 1$, one has by (3.12),

$$\frac{t^\mu G_N^\nu(t) e^{-bG_N(t)}}{t^\nu} \leq \frac{C}{b^{\mu+1}t}.$$

Hence

$$I_2 \leq C\frac{\ln(N+1)}{b^{\mu+1}}.$$

The estimates of I_1 and I_2 imply (3.9). ∎

Throughout this paper, we will use the notation: $p_{n,0,k}(x) = p_{n,k}(x)$ and

$$p_{n,j,k}(x) = (n+1)^j \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} p_{n-j+1,k}(x + t_1 + \cdots + t_j)\, dt_1 \cdots dt_j.$$

We will also use the following associated operator:

$$L_{n,j}f := \sum_{k=0}^{n-j+1} f\left(\frac{k}{n+1}\right) p_{n,j,k}$$

and its iterations $L_{n,j}^N$. We will denote by $\overline{L}_{n,j}$ the operator in which $p_{n,j,k}$ is replaced by $p_{n-j,k}$. It is clear that for some $\alpha_{n,j}$ with $|\alpha_{n,j}-1| \le C_j n^{-1}$ and the Steklov function

$$f_n(x) := (n+1)^j \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} f(x+t_1+\cdots+t_j)\,dt_1\cdots dt_j, \tag{3.13}$$

one has

$$L_{n,j}f_n^{(j-1)} = \alpha_{n,j}\left((K_n f)^{(j-1)}\right)_n \quad \text{or} \quad L_{n,j}f_n^{(j-1)} = \alpha_{n,j}\left((B_{n+1}F)^{(j)}\right)_n,$$

where B_n is the Bernstein operator and $F' = f$.

Let now

$$\psi_v(x) := \sum_{k=0}^{n-j+1} \varphi^{-2v}\left(\frac{k+1}{n+2}\right) p_{n,j,k}(x). \tag{3.14}$$

We have the first iterate inequality as follows.

LEMMA 3.3. *For $j \ge 1$, $v = 0,1,2,\ldots,j$ and $2 \le N \le Cn$,*

$$\overline{L}_{n,j}\left(L_{n,j}^{N-1}\psi_v, x\right) \le C_j \min\left\{\varphi^{-2v}(x), n^v\right\} \ln N.$$

PROOF. It is enough to verify this inequality for $\varphi^{-2v}(x)$ in place of $\min\{\varphi^{-2v}(x), n^v\}$, since $\psi_v(x) \le C_v n^v$. For $v = 0$ the inequality is trivial. We assume in the following that $v \ge 1$. Denote by $\psi_{v,1}$ and $\psi_{v,2}$ the function defined by (3.14) with $\varphi^{-2v}(x)$ being replaced by x^{-v} and $(1-x)^{-v}$, respectively. We have $\psi_v \le 2^v(\psi_{v,1}+\psi_{v,2})$. Furthermore, with $y = 1-x-j/(n+1)$ one gets easily $p_{n,j,k}(x) = p_{n,j,n-j+1-k}(y)$ and $\psi_{v,2}(x) \le \psi_{v,1}(y)$. In this way, we deduce

$$\overline{L}_{n,j}\left(L_{n,j}^N\psi_{v,2}, x\right) \le \overline{L}_{n,j}\left(L_{n,j}^N\psi_{v,1}, 1-x\right).$$

Hence, in order to verify the assertion of this lemma, it is enough to prove it for $\psi_{v,1}$ instead of ψ_v.

To this end, we observe that, by using the binomial formula, we have

$$\begin{aligned}\sum_{k=0}^{n-j+1} & \eta^k p_{n,j,k}(x) \\ &= (n+1)^j \int_0^{1/n+1} \cdots \int_0^{1/n+1} (1-(x+t_1+\cdots+t_j)(1-\eta))^{n-j+1}\,dt_1\cdots dt_j.\end{aligned} \tag{3.15}$$

Making use of the inequality $1+a \le e^a$, one deduces from (3.15)

$$\sum_{k=0}^{n-j+1} \eta^k p_{n,j,k}(x) \le \left(\frac{n+1}{n-j+1}\right)^j e^{-(n-j+1)x(1-\eta)} \left(\frac{1-e^{-\frac{n-j+1}{n+1}(1-\eta)}}{1-\eta}\right)^j. \tag{3.16}$$

On the other hand, as

$$\left(\frac{1}{k+1}\right)^v = \int_0^1 \cdots \int_0^1 \tau^k\,d\tau$$

with $\boldsymbol{\tau}^k = (\tau_1 \cdots \tau_v)^k$ and $d\boldsymbol{\tau} = d\tau_1 \cdots d\tau_v$, we have by (3.16)

$$\psi_{v,1}(x) \le (n+2)^v \left(\frac{n+1}{n-j+1}\right)^j \int_0^1 \cdots \int_0^1 e^{-(n-j+1)x(1-\tau)} \left(\frac{1-e^{-\frac{n-j+1}{n+1}(1-\tau)}}{1-\tau}\right)^j d\tau.$$

The benefit of this estimate is that instead of estimating $L_{n,j}(\psi_{v,1}, x)$ one needs only to do this for $L_{n,j}(e^{-(n-j+1)(1-\tau)\cdot}, x)$. The latter is easy to deal with if we use (3.16) for $e^{-(n-j+1)(1-\tau)/(n+1)}$ instead of η. Setting $H_0(t) = 1-t$, $H_i(t) = 1 - e^{-(n-j+1)/(n+1)\, H_{i-1}(t)}$, $i = 1, 2, \dots$, the above inequality can be rewritten as

$$\psi_{v,1}(x) \le (n+2)^v \left(\frac{n+1}{n-j+1}\right)^j \int_0^1 \cdots \int_0^1 e^{-(n-j+1)xH_0(\tau)} \left(\frac{H_1(\tau)}{H_0(\tau)}\right)^j d\tau. \tag{3.17}$$

In this way, we get recursively

$$\overline{L}_{n,j}\left(L_{n,j}^{N-1}\psi_{v,1}, x\right) \le (n+2)^v \left(\frac{n+1}{n-j+1}\right)^{Nj} \int_0^1 \cdots \int_0^1 \frac{H_N^j(\tau)}{H_0^j(\tau)} e^{-(n-j+1)H_N(\tau)x}\, d\boldsymbol{\tau}.$$

Therefore, in order to complete the proof, it is sufficient to show that for

$$F(u) := \frac{H_N^j(u)}{H_0^j(u)} e^{-(n-j+1)H_N(u)x}, \tag{3.18}$$

one has

$$I := \int_0^1 \cdots \int_0^1 F(\boldsymbol{\tau})\, d\boldsymbol{\tau} \le \frac{C_j}{(nx)^v} \ln(N+1). \tag{3.19}$$

We consider the function F in more detail. Since $y^j e^{-by} \le C_j b^{-k}$ if $0 \le k \le j$ and $y \in [0,1]$ and since $0 \le H_N(u) \le 1$ if $0 \le u \le 1$ we have that for $u \le 1/2$, $F(u) \le C(nx)^{-v}$. Moreover, $\boldsymbol{\tau} = \tau_1 \cdots \tau_v \le 1/2$ if at least one of the τ_i is smaller than $1/2$. Thus, we need only to prove (3.19) over the domain of integration $[1/2, 1] \times \cdots \times [1/2, 1]$. Noticing $\boldsymbol{\tau} = \tau_1 \cdots \tau_v$, the last integral can be rewritten as

$$\int_{1/2}^1 \frac{1}{\tau_1} \int_{\tau_1/2}^{\tau_1} \frac{1}{\tau_2} \cdots \int_{\tau_{v-1}/2}^{\tau_{v-1}} F(\tau_v)\, d\tau_v \cdots d\tau_1.$$

Noticing further that if $\tau_i \le 1/2$ in the above integral, then $\tau_{i+1} \le 1/2$, thus $\tau_v \le 1/2$ and $F(\tau_v) \le C(nx)^{-v}$. Therefore,

$$\begin{aligned} I &\le 2^{v-1} \int_{1/2}^1 \int_{1/2}^{\tau_1} \cdots \int_{1/2}^{\tau_{v-1}} F(\tau_v)\, d\tau_v \cdots d\tau_1 + \frac{C}{(nx)^v} \\ &= \frac{2^{v-1}}{(v-1)!} \int_{1/2}^1 (1-u)^{v-1} F(u)\, du + \frac{C}{(nx)^v}. \end{aligned}$$

Finally, the last integral is

$$\int_{1/2}^1 (1-u)^{v-1} F(u)\, du = \int_0^{1/2} \frac{H_N^j(1-u)}{u^{j-v+1}} e^{-(n-j+1)H_N(1-u)x}\, du.$$

To calculate this integral, we use (3.9) with $G_N(t) = H_N(1-t)$, $\alpha = (n-j+1)/(n+1)$ and $b = (n-j+1)\,x$ to obtain

$$\int_0^{1/2} \frac{H_N^j(1-u)}{u^{j-v+1}} e^{-(n-j+1)H_N(1-u)x}\, du \le C_j \frac{\ln N}{(nx)^v}.$$

The proof is complete. ∎

The second iterate inequality is as follows.

LEMMA 3.4. *For $1 \le v \le 4$ and $0 \le j \le n - v$ there exists a positive constant C_v such that*

$$\sum_{k=0}^{n-v} \frac{p_{n,v+1,k}{}^2\,(j/(n+1))}{(k+1)(n-v+1-k)p_{n,v,k}\,(j/(n+1))} \le \frac{1+C_v n^{-1}}{(j+1)(n-v+1-j)}.$$

PROOF. In [2] (see Lemma 3.2 there) the case $v = 2$ was proved. The proof for other cases is essentially the same. However, as now the situation is more complex, the assertion cannot be deduced from our earlier result.

Denote the term of this sum as $I_{k,v}$. We begin with the case $j = 0$. A routine calculation shows that for all $0 \le u \le (v-1)/(n+1)$

$$(n+1)\int_0^{1/(n+1)} p_{n-v+1,k}(t+u)\,dt \ge \frac{n+1}{k+1}\,p_{n-v+1,k}\left(\frac{1}{n+1}+u\right). \tag{3.20}$$

Using this estimate, Cauchy's inequality yields

$$I_{k,v} \le \frac{(n+1)^{v+2}}{(n+1-v)^2}\int_0^{1/(n+1)}\cdots\int_0^{1/(n+1)} \binom{n-v}{k} A^k B^{n-v-k}\,dt_1\cdots dt_{v+1} \quad (\equiv J_{k,v}),$$

where

$$A = \frac{(t_1+t_2+\cdots+t_{v+1})^2}{t_1+\cdots t_{v-1}+1/(n+1)} \quad\text{and}\quad B = \frac{(1-t_1-t_2-\cdots-t_{v+1})^2}{1-t_1-\cdots-t_{v-1}-1/(n+1)}.$$

We then replace $I_{k,v}$ by $J_{k,v}$ for $k \ge 2$. Using the binomial formula, we get after omitting the term $I_{0,v} - J_{0,v}$,

$$\sum_{k=0}^{n-v} I_{k,v} \le \frac{(n+1)^{v+2}}{(n+1-v)^2}\int_0^{1/(n+1)}\cdots\int_0^{1/(n+1)} (A+B)^{n-v}\,dt_1\cdots dt_{v+1} + (I_{1,v}-J_{1,v}). \tag{3.21}$$

To estimate this integral, we use the inequality

$$\left(1-x^2\right)e^x \le 1+x \le e^x. \tag{3.22}$$

Thus, for $u_i = (n+1)t_i$, $i = 1,\dots,v+1$,

$$(A+B)^{n-v} \le \left(1+\frac{C}{n}\right) e^{\frac{(u_v+u_{v+1}-1)^2}{u_1+\cdots+u_{v-1}+1}}.$$

Therefore, by (3.4) and this inequality, the first term on the right-hand of (3.21) is less than $1.15(1+C/n)(n-v+1)^{-1}$ if $2 \le v \le 4$. If $v = 1$, then this term is less than $1.5(1+C/n)(n-v+1)^{-1}$ as computation shows.

To estimate the second term of (3.21), we use again (3.22) to obtain

$$I_{1,v} \le \frac{1+C/n}{n-v+1}\,\frac{(v+1)^2}{2v}\left(1-e^{-1}\right)^{v+1}\left(1-2e^{-1}\right)$$

and

$$\begin{aligned} J_{1,v} &\ge \frac{1-C/n}{n-v+1}\int_{[0,1]^{v+1}} \frac{(u_1+\cdots+u_{v+1})^2}{u_1+\cdots+u_{v-1}+1} \times \frac{e^{2u_1+\cdots+2u_{v+1}}}{e^{-u_1-\cdots-u_{v-1}-1}}\,du_1\cdots du_{v+1} \\ &\ge \frac{(1-C/n)\,e}{n-v+1}\int_{[0,1]^{v+1}} (u_1+\cdots+u_{v-1}+2u_v+2u_{v+1}-1)\frac{e^{2u_1+\cdots+2u_{v+1}}}{e^{-u_1-\cdots-u_{v-1}}}\,du_1\cdots du_{v+1} \\ &= \frac{(1-C/n)\,e}{4(n-v+1)}\left(1-e^{-2}\right)\left(1-e^{-1}\right)^{v-1}\left((v-1)\left(1+e^{-1}\right)\left(1-2e^{-1}\right)+2-5e^{-2}\right). \end{aligned}$$

We obtain from these two estimates that, for $2 \le v \le 4$,

$$\begin{aligned} I_{1,v} - J_{1,v} &\le \frac{1 + C/n}{n - v + 1} \times \frac{\left(1 - e^{-1}\right)^4}{4} \left(7 - 3e - 19e^{-1} + 25e^{-2}\right) \\ &\le \frac{1 + C/n}{n - v + 1} \times (-0.19), \end{aligned}$$

and

$$I_{1,1} - J_{1,1} \le \frac{1 - C/n}{2(n - v + 1)} \left(1 - e^{-1}\right) \left(3 - e - \frac{19}{2e} + \frac{9}{2e^2}\right) \le \frac{1 - C/n}{n - v + 1} \times (-0.82).$$

Consequently, we get

$$\sum_{k=0}^{n-v} I_{k,v} \le \frac{1 + C/n}{n + 1 - v}.$$

That is the assertion of this lemma in case $j = 0$.

Next we consider the case $1 \le j \le n - v$. Making use of Cauchy's inequality, we have

$$\begin{aligned} &\frac{p^2_{n,v+1,k}\left(j/(n+1)\right)}{p_{n,v,k}\left(j/(n+1)\right)} \\ &\quad \le (n+1)^{v+1} \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} \frac{p^2_{n-v,k}\left(j/(n+1) + t_1 + \cdots + t_{v+1}\right)}{p_{n-v+1,k}\left(j/(n+1) + t_1 + \cdots + t_v\right)} \, dt_1 \cdots dt_{v+1}. \end{aligned}$$

Write now

$$A = \frac{\left(j/(n+1) + t_1 + \cdots + t_{v+1}\right)^2}{\left(j/(n+1) + t_1 + \cdots + t_v\right)} \quad \text{and} \quad B = \frac{\left(1 - j/(n+1) - t_1 - \cdots - t_{v+1}\right)^2}{\left(1 - j/(n+1) - t_1 - \cdots - t_v\right)}.$$

Then,

$$(A + B)^n \le e^{\frac{(n+1)t^2_{v+1}}{\varphi^2(j/(n+1) + t_1 + \cdots + t_v)}}. \tag{3.23}$$

Hence, using Taylor's formula, one gets

$$(A + B)^n \le \begin{cases} \left(1 + \dfrac{C}{n}\right) e^{\frac{(n+1)^2 t^2_{v+1}}{x}}, & 1 \le j \le \dfrac{4n}{5} \\ \left(1 + \dfrac{C}{n}\right) e^{\frac{(n+1)^2 t^2_{v+1}}{n+1-x}}, & \dfrac{4n}{5} \le j \le n - v, \end{cases} \tag{3.24}$$

where $x = j + (n+1)(t_1 + \cdots + t_v)$. Using these estimates, we obtain

$$\begin{aligned} \sum_{k=0}^{n-v} I_{k,v} &\le (n+1)^{v+1} \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} \sum_{k=0}^{n-v} \frac{\binom{n-v}{k} A^k B^{n-k-v} \, dt_1 \cdots dt_{v+1}}{(n - v + 1)(k+1)\left(1 - j/(n+1) - t_1 - \cdots - t_v\right)} \\ &= \frac{(n+1)^{v+1}}{n - v + 1} \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} \int_0^1 \frac{(Au + B)^{n-v}}{1 - j/(n+1) - t_1 - \cdots - t_v} \, du \, dt_1 \cdots dt_v. \end{aligned}$$

Writing $Au + B = A + B - (1 - u)A$ and using (3.23), we deduce for

$$S_v \equiv \sum_{k=0}^{n-v} I_{k,v}$$

$$S_v \le \frac{(n+1)^{v+1}}{n-v+1} \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} \int_0^1 \times \frac{e^{-(1-u)A(n+1)}}{1-j/(n+1)-t_1-\cdots-t_v} e^{\frac{(n+1)t_{v+1}^2}{\varphi^2(j/(n+1)+t_1+\cdots+t_v)}}\, du\, dt_1 \cdots dt_{v+1}.$$

Calculating the integral with respect to u and then taking $u_i = (n+1)t_i$, $i = 1, \dots, v+1$, we get for $y = j + u_1 + \cdots + u_v$

$$S_v \le \begin{cases} \dfrac{1+C/n}{n+1-j-v} \displaystyle\int_0^1 \cdots \int_0^1 \frac{y e^{\frac{u_{v+1}^2}{y}} \left(1 - e^{-\frac{(y+u_{v+1})^2}{y}}\right)}{(y+u_{v+1})^2} du_1 \cdots du_{v+1}, & 1 \le j \le \dfrac{4n}{5}, \\ \dfrac{1+C/n}{j} \displaystyle\int_0^1 \cdots \int_0^1 \frac{e^{\frac{u_{v+1}^2}{n+1-y}}}{n+1-y} du_1 \cdots du_{v+1}, & \dfrac{4n}{5} \le j \le n-v. \end{cases}$$

In what follows, we shall use Lemma 3.2 to estimate these integrals. Now if $j = 1$ and $v = 1$ then by (3.6) of Lemma 3.2

$$\sum_{k=0}^{n-1} I_{k,1} \le \left(1 + \frac{C}{n}\right) \frac{1}{2(n-1)}.$$

If $1 < j \le 4n/5$ and $1 \le v \le 4$, we consider the function

$$J(t) := \int_0^1 \int_0^1 \frac{j+t+t_1}{(j+t+t_1+t_2)^2} e^{\frac{t_2^2}{j+t+t_1}} dt_1\, dt_2.$$

As $J'(t) \le 0$, $t \ge 0$; we have $J(t) \le J(0)$ and, by (3.7) in Lemma 3.2, $J(0) \le 1/(j+1)$. Therefore,

$$\int_0^1 \cdots \int_0^1 \frac{y e^{\frac{u_{v+1}^2}{y}}}{(y+u_{v+1})^2} \left(1 - e^{-\frac{(y+u_{v+1})^2}{y}}\right) du_1 \cdots du_{v+1} \le \frac{1}{j+1}.$$

In case $j = 1$ and $2 \le v \le 4$, the above estimate still holds due to (3.5).

It remains to show the case $4n/5 \le j \le n - v$. We have by using (3.8)

$$\int_0^1 \cdots \int_0^1 \frac{e^{\frac{u_{v+1}^2}{n+1-y}}}{n+1-y} du_1 \cdots du_{v+1} = \int_0^1 \cdots \int_0^1 \frac{e^{\frac{u_{v+1}^2}{n+1-j-v+u_1+\cdots+u_v}}}{n+1-j-v+u_1+\cdots+u_v} du_1 \cdots du_{v+1} \le \frac{1}{n+1-j-v}.$$

Lemma 3.4 follows from these estimates. ∎

The following inequality is analogous to the estimation of the moments of Bernstein polynomials (see [10]).

LEMMA 3.5. *There exist constants $C_{i,j}$ which depend only on i, j such that for $N < Cn$*

$$\left|L_{n,j}^N\left((\cdot - x)^i, x\right)\right| \le C_{i,j} \left(\sqrt{\frac{N}{n}}\varphi(x) + \frac{N}{n}\right)^i. \tag{3.25}$$

PROOF. Write $T_{n,j,\tau}(x) = L_{n,j}((\cdot - x)^\tau, x)$. We have

$$\left|T_{n,j,\tau}^{(\nu)}(x)\right| \le C_{\tau,j} \left(\frac{\varphi(x)}{\sqrt{n}} + \frac{1}{n}\right)^{\tau-\nu}, \qquad 0 \le \nu \le \tau. \tag{3.26}$$

In fact, using (3.13) and the estimates for the moments of Bernstein polynomials, we have (3.26) for $\nu = 0$. On the other hand, the ν^{th} derivative of $L_{n,j}(f)$ is an operator of the same type with $(n-j+1)\cdots(n-j+1-\nu)\int_0^{1/n+1}\cdots\int_0^{1/n+1} f^{(\nu)}(k/(n+1)+t_1+\ldots t_\nu)\,dt_1\cdots dt_\nu$ and $p_{n-\nu,j,k}(x)$ in place of $f(k/(n+1))$ and $p_{n,j,k}(x)$, respectively. This implies that by the binomial formula, namely

$$(t-x)^k = \sum_{\mu=0}^{k}\binom{k}{\mu}(t-u)^\mu(u-x)^{k-\mu},$$

one has

$$\left|T_{n,j,\tau}^{(\nu)}(x)\right| \le C_{\tau,j}\left\{\sum_{i=0}^{\nu} L_{n-i,j}(|\cdot-x|^{\tau-\nu},x)+\frac{1}{n^{\tau-\nu}}\right\}.$$

Combining this with (3.26) for $\nu = 0$, we get (3.26) for all $0 \le \nu \le \tau$.

We verify (3.25) by induction. Applying (3.13) and by the above consideration, one can easily get (3.25) for $i = 1, 2$. Supposing now that (3.25) holds for all $i \le k-1$, we have, with a special choice of u in the above binomial formula,

$$L_{n,j}^{N}\left((\cdot-x)^k,x\right) = \sum_{\mu=0}^{k}\binom{k}{\mu}L_{n,j}^{N-1}\left(T_{n,j,\mu}(\cdot)(\cdot-x)^{k-\mu},x\right).$$

We notice that, by the definition of $L_{n,j}$, $T_{n,j,\mu}(t)$ is a polynomial of t with degree μ. By the Taylor expansion of $T_{n,j,\mu}(t)$ at x, we get then

$$\begin{aligned}L_{n,j}^{N}\left((\cdot-x)^k,x\right) &= \sum_{\mu=0}^{k}\binom{k}{\mu}\frac{T_{n,j,\mu}^{(\mu)}(x)}{\mu!}L_{n,j}^{N-1}\left((\cdot-x)^k,x\right)\\ &\quad+\sum_{\mu=1}^{k}\binom{k}{\mu}\sum_{\nu=0}^{\mu-1}\frac{T_{n,j,\mu}^{(\nu)}(x)}{\nu!}L_{n,j}^{N-1}\left((\cdot-x)^{k-\mu+\nu},x\right).\end{aligned}\tag{3.27}$$

Furthermore, as in case $\nu = \mu-1$, $T_{n,j,\mu}^{(\mu-1)}$ is a linear function, it follows from (3.26) that $|T_{n,j,\mu}^{(\mu-1)}(x)| \le C_{j,\mu}n^{-1}$. Thus,

$$\sum_{\mu=0}^{k}\binom{k}{\mu}\frac{T_{n,j,\mu}^{(\mu)}(x)}{\mu!} = 1+\beta_{n,j,k}$$

with $|\beta_{n,j,k}| \le C_{j,k}n^{-1}$.

Denote the second sum of (3.27) by $\gamma_{n,j,k-1,N-1}(x)$. Then the induction assumption, (3.26) and the above consideration imply

$$\begin{aligned}|\gamma_{n,j,k-1,N-1}(x)| \le C_{j,k-1}&\left\{\frac{1}{n}\left(\sqrt{\frac{N-1}{n}}\varphi(x)+\frac{N-1}{n}\right)^{k-1}\right.\\ &\left.+\left(\frac{\varphi(x)}{\sqrt{n}}+\frac{1}{n}\right)^2\left(\sqrt{\frac{N-1}{n}}\varphi(x)+\frac{N-1}{n}\right)^{k-2}\right\}.\end{aligned}\tag{3.28}$$

Hence, recursively we get

$$\begin{aligned}L_{n,j}^{N}\left((\cdot-x)^k,x\right) &= (1+\beta_{n,j,k})L_{n,j}^{N-1}\left((\cdot-x)^k,x\right)+\gamma_{n,j,k-1,N-1}(x)\\ &= (1+\beta_{n,j,k})^{N-1}L_{n,j}\left((\cdot-x)^k,x\right)+\sum_{\mu=0}^{N-2}(1+\beta_{n,j,k})^{\mu}\gamma_{n,j,k-1,N-1-\mu}(x).\end{aligned}$$

Now as $N \le Cn$, $(1+C_jn^{-1})^N \le C_j'$, the assertion follows from (3.26) and (3.28). ∎

The following lemma is an analogue of Lemma 3.5 of [2]. Its proof is also the same as in [2].

LEMMA 3.6. *For* $v = 1, 2, 3, 4$ *and* $0 \le j \le n - v - 1$

$$\sum_{k=0}^{n-v} \frac{\left((n+1)\int_0^{1/(n+1)} p'_{n,v,k}\left(j/(n+1)+t\right)dt\right)^2}{p_{n,v,k}\left(j/(n+1)\right)} \le C_v n\varphi^{-2}\left(\frac{j+1}{n-v+2}\right).$$

4. PROOF OF THEOREM 3.1

Before proving Theorem 3.1, we verify some identities and inequalities. Following the notations in Section 3, we define for $v = 1, 2, \dots, i = M_0, \dots, M$

$$a_{n,v} := \frac{n(n-1)\cdots(n-v+2)}{(n+1)^{v-1}}$$

and

$$p_{k_i,\dots,k_M,v}(x) := a_{n,v}^{M-i} p_{n-v,k_M}(x) \prod_{\mu=i}^{M-1} p_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right)$$

with the understanding $\prod_{\mu=M}^{M-1} = 1$. We also define the following quantities:

$$l_{j,v}^* := \frac{(n-v+1)\int_0^{1/(n+1)} p'_{n,v,k_j}\left(k_{j+1}/(n+1)+t\right)dt}{p_{n,v,k_j}\left(k_{j+1}/(n+1)\right)}$$

and

$$l_{j,v} := \frac{(n-v+1)\, p_{n,v+1,k_j}\left(k_{j+1}/(n+1)\right)}{(n+1)\, p_{n,v,k_j}\left(k_{j+1}/(n+1)\right)}$$

with $j = M_0, \dots, M-1$. Furthermore,

$$q_{M-1,v} := l_{M-1,v}^*, \quad q_{j,v} := l_{j,v}^* l_{j+1,v} \cdots l_{M-1,v}, \qquad j = M_0, \dots, M-2$$

and

$$q_{k_{M_0},\dots,k_M,v} = \sum_{j=M_0}^{M-1} q_{j,v}.$$

The reader should note that by the definition of the fundamental functions of the Bernstein polynomials (see Section 1) all quantities above are well defined. Moreover, one has $l_{j,v} = 0$ if $k_j \ge n - v + 1$. With these notations, we have the following basic identity: for $f \in C^{v-1}[0,1]$ and $1 \le j \le N-1$, there holds

$$\begin{aligned}\left(K_n^N f\right)^{(v)} = &\sum_{k_1=0}^{n-v+1} \cdots \sum_{k_N=0}^{n-v+1} \frac{(n+1)!}{(n-v+1)!} \\ &\times \int_0^{1/n+1} \cdots \int_0^{1/n+1} f^{(v-1)}\left(\frac{k_1}{n+1} + t_1 + \cdots + t_v\right) dt_1 \cdots dt_v p_{k_1,\dots,k_N,v} q_{j,v}.\end{aligned} \tag{4.1}$$

In fact, for the fundamental functions of the Bernstein polynomials, we have (see [10])

$$p'_{n,k}(x) = n(p_{n-1,k-1}(x) - p_{n-1,k}(x)). \tag{4.2}$$

Using this and Abel's transformation, we get

$$(K_n f)^{(v-1)} = \frac{(n+1)!}{(n-v+1)!} \times \sum_{k=0}^{n-v+1} \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} f^{(v-1)}\left(\frac{k}{(n+1)} + t_1 + \cdots + t_v\right) dt_1 \cdots dt_v p_{n-v+1,k}(x).$$

Applying this formula to $K_n^{N-1} f, \ldots, K_n f$ instead of the function f there and working out, we deduce

$$(K_n^N f)^{(v-1)} = a_{n,v}^{N-1} \sum_{k_1=0}^{n-v+1} \cdots \sum_{k_N=0}^{n-v+1} \frac{(n+1)!}{(n-v+1)!} \times \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} f^{(v-1)}\left(\frac{k_1}{n+1} + t_1 + \cdots + t_v\right) dt_1 \cdots dt_v p_{n-v+1,k_N} \prod_{\mu=1}^{N-1} p_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right).$$

We then take derivatives on both sides of the above. Now consider the terms of the right side. Those which depend on k_N are of course

$$p'_{n-v+1,k_N}(x) p_{n,v,k_{N-1}}\left(\frac{k_N}{n+1}\right).$$

Hence, by (4.2) and the Abel transformation with respect to k_N, these terms change to

$$(n-v+1)\, p_{n-v,k_N}(x) \int_0^{1/(n+1)} p'_{n,v,k_{N-1}}\left(\frac{k_N}{n+1} + t\right) dt = p_{n-v,k_N}(x) p_{n,v,k_{N-1}}\left(\frac{k_N}{n+1}\right) q_{N-1,v}.$$

In other words, (4.1) holds for $j = N-1$. Let us deal next with the general case. Suppose we have proved (4.1) for $j = \mu$. In order to show the case $j = \mu - 1$, we note that (4.2) and the Abel transformation imply in our notations

$$\sum_{k=0}^{n-v+1} g\left(\frac{k}{n+1}\right) \int_0^{\frac{1}{n+1}} p'_{n,v,k}(x+t)\, dt = \sum_{k=0}^{n-v+1} \int_0^{\frac{1}{n}} g'\left(\frac{k}{n+1} + t\right) dt \, \frac{n-v+1}{n+1} p_{n,v+1,k}(x).$$

That is $g(k/n+1)$ replaced by

$$\int_0^{1/(n+1)} g'\left(\frac{k}{n+1} + t\right) dt$$

and

$$\int_0^{1/(n+1)} p'_{n,v,k}(x+t)\, dt$$

by

$$\frac{n-v+1}{n+1} p_{n,v+1,k}(x),$$

respectively. Now we observe that the expression $p_{k_1,\ldots,k_N}(x) q_{\mu,v}$ has only the factor

$$p_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}\right) p_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right) l_\mu^* = p_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}\right) (n-v+1) \int_0^{1/(n+1)} p'_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1} + t\right) dt$$

depending on k_μ. Thus, the Abel transformation with respect to k_μ is simply to replace

$$(n-v+1)p_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}\right) \quad \text{by} \quad \int_0^{1/(n+1)} p'_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}+t\right)dt$$

and

$$\int_0^{1/(n+1)} p'_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}+t\right)dt \quad \text{by} \quad \frac{n-v+1}{n+1}p_{n,v+1,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right),$$

respectively, and therefore

$$p_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}\right)p_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right)l^*_\mu \quad \text{by} \quad p_{n,v,k_{\mu-1}}\left(\frac{k_\mu}{n+1}\right)p_{n,v,k_\mu}\left(\frac{k_{\mu+1}}{n+1}\right)l^*_{\mu-1}l_\mu.$$

That is nothing but the expression $p_{k_1,\dots,k_N}(x)q_{\mu,v}$ of (4.1) changes to $p_{k_1,\dots,k_N}(x)q_{\mu-1,v}$. Thus, (4.1) holds also for $j=\mu-1$. In this way, we get (4.1) for all $1\le j\le N-1$. After we proved (4.1), we then take the sum on the both sides of (4.1) with respect to $j=1,\dots,N-1$ to obtain

$$(N-1)\left(K_n^N f\right)^{(v)} = \sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}\frac{(n+1)!}{(n-v+1)!}$$
$$\times\int_0^{1/(n+1)}\cdots\int_0^{1/(n+1)} f^{(v-1)}\left(\frac{k_1}{n+1}+t_1+\cdots+t_v\right)dt_1\cdots dt_v p_{k_1,\dots,k_N,v}q_{k_1,\dots,k_N,v}. \quad (4.3)$$

We note that (4.3) is in fact an important step to get the lower estimate for some operators (see also [2]).

Under these new notations, Lemma 3.3 means that, for $1\le N\le Cn$ and $0\le i\le 2v$,

$$\sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}\varphi^{-i}\left(\frac{k_1+1}{n+2}\right)p_{k_1,\dots,k_N,v}(x)\le C_v\min\left\{\varphi^{-i}(x),n^{i/2}\right\}\ln(N+1). \quad (4.4)$$

Moreover, by Lemmas 3.3 and 3.6,

$$\sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}p_{k_1,\dots,k_N,v}(x)q^2_{j,v}\le C_v n\varphi^{-2}(x), \quad 1\le v\le 4;\quad 1\le j\le N-1,$$

and by the definition of $q_{j,v}$

$$\sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}p_{k_1,\dots,k_N,v}(x)q_{j,v}q_{l,v}=0, \qquad j\ne l \qquad j,l=1,2,\dots,N-1.$$

Thus, for $1\le v\le 4$,

$$\sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}p_{k_1,\dots,k_N,v}(x)q^2_{k_1,\dots,k_N,v}\le CNn\varphi^{-2}(x). \quad (4.5)$$

Finally, Lemma 3.5 implies for $1\le N\le Cn$

$$\sum_{k_1=0}^{n-v+1}\cdots\sum_{k_N=0}^{n-v+1}\left|\frac{k_1}{n+1}-x\right|^j p_{k_1,\dots,k_N,v}(x)\le C_{j,v}\left(\sqrt{\frac{N}{n}}\varphi(x)+\frac{N}{n}\right)^j. \quad (4.6)$$

We need some more inequalities. Noticing $(K_n^N f)^{(v-1)}=0$ if $f\in\Pi_{v-2}$, we can replace $f^{(v-2)}(k_1/(n+1)+t_1+\cdots+t_v)$ in (4.3) by $\int_y^{k_1/(n+1)+t_1+\cdots+t_v} f^{(v-1)}(u)\,du$ for any $y\in[0,1]$.

In what follows, we shall take $K_n^N f$ in place of f in (4.3) and then substitute the expression of $(K_n^N f)^{(v-1)}$ by (4.3), in which

$$f^{(v-2)}\left(\frac{k_1}{n+1}+t_1+\cdots+t_{v-1}\right)$$

is replaced by

$$\int_x^{k_1/(n+1)+t_1+\cdots+t_{v-1}} f^{(v-1)}(u)\,du.$$

In this way, we obtain

$$\begin{aligned}(N-1)^2&\left(K_n^{2N}f\right)^{(v)}\\ &= a_{n,v}\int_0^{1/(n+1)}\cdots\int_0^{1/(n+1)}\sum_1\sum_2 I_{k_1}\left(f^{(v-1)}\right)P_{1,v}Q_{1,v}P_{2,v}Q_{2,v}\,dt_1\cdots dt_v,\end{aligned} \tag{4.7}$$

where $a_{n,v}:=(n+1)\cdots(n-v+2)$, $\sum_1 := \sum_{k_1=0}^{n-v+2}\cdots\sum_{k_N=0}^{n-v+2}$, $\sum_2 := \sum_{k_{N+1}=0}^{n-v+1}\cdots\sum_{k_{2N}=0}^{n-v+1}$,

$$\begin{aligned}I_{k_1}(f^{(v-1)})&:=(n+1)\cdots(n-v+3)\int_0^{\frac{1}{n+1}}\cdots\int_0^{\frac{1}{n+1}}\int_x^{\frac{k_1}{n+1}+u_2+\cdots+u_v} f^{(v-1)}(u_1)\,du_1\cdots du_v,\\ P_{1,v}&:=p_{k_1,\ldots,k_N,v-1}\left(\frac{k_{N+1}}{n+1}+t_1+\cdots+t_v\right),\qquad Q_{1,v}:=q_{k_1,\ldots,k_N,v-1},\\ P_{2,v}&:=p_{k_{N+1},\ldots,k_{2N},v}(x),\qquad Q_{2,v}:=q_{k_{N+1},\ldots,k_{2N},v}.\end{aligned}$$

Now we are in the position to prove the following two inequalities: for $0\le j$, $0\le i\le \min\{v-1,v/2\}$, $2\le v\le 4$ and $2\le N\le CN$ one has

$$\begin{aligned}a_{n,v}&\int_0^{1/(n+1)}\cdots\int_0^{1/n+1}\sum_1\sum_2\left|\frac{k_1}{n+1}-x\right|^j P_{1,v}\cdot|Q_{1,v}|\cdot P_{2,v}\cdot|Q_{2,v}|\,dt_1\cdots dt_v\\ &\le C_j\left(\sqrt{\frac{N}{n}}\varphi(x)+\frac{N}{n}\right)^j\varphi^{-1}(x)\min\left\{\varphi^{-1}(x),\sqrt{n}\right\}nN(\ln N)^{1/4},\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}a_{n,v}&\int_0^{1/n+1}\cdots\int_0^{1/n+1}\sum_1\sum_2\varphi^{-i}\left(\frac{k_1+1}{n+1}\right)P_{1,v}\cdot|Q_{1,v}|\cdot P_{2,v}\cdot|Q_{2,v}|\,dt_1\cdots dt_v\\ &\le C_j\varphi^{-2-i}(x)nN\ln N.\end{aligned}\tag{4.9}$$

Indeed, we have

$$\begin{aligned}\sum_1\sum_2&\left|\frac{k_1}{n+1}-x\right|^j P_{1,v}\cdot|Q_{1,v}|\cdot P_{2,v}\cdot|Q_{2,v}|\\ &\le\left(\sum_1\sum_2\left|\frac{k_1}{n+1}-x\right|^{4j}P_{1,v}P_{2,v}\right)^{1/4}\left(\sum_1\sum_2 P_{1,v}\cdot P_{2,v}\cdot|Q_{2,v}|^2\right)^{1/4}\\ &\quad\times\left(\sum_1\sum_2 P_{1,v}\cdot|Q_{1,v}|^2\cdot P_{2,v}\cdot|Q_{2,v}|\right)^{1/2}.\end{aligned}$$

To estimate the first factor on the right-hand side of the above, we write

$$\left|\frac{k_1}{n+1}-x\right|^{4j}\le 2^{4j}\left\{\left|\frac{k_1}{n+1}-\left(\frac{k_N+1}{n+1}+t_1+\cdots+t_v\right)\right|^{4j}+\left|\frac{k_N+1}{n+1}+t_1+\cdots+t_v-x\right|^{4j}\right\}.$$

Thus, by (4.6) we get

$$\sum_1 \sum_2 \left|\frac{k_1}{n+1} - x\right|^{4j} P_{1,v}P_{2,v}$$
$$\le C_j \left\{ \sum_2 \left(\sqrt{\frac{N}{n}}\varphi\left(\frac{k_{N+1}}{n+1} + t_1 + \cdots + t_v\right) + \frac{N}{n}\right)^{4j} P_{2,v} + + \left(\sqrt{\frac{N}{n}}\varphi(x) + \frac{N}{n}\right)^{4j} \right\}.$$

On the other hand, as

$$\left(\sqrt{\frac{N}{n}}\varphi(t) + \frac{N}{n}\right)^{4j} \le C_j \left\{ \left(\sqrt{\frac{N}{n}}\varphi(x) + \frac{N}{n}\right)^{4j} + \left(\frac{N}{n}|t-x|\right)^{2j} \right\},$$

we see the last sum can be estimated by the second term due to (4.6). In short, we obtain

$$\sum_1 \sum_2 \left|\frac{k_1}{n+1} - x\right|^{4j} P_{1,v}P_{2,v} \le C_j \left(\sqrt{\frac{N}{n}}\varphi(x) + \frac{N}{n}\right)^{4j}.$$

Moreover, (4.5) yields

$$\sum_1 \sum_2 P_{1,v} \cdot P_{2,v} \cdot |Q_{2,v}|^2 = \sum_2 P_{2,v} \cdot |Q_{2,v}|^2 \le CnN\varphi^{-2}(x)$$

and

$$a_{n,v} \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} \sum_1 P_{1,v}|Q_{1,v}|^2 \, dt_1 \cdots dt_v \le CnN\varphi^{-2}\left(\frac{k_{N+1}+1}{n+2}\right).$$

Using (4.4) and (4.5) again, the last two inequalities imply

$$a_{n,v} \int_0^{1/(n+1)} \cdots \int_0^{1/(n+1)} \sum_1 \sum_2 P_{1,v} \cdot |Q_{1,v}|^2 \cdot P_{2,v} \cdot |Q_{2,v}| \, dt_1 \cdots dt_v$$
$$\le CnN \left(\sum_2 \varphi^{-4}\left(\frac{k_{N+1}+1}{n+2}\right) P_{2,v}\right)^{1/2} \left(\sum_2 P_{2,v}|Q_{2,v}|^2\right)^{1/2}$$
$$\le C\varphi^{-1}(x) \min\left\{\varphi^{-2}(x), n\right\} (nN)^{3/2} (\ln N)^{1/2}.$$

The inequality (4.8) follows from these estimates. (4.9) can be verified in the same way.

Now we are ready to prove Theorem 3.1. First we prove (3.2) and (3.3).

PROOF OF (3.2) AND (3.3). Both inequalities follow from the three inequalities below: for $2 \le N \le Cn$, one has

$$\left\|\varphi^2 \left(K_n^N f\right)^{(4)}\right\|_p \le C \left\|\varphi f^{(3)}\right\|_p \sqrt{\frac{n}{N}} \ln N, \tag{4.10}$$

$$\left\|\varphi \left(K_n^N f\right)^{(3)}\right\|_p \le C \left\|f^{(2)}\right\|_p \sqrt{\frac{n}{N}} \ln N, \tag{4.11}$$

and, for $j = 2, 3, 4$,

$$\left\|\left(K_n^N f\right)^{(j)}\right\|_p \le C \left\|f^{(j-1)}\right\|_p \frac{n}{\sqrt{N}} \ln N. \tag{4.12}$$

In fact, if they are valid, then, e.g., to prove (3.2), we may use (4.10) for $K_n^{2N} f$ instead of f and use (4.11) for $K_n^N f$, and finally (4.12). In this way, we get (3.2) for $3N$, which obviously implies (3.2) for any $2 \le N \le Cn$.

All three inequalities can be deduced by using (4.3)–(4.9). Here we verify only the most complex one, namely, (4.10). Furthermore, it is enough to prove it for $p = 1$ and $p = \infty$ in view of the Riesz-Thorin theorem (see [11]), since the operator defined by

$$L\left(\varphi f^{(3)}\right) := \varphi^2 \left(K_n^N f\right)^{(4)}$$

is linear in $\varphi f^{(3)}$.

To show (4.10) in case $p = \infty$ we employ (4.3) with $v = 4$ to get

$$N\left|\left(K_n^N f\right)^{(4)}\right| \le \left\|\varphi f^{(3)}\right\|_\infty \sum_{k_1=0}^{n-3} \cdots \sum_{k_N=0}^{n-3} \varphi^{-1}\left(\frac{k_1+1}{n+2}\right) p_{k_1,\dots,k_N,4} |q_{k_1,\dots,k_N,4}|.$$

To estimate the last sum, we use Cauchy's inequality, (4.4) and (4.5) to obtain

$$\sum_{k_1=0}^{n-3} \cdots \sum_{k_N=0}^{n-3} \varphi^{-1}\left(\frac{k_1+1}{n+2}\right) p_{k_1,\dots,k_N,4} |q_{k_1,\dots,k_N,4}| \le C\varphi^{-2}(x)(nN\ln N)^{1/2}.$$

Thus,

$$N\left|\left(K_n^N f\right)^{(4)}\right| \le C\left\|\varphi f^{(3)}\right\|_\infty \varphi^{-2}(x)(nN\ln N)^{1/2},$$

which proves (4.10) in case $p = \infty$.

Of course, the more complicated case is $p = 1$. We use the approach first used in [12]. Its modification was also used to give estimates for other operators (see [4]). What we apply here is essentially the modified form from [4, p. 146–147].

We define

$$F(l,x) := \left\{u : |u-x| \le (l+1)\left(\sqrt{\frac{N}{n}}\varphi(x) + \frac{N}{n}\right)\right\}, \qquad G(l,u) := \{x : x \in [0,1], u \in F(l,x)\}.$$

Thus, if k_1 satisfies

$$l \le \left|\frac{k_1}{n+1} - x\right| \left(\frac{\sqrt{N}}{n}\varphi(x) + \frac{N}{n}\right)^{-1} < l+1, \tag{4.13}$$

then for $I_{k_1}(f^{(3)})$ of (4.7) one has

$$I_{k_1}\left(f^{(3)}\right) \le C\left(\varphi^{-1}\left(\frac{k_1+1}{n+2}\right) + \varphi^{-1}(x)\right) \int_{F(l,x)} \varphi(u)\left|f^{(3)}(u)\right| du.$$

Next we divide $\sum_{k_1=0}^{n-2}$ according to (4.13). We obtain

$$\sum_1 \sum_2 I_k\left(f^{(3)}\right) P_{1,4} \cdot |Q_{1,4}| \cdot P_{2,4} \cdot |Q_{2,4}| \le C \sum_{l=0}^{\infty} \frac{1}{(l+1)^6} \int_{F(l,x)} \varphi(u)\left|f^{(3)}(u)\right| du$$

$$\times \sum_1 \sum_2 \left(\varphi^{-1}\left(\frac{k_1+1}{n+2}\right) + \varphi^{-1}(x)\right)\left(1 + \frac{\left|\frac{k_1}{n+1} - x\right|^6}{\left(\frac{\sqrt{N}}{n}\varphi(x) + \frac{N}{n}\right)^6}\right) P_{1,4} \cdot |Q_{1,4}| \cdot P_{2,4} \cdot |Q_{2,4}|.$$

It follows from (4.8) and (4.9) that

$$a_{n,4}\int_0^{1/(n+1)}\cdots\int_0^{1/(n+1)}\sum_1\sum_2\left(\varphi^{-1}\left(\frac{k_1+1}{n+2}\right)+\varphi^{-1}(x)\right)$$
$$\times\left(1+\frac{\left|\dfrac{k_1}{n+1}-x\right|^6}{\left(\dfrac{\sqrt{N}}{n}\varphi(x)+\dfrac{N}{n}\right)^6}\right)P_{1,4}\cdot|Q_{1,4}|\cdot P_{2,4}\cdot|Q_{2,4}|\,dt_1\cdots dt_4\le C\varphi^{-3}(x)nN\ln N.$$

Therefore, we get by (4.7)

$$N^2\left|\left(K_n^{2N}f(x)\right)^{(4)}\right|\le C\sum_{l=0}^{\infty}\frac{1}{(l+1)^6}\int\limits_{F(l,x)}\varphi(u)\left|f^{(3)}(u)\right|du\varphi^{-3}(x)nN\ln N.$$

Finally, using the estimate in [4, p. 147], we obtain

$$\left\|\varphi^2\left(K_n^{2N}f\right)^{(4)}\right\|_1\le C\frac{n\ln N}{N}\sum_{l=0}^{\infty}\frac{1}{(l+1)^6}\int_0^1\varphi(u)\left|f^{(3)}(u)\right|\int\limits_{G(l,u)}\varphi^{-1}(x)\,dx\,du$$
$$\le C\sqrt{\frac{n}{N}}(\ln N)\left\|\varphi f^{(3)}\right\|_1.$$

This completes the proof of (4.10) in case $p=1$. ∎

PROOF OF (3.1). Straightforward calculation shows that for $f\in\Pi_n$,

$$P(D)K_nf-K_nP(D)f=\frac{1}{12(n+1)^2}K_nP(D)f^{(2)}+$$
$$+(n+1)\sum_{k=0}^{n}\Delta_{1/(n+1)}\left\{\varphi^2\left(\frac{k}{n+1}\right)R_n\left(\frac{k}{n+1}\right)\right\}p_{n,k},\tag{4.14}$$

where

$$R_n(t)=\frac{(n+1)^2}{2}\int_0^{1/(n+1)}\int_0^{1/(n+1)}\int_t^{t+t_1-t_2}(t+t_1-t_2-u)^2f^{(4)}(u)\,du\,dt_1\,dt_2.$$

Using (2.15) we deduce from (4.14) that

$$\left\|\varphi^2(P(D)K_nf-K_n(P(D)f)''\right\|_p+\|(P(D)K_nf-K_nP(D)f)'\|_p\le C\Phi_{n,p}(f)\tag{4.15}$$

and $\Phi_{n,p}(K_n^jf)\le C_j\Phi_{n,p}(f)$. Thus for a polynomial f

$$\left\|P^2(D)K_n^Nf-P(D)K_n^NP(D)f\right\|_p$$
$$\le\sum_{j=0}^{N-1}\left\|P(D)\left\{K_n^jP(D)K_n^{N-j}f-K_n^{j+1}P(D)K_n^{N-j-1}f\right\}\right\|_p\le C_N\Phi_{n,p}(f).$$

Replacing f in (4.14) by $K_n^{N-v-1}(P(D)f)$ and using (2.15) again we obtain for $1\le j\le N-1$

$$\left\|P(D)K_n^NP(D)f-K_n^{N-j}P(D)K_n^jP(D)f\right\|_p$$
$$\le\sum_{v=0}^{N-1-j}\left\|K_n^vP(D)K_n^{N-v}P(D)f-K_n^{v+1}P(D)K_n^{N-v-1}P(D)f\right\|_p\le C_N\Phi_{n,p}(f).$$

Thus, it follows from the above two inequalities that

$$\left\| P^2(D)K_n^N f - \frac{1}{N-1} \sum_{j=1}^{N-1} K_n^{N-j} P(D) K_n^j P(D) f \right\|_p \le C_N \Phi_{n,p}(f).$$

Therefore, in order to complete the proof of (3.1) we need to verify

$$\left\| \sum_{j=1}^{N-1} K_n^{N-j} P(D) K_n^j f \right\|_p \le Cn\sqrt{N}\|f\|_p. \tag{4.16}$$

Recalling the definition of the iterates of K_n, we deduce with

$$l_j := \frac{(n+1)\int_0^{1/(n+1)} P(D)\, p_{n,k_j}\left(k_{j+1}/(n+1)+t\right) dt}{p_{n,1,k_j}\left(k_{j+1}/(n+1)\right)}$$

and $\sum := \sum_{k_1=0}^n \cdots \sum_{k_N=0}^n$,

$$\sum_{j=1}^{N-1} K_n^{N-j} P(D) K_n^j f = \sum (n+1) \int_0^{1/(n+1)} f\left(\frac{k_1}{n+1}+t\right) dt\, p_{k_1,\ldots,k_N,1} \sum_{j=1}^{N-1} l_j. \tag{4.17}$$

The Riesz-Thorin theorem shows that to verify (4.16) for all $1 \le p \le \infty$ it is enough to do this for $p=1$ and $p=\infty$. By (4.17) it suffices to verify

$$\left\| \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \left| \sum_{j=1}^{N-1} l_j \right| \right\|_1 \le C\sqrt{N}, \tag{4.18}$$

and

$$\left\| \sum_{k_1=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \left| \sum_{j=1}^{N-1} l_j \right| \right\|_\infty \le C\sqrt{N}n.$$

As the proofs of the two inequalities are analogous, we show here the more difficult one, namely (4.18). Cauchy's inequality implies

$$\left\| \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \left| \sum_{j=1}^{N-1} l_j \right| \right\|_1 \le \left(\frac{1}{n+1}\right)^{1/2} \left\| \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \left(\sum_{j=1}^{N-1} l_j \right)^2 \right\|_1^{1/2},$$

as

$$\left\| \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \right\|_1 = \frac{1}{n+1}.$$

On the other hand, it is not hard to see that

$$\int_0^1 \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} l_j l_i \, dx = 0, \qquad i \ne j.$$

We obtain

$$\left\| \sum_{k_2=0}^n \cdots \sum_{k_N=0}^n p_{k_1,\ldots,k_N,1} \left(\sum_{j=1}^{N-1} l_j \right)^2 \right\|_1 = \sum_{j=1}^{N-1} \left(\sum_{k_2=0}^n \cdots \sum_{k_N=0}^n \int_0^1 p_{k_1,\ldots,k_N,1} l_j^2 \, dx \right).$$

To complete the proof, it is therefore sufficient to show that the sum in parentheses of the right-hand side of the above is not larger than Cn. To this end, we simplify this sum using $\int_0^1 p_{n,k}(t)\,dt = (n+1)^{-1}$. We see that all we need is to verify, for $0 \le l \le n$,

$$\sum_{k=0}^{n} \frac{(n+1)^2 \left(\int_0^{1/(n+1)} P(D) p_{n,l}\left(k/(n+1)+t\right)\,dt \right)^2}{p_{n,1,l}\left(k/(n+1)\right)} \le Cn^2.$$

But this is almost immediate if we divide this sum into $\sum_{k=1}^{n-1}$ and the remaining part and use the expression of $P(D)p_{n,k}(t)$ for the first one and some direct calculations for the remaining part. ∎

5. THE PROOFS OF THEOREMS 1.1 AND 1.2

PROOF OF THEOREM 1.1. The upper estimate follows from Theorem 2.2 and the definition of the K-functional.

After proving Theorem 3.1, the lower estimate is not too difficult to obtain. We notice that by the definition of our K-functional one has

$$K\left(f, n^{-1/2}\right)_p \le \|f - K_n f\|_p + \frac{1}{n}\|P(D)\,K_n f\|_p. \tag{5.1}$$

Thus, all we have to do is to show that there exists a positive constant C such that for all $n = 1, 2, \dots$

$$\frac{1}{n}\|P(D)K_n f\|_p \le C\|f - K_n f\|_p. \tag{5.2}$$

For this purpose, we prove next the following two inequalities: for $n \ge N'$ one has

$$\frac{1}{n}\|P(D)K_n f\|_p \le C\|f - K_n f\|_p + \frac{C}{n^2}\left\|\varphi^4 (K_n f)^{(4)}\right\|_p \tag{5.3}$$

and

$$\frac{1}{n^{3/2}}\|\varphi(P(D)K_n f)'\|_p \le C\|f - K_n f\|_p + \frac{C}{n^{5/2}}\left\|\varphi^5 (K_n f)^{(5)}\right\|_p. \tag{5.4}$$

To verify (5.3), we note that by using Lemma 2.1 (see (2.1)) we get with $g = K_n f$

$$\frac{1}{2(n+1)}\|P(D)K_n f\|_p \le \|f - K_n f\|_p + C\left\{\left\|\varphi^4(K_n f)^{(4)}\right\|_p + \left\|(K_n f)^{(2)}\right\|_p + \|K_n f\|_p\right\} n^{-2}.$$

Now, if $1 \le p < \infty$, then (see [4, p. 135])

$$\|g''\|_p \le C\left\{\left\|\varphi^4 g^{(4)}\right\|_p + \|g\|_p\right\}.$$

If $p = \infty$, then by (3.3) of Theorem 3.1 for $n \ge N$ (note $\|(K_n f)'\|_\infty \le C\|P(D)K_n f\|_\infty$)

$$\begin{aligned} \frac{1}{n^2}\left\|(K_n f)^{(2)}\right\|_\infty &\le \frac{1}{n^2}\left\|\left(K_n f - K_n^N f\right)''\right\|_\infty + \frac{1}{n^2}\left\|\left(K_n^N f\right)''\right\|_\infty \\ &\le C_N\|f - K_n f\|_\infty + \frac{C\ln N}{n\sqrt{N}}\|P(D)K_n f\|_\infty. \end{aligned}$$

Hence, in all cases, we get for proper N' fixed and $n \ge N'$

$$\frac{1}{2(n+1)}\|P(D)K_n f\|_p \le C\|f - K_n f\|_p + C\left\{\left\|\varphi^4(K_n f)^{(4)}\right\|_p + \|K_n f\|_p\right\} n^{-2}. \tag{5.5}$$

As $K_n(f-a) = K_n(f) - a$ for any constant a, one can replace $\|K_nf\|_p$ in (5.5) by $\|(K_nf)'\|_p$. But since $K_nf \in \Pi_n$ we have (see [4, p. 91])

$$\|(K_nf)'\|_p \le C\|(K_nf)'\|_{L_p[1/n^2,1-1/n^2]}.$$

Hence, for $F := (\varphi^2(K_nf)')'$ one gets

$$\|(K_nf)'\|_{L_p[1/n^2,1-1/n^2]} \le C\ln n\|F\|_p = C\ln n\|P(D)K_nf\|_p.$$

Combining this with (5.5) we get (5.3). The proof of (5.4) is similar to the proof of (5.3). The only difference is instead of using (2.1) there we shall use (2.2).

In what follows, we shall verify (5.2) for large n. More clearly, we will show that there exist C and N_0 so that for all $n \ge N_0$ one has (5.2). We consider the last term of (5.3). Using (2.15) and (2.16) this can be estimated in the following way:

$$\frac{1}{n^2}\left\|\varphi^4(K_nf)^{(4)}\right\|_p \le CN\|f-K_nf\|_p + Cn^{-3/2}\left\|\varphi\left(P(D)K_n^{N+3}f\right)'\right\|_p$$

and using (5.4) and (2.17) one also has

$$\frac{1}{n^{3/2}}\left\|\varphi\left(P(D)K_n^{N+3}f\right)'\right\|_p \le C\|f-K_nf\|_p + \frac{C}{n^2}\left\|P^2(D)K_n^{N+1}f\right\|_p.$$

Now it follows from (5.3) and the above that

$$\frac{1}{n}\|P(D)K_nf\|_p \le CN\|f-K_nf\|_p + \frac{C}{n^2}\left\|P^2(D)K_n^{N+1}f\right\|_p. \tag{5.6}$$

We then estimate the last term of (5.6) by using (3.1) to obtain

$$\frac{C}{n^2}\left\|P^2(D)K_n^{N+1}f\right\|_p \le \frac{C}{n\sqrt{N}}\|P(D)K_nf\|_p + \frac{C_N}{n^2}\Phi_{n,p}(K_nf).$$

Thus for N large enough such that $C/\sqrt{N} < 1/8$ (say $N = N''$), we obtain from (5.6) and the last inequality that for $n \ge \max\{N', N''\} =: N_0$

$$\frac{1}{n}\|P(D)K_nf\|_p \le C_{N_0}\|f-K_nf\|_p + \frac{C_{N_0}}{n^2}\Phi_{n,p}(K_nf). \tag{5.7}$$

Thus, we must show that

$$n^{-2}\Phi_{n,p}(K_nf) \le C\|f-K_nf\|_p.$$

Recalling the definition of $\Phi_{n,p}$ (see the definition in Theorem 3.1), we have to verify

$$n^{-3}\left\|\varphi^2(K_nf)^{(4)}\right\|_p + n^{-4}\left\|(K_nf)^{(4)}\right\|_p + n^{-4}\|(K_nf)'\|_p \le C\|f-K_nf\|_p.$$

To this end, we write K_nf as a sum of terms of the form $K_n^if - K_n^{i+1}f$ with $i \le n$ and apply (3.2) of Theorem 3.1 to obtain

$$\left\|\varphi^2(K_nf)^{(4)}\right\|_p \le Cn^3\|f-K_nf\|_p + C(\ln n)^3 n^{1/2}\|(K_nf)'\|_p. \tag{5.8}$$

Similarly applying (3.3) of Theorem 3.1 one gets

$$\left\|(K_nf)^{(4)}\right\|_p \le Cn^4\|f-K_nf\|_p + C(\ln n)^3 n^{3/2}\|(K_nf)'\|_p. \tag{5.9}$$

Hence, in order to complete the proof in case $n \geq N_0$, it remains to verify

$$n^{-5/2}(\ln n)^3\|(K_nf)'\|_p \leq C\|f - K_nf\|_p. \tag{5.10}$$

The following fact can be easily verified:

$$\|(K_n^if)'\|_\infty \leq Cn\left(\frac{n}{n+1}\right)^{i-1}\|f\|_\infty, \qquad \forall i \geq 1. \tag{5.11}$$

On the other hand, (2.15) implies (choose $i = \tau = 1$ there)

$$\left\|\left(K_nf - K_n^2f\right)'\right\|_p \leq Cn\|f - K_nf\|_p \quad \text{and} \quad \left\|\left(K_n^if\right)'\right\|_1 \leq Cn\|f\|_1.$$

By the first inequality, it is clear that we need only to prove (5.10) for K_n^4 on the left-hand side instead of K_n there. Moreover, using (5.11) and the last inequality, we have via the Riesz-Thorin Theorem, for $1 < p \leq \infty$,

$$\begin{aligned}\left\|\left(K_n^4f\right)'\right\|_p &\leq Cn\sum_{i=2}^{\infty}\left(\frac{n}{n+1}\right)^{i(1-1/p)}\left\|K_nf - K_n^2f\right\|_p\\ &\leq \frac{Cn}{1-(n/(n+1))^{1-1/p}}\|f - K_nf\|_p,\end{aligned}$$

which obviously implies (5.10) in case $p \geq 3/2$. If $1 \leq p \leq 3/2$, then since $\|K_nf\|_\infty \leq Cn\|f\|_p$, using the above estimate we have

$$\begin{aligned}\left\|\left(K_n^4f\right)'\right\|_p \leq \left\|\left(K_n^4f\right)'\right\|_{3/2} &\leq Cn^2\left\|K_nf - K_n^2f\right\|_{3/2}\\ &\leq Cn^2\|f - K_nf\|_p^{2/3}\left\|K_nf - K_n^2f\right\|_\infty^{(2/3)(3/2-p)}\\ &\leq Cn^{2+(2/3)(3/2-p)}\|f - K_nf\|_p,\end{aligned}$$

which implies (5.10) in case $1 \leq p \leq 3/2$ as $3 - 2p/3 < 5/2$.

Next we deal with the case $1 \leq p \leq \infty$ and $1 \leq n \leq N_0$. Obviously, by (2.15), one has

$$\left\|\varphi^2\left(K_n^3f\right)''\right\|_p \leq C\sqrt{n}\left\|\varphi\left(K_n^2f\right)'\right\|_p \leq Cn\left\|\left(K_n^2f\right)'\right\|_\infty,$$

and therefore

$$\|P(D)K_nf\|_p \leq Cn\|f - K_nf\|_p + \left\|P(D)K_n^3f\right\|_p \leq Cn\|f - K_nf\|_p + Cn\left\|\left(K_n^2f\right)'\right\|_\infty.$$

Writing K_n^2f as a sum of terms of the form $K_n^{i+1}f - K_n^if$ and using (5.11), we get (notice again $\|K_nf\|_\infty \leq Cn\|f\|_p$)

$$\begin{aligned}\left\|\left(K_n^2f\right)'\right\|_\infty &\leq Cn\sum_{i=0}^{\infty}\left(\frac{n}{n+1}\right)^i\left\|K_nf - K_n^2f\right\|_\infty \leq Cn^2\|K_n(f - K_nf)\|_\infty\\ &\leq Cn^3\|f - K_nf\|_p \leq CN_0^3\|f - K_nf\|_p.\end{aligned}$$

Hence, (5.2) holds. The proof is complete. ∎

PROOF OF THEOREM 1.2. We have, for $1 < p < \infty$ and arbitrary $g \in C^2[0,1]$,

$$\|P(D)g\|_p \sim \left\|\varphi^2g''\right\|_p + \|g'\|_p. \tag{5.12}$$

On the other hand (see [4, p. 135]), for any constant c,

$$\|g'\|_p \le C\left(\left\|\varphi^2 g''\right\|_p + \|g-c\|_p\right).$$

Thus,

$$\|f-g\|_p + t^2\|P(D)g\|_p \le C\left\{\|f-g\|_p + t^2\left\|\varphi^2 g''\right\|_p + t^2 E_0(f)_p\right\}.$$

We get, by using the equivalence of the modulus of smoothness and the K-functional,

$$K(f,t)_p \le C\left\{\omega_\varphi^2(f,t)_p + t^2 E_0(f)_p\right\}.$$

But, in case $1 < p \le \infty$,

$$\left\|\varphi^2 g''\right\|_p \le C\|P(D)g\|_p, \qquad \|g'\|_p \le C\|P(D)g\|_p.$$

Therefore, again using the equivalence of the modulus of smoothness and the K-functional, we obtain

$$\omega_\varphi^2(f,t)_p \le CK(f,t)_p, \qquad \omega(f,t^2)_p \le CK(f,t)_p.$$

Obviously,

$$E_0(f)_p \le C\frac{\omega(f,1)_p}{1} \le C\frac{\omega(f,t^2)_p}{t^2}.$$

Hence in the case $1 < p < \infty$ there also holds

$$\omega_\varphi^2(f,t)_p + t^2 E_0(f)_p \le CK(f,t)_p.$$

For $p=\infty$, we get from the above

$$\omega_\varphi^2(f,t)_\infty + \omega(f,t^2)_\infty \le CK(f,t)_\infty.$$

On the other hand,

$$K(f,t)_\infty \le C\inf\left\{\|f-g\|_\infty + t^2\left\|\varphi^2 g''\right\|_\infty + t^2\|g'\|_\infty\right\}.$$

For some $n \sim t^{-2}$ choosing $g = B_n(f)$ (the Bernstein polynomial of f), we get for arbitrary $h \in C^1[0,1]$

$$\begin{aligned} K(f,t)_\infty &\le C\left(\omega_\varphi^2(f,t)_\infty + t^2\|B_n' f\|_\infty\right) \\ &\le C\left(\omega_\varphi^2(f,t)_\infty + t^2\|B_n'(f-h) + B_n'(h)\|_\infty\right) \\ &\le C\left(\omega_\varphi^2(f,t)_\infty + t^2 n\|f-h\|_\infty + t^2\|h'\|_\infty\right). \end{aligned}$$

Here in the first inequality, we have used (1.1) and in the last step the simple facts: $\|(B_n f)'\| \le \|f'\|$ and $\|(B_n f)'\| \le 2n\|f\|$. Taking the infimum over $h \in C^1[0,1]$, we get

$$K(f,t)_\infty \le C\left(\omega_\varphi^2(f,t)_\infty + \omega\left(f,t^2\right)_\infty\right).$$

The proof of Theorem 2.2 is thus complete. ∎

REFERENCES

1. Z. Ditzian and K.G. Ivanov, Strong converse inequalities, *J. d'Analyse Mat.* **61**, 61–111, (1993).
2. H.-B. Knoop and X.-l. Zhou, *The Lower Estimate for Linear Positive Operators*, Schriftenreihe des Fachbereichs Mathematik, Duisburg, SM-DU-201, (1992).
3. V. Totik, Approximation by Bernstein polynomials, Manuscript, (1992).
4. Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer, New York, (1987).
5. V. Totik, Approximation by algebraic polynomials, In *Approximation Theory VII*, (Edited by E.W. Cheney, C.K. Chui, and L.L. Schumaker), pp. 227–249, Academic Press, New York, (1992).
6. Z. Ditzian and X.-l. Zhou, Kantorovich-Bernstein polynomials, *Constr. Approx.* **6**, 421–435, (1990).
7. H. Berens and Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights, In *Approximation Theory and Functional Analysis*, (Edited by C.K. Chui), pp. 25–46, Academic Press, New York, (1990).
8. M.M. Derriennic, On multivariate approximation by Bernstein-type polynomials, *J. Approx. Theory* **45**, 155–166, (1985).
9. H.H. Gonska and X.-l. Zhou, A global inverse theorem on simultaneous approximation by Bernstein-Durrmeyer operators, *J. Approx. Theory* **67**, 284–302, (1991).
10. G.G. Lorentz, Bernstein Polynomials, In *Mathematical Expositions*, No. 8, University of Toronto Press, Toronto, (1953).
11. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, New York, (1976).
12. R. Bojanic and O. Shisha, Degree of L^1 approximation to integrable functions by modified Bernstein polynomials, *J. Approx. Theory* **13**, 66–72, (1975).