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On dimensional rigidity of bar-and-joint frameworks

Abdo Y. Alfakih*,[1]

Department of Mathematics and Statistics, University of Windsor, Windsor, Ont. Canada N9B 3P4

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Abstract

Let $V = \{1, 2, \ldots, n\}$. A mapping $p : V \to \Re^r$, where $p^1, \ldots, p^n$ are not contained in a proper hyper-plane is called an *r-configuration*. Let $G = (V, E)$ be a simple connected graph on n vertices. Then an r-configuration p together with graph G, where adjacent vertices of G are constrained to stay the same distance apart, is called a *bar-and-joint framework* (or a *framework*) in $\Re^r$, and is denoted by $G(p)$. In this paper we introduce the notion of dimensional rigidity of frameworks, and we study the problem of determining whether or not a given $G(p)$ is dimensionally rigid. A given framework $G(p)$ in $\Re^r$ is said to be dimensionally rigid iff there does not exist a framework $G(q)$ in $\Re^s$ for $s \geqslant r + 1$, such that $\|q^i - q^j\|^2 = \|p^i - p^j\|^2$ for all $(i, j) \in E$. We present necessary and sufficient conditions for $G(p)$ to be dimensionally rigid, and we formulate the problem of checking the validity of these conditions as a semidefinite programming (SDP) problem. The case where the points $p^1, \ldots, p^n$ of the given r-configuration are in general position, is also investigated.

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1. Introduction

Let $V = \{1, 2, \ldots, n\}$ be a finite set. A mapping $p : V \to \Re^r$, where $p^1, \ldots, p^n$ are not contained in a proper hyper-plane is called a *configuration* in $\Re^r$ (or an *r-configuration*). Let $G = (V, E)$ be a simple connected graph on n vertices, i.e., G has no loops or multiple edges. Then an r-configuration p together with graph G, where adjacent vertices of G are constrained to stay the same distance apart, is called a *bar-and-joint framework* (or a *framework*) in $\Re^r$, and is denoted by $G(p)$. Let $G(p)$ be a given framework. Then each edge (i, j) of G can be viewed as a rigid bar of length equal to $\|p^i - p^j\|$, where $\|.\|$ denotes the Euclidean norm; and each node of G can be viewed as a joint. Furthermore, edges of G can freely rotate around their end nodes, and we assume that two edges may cross each other at a point other than a node. An example of two frameworks in $\Re^2$ is given Fig. 1, where the nodes (joints) are represented by little circles, while the edges (bars) are represented by straight lines.

* Corresponding author.
E-mail address: alfakih@uwindsor.ca.
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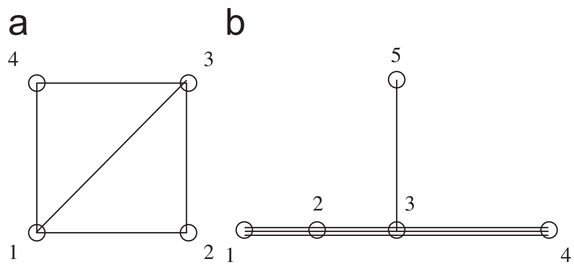

Fig. 1. An example of two frameworks in $\Re^2$. Framework $G_1(p)$ in (a), where $\bar{E}_1 = \{(2, 4)\}$, is rigid and dimensionally flexible; while framework $G_2(q)$ in (b), where $\bar{E}_2 = \{(1, 5), (2, 5), (4, 5)\}$, is flexible and dimensionally rigid.

Two frameworks $G(p)$ in $\Re^r$ and $G(q)$ in $\Re^s$ are said to be *equivalent*[2] if and only if $\|q^i - q^j\|^2 = \|p^i - p^j\|^2$ for all $(i, j) \in E$. Let p be an r-configuration. We will find it convenient to represent the points $p^1, \ldots, p^n$ in the form of an $n \times r$ matrix

$$P = \begin{bmatrix} p^{1\mathrm{T}} \\ \vdots \\ p^{n\mathrm{T}} \end{bmatrix},$$

and we will use the terms "framework $G(p)$" and "framework $G(P)$" interchangeably.

For each framework $G(P)$ in $\Re^r$, the $n \times n$ matrix $D = (d_{ij}) = \|p^i - p^j\|^2$ is called the *Euclidean distance matrix (EDM)* defined by P. Two r-configurations P and P' are said to be *congruent* iff P and P' define the same EDM. Thus, configurations obtained from each other by applying a rigid motion, such as a translation or a rotation, are congruent. In this paper, we do not distinguish between congruent configurations. Hence, without loss of generality, we will assume that the origin is the centroid of the points $p^1, \ldots, p^n$. i.e., $P^{\mathrm{T}} e = 0$, where e is the vector of all ones in $\Re^n$.

Two of the most studied problems concerning bar-and-joint frameworks are those of rigidity and generic rigidity. Given a framework $G(P)$ in $\Re^r$, the rigidity problem asks whether $G(P)$ is rigid or flexible [1,4,5,10,11,13,16]. $G(P)$ in $\Re^r$ is said to be *flexible*, if there exists a differentiable function $\gamma(t) : t \in [0, 1] \to \Re^{n\times r}$ such that $\gamma(0) = P$, $G(\gamma(t))$ is equivalent to $G(P)$, and $\gamma(t)$ is not congruent to P for all t, $0 < t \leqslant 1$. A framework $G(P)$ is said to be *rigid* iff it is not flexible.

In this paper, we introduce the notion of dimensional rigidity; and we study the problem of determining whether or not a given framework $G(p)$ in $\Re^r$ is dimensionally rigid. A given framework $G(p)$ in $\Re^r$ is said to be *dimensionally rigid* if and only if there does not exist a framework $G(q)$ in $\Re^s$, equivalent to $G(p)$, for $s \geqslant r+1$. A framework $G(p)$ is said to be *dimensionally flexible* if and only if it is not dimensionally rigid. Fig. 1 shows two examples of frameworks, one is dimensionally flexible while the other is dimensionally rigid. We present necessary and sufficient conditions for the dimensional rigidity of a given framework $G(P)$ in terms of Z, the Gale matrix corresponding to P; and we show that if $G(P)$ is both rigid and dimensionally rigid, then $G(P)$ is unique. We also formulate the problem of checking the validity of these conditions as a semidefinite program.

We denote by $\mathscr{S}_n$ the space of $n \times n$ real symmetric matrices. The inner product on $\mathscr{S}_n$ is given by

$$\langle A, B\rangle := \mathrm{tr}(AB),$$

where tr denotes the trace. Positive semidefiniteness (positive definiteness) of a symmetric matrix A is denoted by $A \succeq 0$ $(A \succ 0)$. For a matrix A in $\mathscr{S}_n$, diag(A) denotes the n-vector formed from the diagonal entries of A. We denote by e the vector of all ones in $\Re^n$; and we denote by E^{ij} the $n \times n$ symmetric matrix with ones in the (i, j)th and (j, i)th entries and zeros elsewhere. The $n \times n$ identity matrix will be denoted by I_n. Finally, " $\setminus$ " denotes the set theoretic difference.

[2] Many authors use the term "equivalent" only for frameworks in the same Euclidean space.

2. Preliminaries

In this section, we present some results on Euclidean distance matrices (EDMs) and Gale transform, which will be used in the paper. In particular, given a framework $G(p)$ in $\Re^r$, we present a characterization of the set of all frameworks $G(q)$ that are equivalent $G(p)$. We begin by reviewing some basic definitions and results on EDMs.

An $n \times n$ matrix $D = (d_{ij})$ is said to be a *EDM* if and only if there exist points $p^1, p^2, \ldots, p^n$ in some Euclidean space such that $d_{ij} = \|p^i - p^j\|^2$ for all $i, j = 1, \ldots, n$. The dimension of the affine subspace spanned by $p^1, \ldots, p^n$ is called the *embedding dimension* of D.

It is well known [7,9,14] that a symmetric $n \times n$ matrix D with zero diagonal is EDM if and only if D is negative semidefinite on the subspace

$$M := \{x \in \Re^n : e^{\mathrm{T}} x = 0\}.$$

Let V be the $n \times (n-1)$ matrix whose columns form an orthonormal basis of M; that is, V satisfies

$$V^{\mathrm{T}} e = 0, \quad V^{\mathrm{T}} V = I_{n-1}. \tag{1}$$

Then the orthogonal projection on M, denoted by J, is given by $J := VV^{\mathrm{T}} = I_n - ee^{\mathrm{T}}/n$. Hence, it readily follows that if D is a symmetric matrix with zero diagonal, then

$$D \text{ is EDM iff } \quad \mathscr{T}(D) := -\tfrac{1}{2} JDJ \succeq 0. \tag{2}$$

Furthermore, it is also well known that the embedding dimension of D is given by the rank of $\mathscr{T}(D)$.

There are many equivalent ways to represent a given r-configuration P of framework $G(p)$. For example, an r-configuration P can be represented by the EDM matrix D defined by P. Recall that $D = (d_{ij}) = \|p^i - p^j\|^2$, which is equivalent to

$$D = \mathscr{K}(PP^{\mathrm{T}}) := \mathrm{diag}(PP^{\mathrm{T}})e^{\mathrm{T}} + e(\mathrm{diag}(PP^{\mathrm{T}}))^{\mathrm{T}} - 2PP^{\mathrm{T}}. \tag{3}$$

Let $\mathscr{S}_H$ and $\mathscr{S}_C$ denote the subspaces of $\mathscr{S}_n$ defined by

$$\mathscr{S}_H := \{B \in \mathscr{S}_n : \mathrm{diag}(B) = 0\},$$
$$\mathscr{S}_C := \{B \in \mathscr{S}_n : Be = 0\}.$$

Then it can be easily verified that the linear operators $\mathscr{T} : \mathscr{S}_H \to \mathscr{S}_C$ and $\mathscr{K} : \mathscr{S}_C \to \mathscr{S}_H$ are mutually inverse [7]. Thus, given an EDM D with embedding dimension r, the r-configuration P generating D can be recovered as follows. Let $B = \mathscr{T}(D)$. Then $B \succeq 0$, $Be = 0$, and rank $B = r$. Thus, P is obtained by factorizing B as $B = PP^{\mathrm{T}}$. Note that $P^{\mathrm{T}} e = 0$ since $Be = 0$. Also, note that the factorization of B into PP^{T} is not unique. However, if $B = PP^{\mathrm{T}} = P'P'^{\mathrm{T}}$, then the two r-configurations P and P' are congruent. Thus, P and D uniquely determine each other.

Next, we present a third equivalent representation of an r-configuration P, which happens to be the most convenient for our purposes. Recall that $\mathscr{S}_{n-1}$ denote the space of symmetric matrices of order $n-1$; and consider the two linear operators $\mathscr{K}_V : \mathscr{S}_{n-1} \to \mathscr{S}_H$ and $\mathscr{T}_V : \mathscr{S}_H \to \mathscr{S}_{n-1}$ such that

$$\mathscr{K}_V(X) := \mathscr{K}(VXV^{\mathrm{T}}), \tag{4}$$

and

$$\mathscr{T}_V(B) := V^{\mathrm{T}} \mathscr{T}(B) V = -\tfrac{1}{2} V^{\mathrm{T}} B V, \tag{5}$$

where V is the $n \times (n-1)$ matrix defined in (1). Then we have the following lemma.

Lemma 2.1 (*Alfakih et al. [3]*). *The operators $\mathscr{T}_V$ and $\mathscr{K}_V$ are mutually inverse; and D in $\mathscr{S}_H$ is a EDM of embedding dimension r if and only if $\mathscr{T}_V(D) \succeq 0$ and rank $\mathscr{T}_V(D) = r$.*

Therefore, let D be the EDM matrix defined by the r-configuration P and let $X = \mathcal{T}_V(D)$. Then, D and X uniquely determine each other. Furthermore, we have the following relations:

$$\begin{aligned} X &= -\tfrac{1}{2} V^{\mathrm{T}} D V = V^{\mathrm{T}} P P^{\mathrm{T}} V, \\ D &= \mathcal{K}_V(X) = \mathcal{K}(P P^{\mathrm{T}}), \\ P P^{\mathrm{T}} &= V X V^{\mathrm{T}} = \mathcal{T}(D). \end{aligned} \tag{6}$$

Hence, P, D and X uniquely determine one another. In a slight abuse of notation, we will use the term r-configuration to refer to X as well as to P. Thus, the terms "framework $G(P)$" and "framework $G(X)$" can be used interchangeably.

Given framework $G(P_1)$ in $\Re^r$, let $\bar{G} = (V, \bar{E})$ denote the complement graph of G. i.e., $\bar{G}$ has the same set of nodes V, and edge set $\bar{E} = (V \times V) \backslash E$. Let $\bar{m}$ be the cardinality of $\bar{E}$. To avoid trivialities, assume that G is not a complete graph, thus $\bar{m} \geqslant 1$. For each edge of the complement graph $\bar{G}$ define the matrix

$$M^{ij} := \mathcal{T}_V(E^{ij}) = -\tfrac{1}{2} V^{\mathrm{T}} E^{ij} V, \tag{7}$$

where E^{ij} is the $n \times n$ matrix with ones in the (i, j)th and (j, i)th entries and zeros elsewhere. Let $X_1 = V^{\mathrm{T}} P_1 P_1^{\mathrm{T}} V$, and let

$$\Omega = \left\{ y \in \Re^{\bar{m}} : X(y) := X_1 + \sum_{(i,j) \in \bar{E}} y_{ij} M^{ij} \succeq 0 \right\}. \tag{8}$$

Then it was shown in [1] that the set of all frameworks $G(q)$ in $\Re^r$ that are equivalent to $G(P_1)$ is given by

$$\{G(X(y)) : y \in \Omega \quad \text{and rank } X(y) = r\}; \tag{9}$$

and that the set of all frameworks $G(q)$ in $\Re^s$, equivalent to $G(P_1)$, for some s, $1 \leqslant s \leqslant n-1$, is given by

$$\{G(X(y)) : y \in \Omega\}. \tag{10}$$

Note that Ω is a closed convex set that contains the origin. Furthermore, Ω is bounded since the graph G is connected.

Let $G(P)$ be a given framework in $\Re^r$. Consider the $(r+1) \times n$ matrix

$$\begin{bmatrix} P^{\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} p^1 & p^2 & \dots & p^n \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Recall that $p^1, \dots, p^n$ are not contained in a proper hyper-plane in $\Re^r$; that is, the affine subspace spanned by $p^1, \dots, p^n$ has dimension r. Then $r \leqslant n-1$, and the matrix $\begin{bmatrix} P^{\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}$ has full row rank. Let $\bar{r} = n - 1 - r$. If $\bar{r} = 0$, then framework $G(P)$ is obviously dimensionally rigid since the points $p^1, \dots, p^n$ are affinely independent. Therefore, without loss of generality, we assume that $\bar{r} \geqslant 1$. Let Λ be the $n \times \bar{r}$ matrix, whose columns form a basis for the null space of $\begin{bmatrix} P^{\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}$. Λ is called a *Gale matrix* corresponding to P; and the ith row of Λ, considered as a vector in $\Re^{\bar{r}}$, is called a *Gale transform* of p^i [8]. Note that Λ is not unique. In fact, for any nonsingular $\bar{r} \times \bar{r}$ matrix Q, the columns of ΛQ span the null space of $\begin{bmatrix} P^{\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}$. Hence, ΛQ is also a Gale matrix. We will exploit this property to define a special Gale matrix Z which is more sparse than Λ and more convenient for our purposes.

Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is $\bar{r} \times \bar{r}$ and Λ_2 is $(r+1) \times \bar{r}$. Since Λ has full column rank, we can assume without loss of generality that Λ_1 is nonsingular. Then Z is defined as

$$Z := \Lambda \Lambda_1^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 \Lambda_1^{-1} \end{bmatrix}. \tag{11}$$

Let ${z^i}^{\mathrm{T}}$ denote the ith row of Z; i.e.,

$$Z := \begin{bmatrix} {z^1}^{\mathrm{T}} \\ {z^2}^{\mathrm{T}} \\ \vdots \\ {z^n}^{\mathrm{T}} \end{bmatrix}.$$

Note that $z^1, z^2, \ldots, z^{\bar{r}}$, the Gale transforms of $p^1, p^2, \ldots, p^{\bar{r}}$, respectively, are simply the unit vectors in $\Re^{\bar{r}}$.

The following lemma enables us to express the necessary and sufficient conditions for the dimensional rigidity of a given framework $G(P)$ in terms of Z, the Gale matrix corresponding to P.

Lemma 2.2 (*Alfakih [2]*). *Let $G(P)$ be a given framework in $\Re^r$, and let Z be the Gale matrix corresponding to P. Further, let U and W be the matrices whose columns form orthonormal bases of the null space and the range space of $X = V^{\mathrm{T}} P P^{\mathrm{T}} V$, respectively. Then,*

1. *$VU = ZQ$ for some nonsingular matrix Q, i.e., VU is a Gale matrix.*
2. *$VW = PQ'$ for some nonsingular matrix Q'.*

Proof. Statement 1 holds since $XU=0$ iff $P^{\mathrm{T}}VU=0$, and since $e^{\mathrm{T}}VU=0$. Moreover, since $Z^{\mathrm{T}}VW=Q^{-\mathrm{T}}U^{\mathrm{T}}V^{\mathrm{T}}VW=0$ and since $e^{\mathrm{T}}VW=0$, statement 2 also holds. □

The following Farkas-type lemma is a special case of a known result (see [17, p. 171]). A proof is given for completeness.

Lemma 2.3. *Let $G(P)$ be a framework in $\Re^r$, and let M^{ij}, for $(i,j) \in \bar{E}$, be the matrices defined in (7). Let U be the $(n-1)\times\bar{r}$ matrix whose columns form an orthonormal basis for the null space of $X=V^{\mathrm{T}}PP^{\mathrm{T}}V$. Then the following two statements are equivalent.*

1. *There does not exist an $\bar{r}\times\bar{r}$ positive definite Ψ such that $\langle \Psi, U^{\mathrm{T}}M^{ij}U\rangle = 0$ for all $(i,j)\in\bar{E}$.*
2. *There exists a nonzero $\hat{y}\in\Re^{|\bar{E}|}$ such that $\sum_{(i,j)\in\bar{E}} \hat{y}_{ij}U^{\mathrm{T}}M^{ij}U$ is a nonzero $\bar{r}\times\bar{r}$ positive semidefinite matrix.*

Proof. Assume statement 1 holds, and let

$$\mathscr{L} = \{B \in \mathscr{S}_{\bar{r}} : \langle B, U^{\mathrm{T}}M^{ij}U\rangle = 0 \quad \text{for all } (i,j)\in\bar{E}\}.$$

Let $\mathscr{P}_{\bar{r}}$ denote the cone of $\bar{r}\times\bar{r}$ positive semidefinite matrices. Then $\mathscr{L}\cap$ interior of $\mathscr{P}_{\bar{r}}=\emptyset$. By the separation theorem [12, p. 96], there exists a nonzero $Y\in\mathscr{S}_{\bar{r}}$ such that $\langle Y, B\rangle=0$ for all $B\in\mathscr{L}$ and $\langle Y, C\rangle \geqslant 0$ for all C in the interior of $\mathscr{P}_{\bar{r}}$. i.e., for all $C\succ 0$. Therefore, $Y\succeq 0$ and $Y=\sum_{(i,j)\in\bar{E}}\hat{y}_{ij}U^{\mathrm{T}}M^{ij}U$ for some nonzero $\hat{y}\in\Re^{|\bar{E}|}$. Hence, statement 2 holds.

Now assume that statement 1 does not hold. If statement 2 holds, let $Y=\sum_{(i,j)\in\bar{E}}\hat{y}_{ij}U^{\mathrm{T}}M^{ij}U$. Then, on one hand $\langle\Psi, Y\rangle > 0$ since $\Psi\succ 0$ and $Y\succeq 0$, $Y\neq 0$. On the other hand $\langle \Psi, Y\rangle = \sum_{(i,j)\in\bar{E}}\hat{y}_{ij}\langle\Psi, U^{\mathrm{T}}M^{ij}U\rangle = 0$, hence we have a contradiction. Thus, statement 2 cannot hold and the result follows. □

3. Main results

In this section we present the main results of the paper.

Theorem 3.1. *Let $G(P_1)$ be a given framework in $\Re^r$ for some $r\leqslant n-2$, and let $\bar{r}$ be the nullity of $X_1=V^{\mathrm{T}}P_1P_1^{\mathrm{T}}V$. Further, let M^{ij}'s be the matrices defined in (7); and let U and W be the matrices whose columns form orthonormal bases for the null space and the range space of X_1, respectively. If the following condition holds*:

$$\exists\ \bar{r}\times\bar{r} \text{ matrix } \Psi\succ 0 : \langle\Psi, U^{\mathrm{T}}M^{ij}U\rangle = 0 \quad \forall (i,j)\in\bar{E}, \tag{12}$$

then $G(P_1)$ is dimensionally rigid. Otherwise, if (12) *does not hold, then $G(P_1)$ is dimensionally flexible iff*

$$\text{null space of } U^{\mathrm{T}}\mathscr{M}(\hat{y})U \subseteq \text{null space of } W^{\mathrm{T}}\mathscr{M}(\hat{y})U, \tag{13}$$

for some nonzero $\hat{y} \in \Re^{\bar{m}}$ such that $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$ is nonzero positive semidefinite, where $\mathscr{M}(\hat{y}) = \sum_{(i,j)\in\bar{E}} \hat{y}_{ij} M^{ij}$.

Proof. Let $Q = [WU]$. Then for some nonzero $\hat{y} \in \Re^{\bar{m}}$, $X_1 + \mathscr{M}(\hat{y}) \succeq 0$ if and only if $Q^{\mathrm{T}}(X_1 + \mathscr{M}(\hat{y}))Q \succeq 0$. But,

$$Q^{\mathrm{T}}(X_1 + \mathscr{M}(\hat{y}))Q = \begin{bmatrix} \Lambda + W^{\mathrm{T}}\mathscr{M}(\hat{y})W & W^{\mathrm{T}}\mathscr{M}(\hat{y})U \\ U^{\mathrm{T}}\mathscr{M}(\hat{y})W & U^{\mathrm{T}}\mathscr{M}(\hat{y})U \end{bmatrix},$$

where Λ is the diagonal matrix of the positive eigenvalues of X_1. Thus, $U^{\mathrm{T}}\mathscr{M}(\hat{y})U \succeq 0$ is a necessary condition for $X_1 + \mathscr{M}(\hat{y})$ to be positive semidefinite.

Now assume that Condition (12) holds and suppose that $G(P_1)$ is dimensionally flexible. Then by (9), there exists a nonzero $\hat{y} \in \Re^{\bar{m}}$ such that $X(\hat{y}) = X_1 + \mathscr{M}(\hat{y}) \succeq 0$ and rank $X(\hat{y}) \geqslant r+1$. Since Λ is $r \times r$, this implies that $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$ is a nonzero positive semidefinite matrix. But this contradicts Lemma 2.3. Thus, $G(P_1)$ is dimensionally rigid.

On the other hand, assume that Condition (12) fails to hold. Then, $G(P_1)$ is dimensionally flexible iff there exists a nonzero $\hat{y}$ such that $X(\hat{y}) = X_1 + \mathscr{M}(\hat{y}) \succeq 0$ and rank $X(\hat{y}) \geqslant r+1$. But this holds if and only if $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$ is nonzero positive semidefinite, and null space of $U^{\mathrm{T}}\mathscr{M}(\hat{y})U \subseteq$ null space of $W^{\mathrm{T}}\mathscr{M}(\hat{y})U$. Thus, the result follows. □

Corollary 3.1. *Let $G(P)$ be a given framework in $\Re^r$ for some $r \leqslant n-2$. If $G(P)$ is both rigid and dimensionally rigid, then $G(P)$ is unique.*

Proof. Let matrices Λ, W, U and X be as in Theorem 3.1, and assume that $G(P)$ is both rigid and dimensionally rigid. Now suppose that $G(P)$ is not unique. Then there exists a framework $G(q)$ in $\Re^s$, which is equivalent to $G(P)$, for some s, $1 \leqslant s \leqslant n-1$. Therefore, there exists a nonzero $\hat{y}$ in $\Re^{\bar{m}}$ such that $X(\hat{y}) = X_1 + \mathscr{M}(\hat{y}) \succeq 0$. i.e.,

$$\begin{bmatrix} \Lambda + W^{\mathrm{T}}\mathscr{M}(\hat{y})W & W^{\mathrm{T}}\mathscr{M}(\hat{y})U \\ U^{\mathrm{T}}\mathscr{M}(\hat{y})W & U^{\mathrm{T}}\mathscr{M}(\hat{y})U \end{bmatrix} \succeq 0.$$

Thus, $U^{\mathrm{T}}\mathscr{M}(\hat{y})U \succeq 0$ and null space of $U^{\mathrm{T}}\mathscr{M}(\hat{y})U \subseteq$ null space of $W^{\mathrm{T}}\mathscr{M}(\hat{y})U$. Now if $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$ is nonzero, we have a contradiction since $G(P)$ is dimensionally rigid. Therefore, both matrices $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$ and $W^{\mathrm{T}}\mathscr{M}(\hat{y})U$ must be zero. Hence, there exists a sufficiently small $\alpha > 0$ such that $X(t\hat{y}) = X_1 + \mathscr{M}(t\hat{y}) \succeq 0$ and rank $X(t\hat{y}) = r$ for all $t \in [0, \alpha]$, which implies that $G(P)$ is flexible, a contradiction. Thus, $G(P)$ is unique. □

A remark is in order here. A given framework $G(P)$ in $\Re^r$ may have an equivalent framework $G(q)$ in $\Re^s$ for some $s \neq r$, but not in $\Re^r$. That is, $G(P)$ is unique in $\Re^r$. Such a framework is often called "globally rigid" or "uniquely rigid" [6]. The above corollary establishes a sufficient condition (which, obviously, is also necessary) for the uniqueness of $G(P)$ not only in $\Re^r$, but in all Euclidean spaces. In light of Lemma 2.2, we also have the following corollary.

Corollary 3.2. *Let $G(P)$ be a given framework in $\Re^r$ for some $r \leqslant n-2$, and let Z be the Gale matrix corresponding to P. Further, let $\bar{r}$ be the nullity of $X = V^{\mathrm{T}}PP^{\mathrm{T}}V$. If the following condition holds*:

$$\exists \bar{r} \times \bar{r} \text{ matrix } \Psi \succ 0 : {z^i}^{\mathrm{T}} \Psi z^j = 0 \quad \forall (i,j) \in \bar{E}, \tag{14}$$

then $G(P)$ is dimensionally rigid. Otherwise, if (14) *does not hold, then $G(P)$ is dimensionally flexible iff*

$$\text{null space of } Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z \subseteq \text{null space of } P^{\mathrm{T}}\mathscr{E}(\hat{y})Z, \tag{15}$$

for some nonzero $\hat{y} \in \Re^{\bar{m}}$ such that $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ is nonzero positive semidefinite, where $\mathscr{E}(\hat{y}) = \sum_{(i,j)\in\bar{E}} \hat{y}_{ij} E^{ij}$.

Proof. It follows from (7) that $U^{\mathrm{T}}M^{ij}U = -\frac{1}{2}U^{\mathrm{T}}V^{\mathrm{T}}E^{ij}VU$. But from Lemma 2.2, we have that $VU = ZQ$ for some nonsingular Q. Thus, Condition (12) is equivalent to Condition (14). Now assume that (13) holds and let u be a nonzero vector in the null space of $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$. As in Lemma 2.2, let $VU = ZQ$ and $VW = PQ'$. Then $Q^{-1}u$ belongs to the null space of $U^{\mathrm{T}}\mathscr{M}(\hat{y})U$, which implies that $Q^{-1}u$ also belongs to the null space of $W^{\mathrm{T}}\mathscr{M}(\hat{y})U$. i.e.,

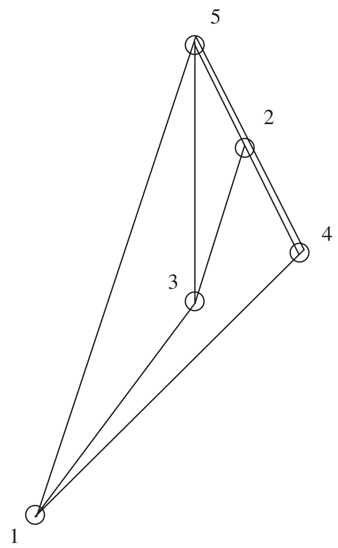

Fig. 2. The framework $G(P)$ in $\Re^2$ of Example 3.1. $G(P)$ is dimensionally rigid, and Conditions (14) and (15) both fail to hold in this case. Note that the points p^2, p^4, and p^5 are collinear; i.e., P is not in general position.

$Q'^{\mathrm{T}} P^{\mathrm{T}} \mathscr{E}(\hat{y}) Z u = 0$. Hence, u belongs to the null space of $P^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ since Q' is nonsingular. Therefore, (15) holds. Similarly we can show that (15) implies (13). Thus, Conditions (13) and (15) are equivalent and the result follows since $U^{\mathrm{T}}\mathscr{M}(\hat{y})U = Q^{\mathrm{T}} Z^{\mathrm{T}} \mathscr{E}(\hat{y}) Z Q$ □

Note that if Condition (14) fails to hold, then Lemma 2.3 guarantees the existence of a nonzero $\hat{y}$ such that $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ is nonzero positive semidefinite. However, Condition (15) may not hold in some degenerate cases. The following example shows a case where Conditions (14) and (15) both fail to hold at the same time.

Example 3.1. Consider the following framework $G(P)$ in $\Re^2$ (see Fig. 2), where $\bar{G} = (V = \{1, 2, 3, 4, 5\}, \bar{E} = \{(1, 2), (3, 4)\})$; and where P and its corresponding Gale matrix Z are

$$P = \begin{bmatrix} -3 & -5 \\ 1 & 2 \\ 0 & -1 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 0 \\ 3/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Then Condition (14) does not hold since $z^2 + 2z^4 = -z^3 = 3z^1$. On the other hand, $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ is nonzero positive semidefinite implies that $\hat{y}_{12} = 1$ and $\hat{y}_{34} = -2/3$. But, null space of $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \not\subseteq$ null space of $P^{\mathrm{T}}\mathscr{E}(\hat{y})Z = \begin{bmatrix} 5 & -3 \\ 3 & -16/3 \end{bmatrix}$. Thus, Condition (15) also does not hold and $G(P)$ is dimensionally rigid.

A case where Condition (15) is known to hold whenever Condition (14) fails to hold, is presented next. Thus in this case, Condition (14) is both sufficient and necessary for a given framework to be dimensionally rigid.

Corollary 3.3. *Let $G(P)$ be a given framework in $\Re^{n-2}$. Then Condition* (14) *is necessary and sufficient for $G(P)$ to be dimensionally rigid.*

Proof. Assume that Condition (14) does not hold. Then by Lemma 2.3 there exists a nonzero $\hat{y}$ such that $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ is nonzero positive semidefinite. But in this case $\bar{r} = 1$ since P is an $(n-2)$-configuration. Therefore, $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$ is a positive number. Hence, Condition (15) trivially holds. This establishes the necessity of Condition (14) and the result follows. □

In what follows, we discuss the case where the r-configuration P of a given framework $G(P)$ is in general position. An r-configuration P is said to be in *general position* iff no $r+1$ of the points $p^1, \ldots, p^n$ are affinely dependent. For example, a configuration P in the plane is in general position if no three of the points $p^1, \ldots, p^n$ lie on a straight line. The following lemma characterizes P in general position in terms of its corresponding Gale matrix Z.

Lemma 3.1. *Let $G(P)$ be a framework in $\Re^r$, and let Z be the $n \times \bar{r}$ Gale matrix corresponding to P. Then, P is in general position if and only if every $\bar{r} \times \bar{r}$ sub-matrix of Z is nonsingular.*

Proof. Assume $\bar{r} \leqslant r$. The proof of the case where $\bar{r} \geqslant r+1$ is similar. Let $\bar{Z}$ be any $\bar{r} \times \bar{r}$ sub-matrix of Z, and without loss of generality, assume that it is the sub-matrix defined by the rows $\bar{r}+1, \bar{r}+2, \ldots, 2\bar{r}$. Then, $\bar{Z}$ is singular if and only if there exists a nonzero $\lambda \in \Re^{\bar{r}}$ such that $\bar{Z}\lambda = 0$. Clearly, $Z\lambda$ is in the null space of $\begin{bmatrix} P^{\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}$. Furthermore, $\bar{Z}\lambda = 0$ if and only if the components $(Z\lambda)_{\bar{r}+1} = (Z\lambda)_{\bar{r}+2} = \ldots (Z\lambda)_{2\bar{r}} = 0$. Now since $Z\lambda \neq 0$, this last statement holds if and only if the following $r+1$ points $p^1, p^2, \ldots, p^{\bar{r}}, p^{2\bar{r}+1}, \ldots, p^n$ are affinely dependent; i.e., P is not in general position. $\square$

Let $p^1, \ldots, p^n \in \Re^r$ be in general position and let $z^1, \ldots, z^n \in \Re^{\bar{r}}$ be the Gale transform of $p^1, \ldots, p^n$, respectively. Then in light of Lemma 3.1, $z^{i_1}, z^{i_2}, \ldots, z^{i_{\bar{r}}}$ are linearly independent for any $\{i_1, i_2, \ldots, i_{\bar{r}}\} \subset \{1, 2, \ldots, n\}$. Let $\delta(G)$ denote the minimum degree of the vertices of graph G. Then we have the following result.

Theorem 3.2. *Let $G(P)$ be a given framework in $\Re^r$ for some $r \leqslant n-2$, and let $\bar{r}$ be the nullity of $X = V^{\mathrm{T}} P P^{\mathrm{T}} V$. Assume that P is in general position. If $\delta(G) \leqslant r$, then $G(P)$ is dimensionally flexible.*

Proof. Assume $\delta(G) \leqslant r$ and let i_0 be a vertex of G such that $\deg(i_0) = \delta(G)$. Let $i_1, i_2, \ldots, i_{\bar{r}}$ be the nodes of G not adjacent to i_0. Since $p^1, p^2, \ldots, p^n$ are in general position, it follows from Lemma 3.1 that $z^{i_0} \neq 0$ and that $z^{i_1}, z^{i_2}, \ldots, z^{i_{\bar{r}}}$ are linearly independent, hence $z^{i_1}, \ldots, z^{i_{\bar{r}}}$ form a basis in $\Re^{\bar{r}}$. Therefore, there exists $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{\bar{r}}$, not all of which are zero, such that $z^{i_0} = \sum_{k=1}^{\bar{r}} \hat{x}_k z^{i_k}$.

Let Ψ be an $\bar{r} \times \bar{r}$ matrix such that ${z^{i_0}}^{\mathrm{T}} \Psi z^{i_k} = 0$ for all $k = 1, \ldots, \bar{r}$. Then, $\sum_{k=1}^{\bar{r}} \hat{x}_k {z^{i_0}}^{\mathrm{T}} \Psi z^{i_k} = 0 = {z^{i_0}}^{\mathrm{T}} \Psi z^{i_0}$. Hence, Ψ is singular and thus it cannot be positive definite. On the other hand, let $\hat{y} \in \Re^{\bar{m}}$ such that $\hat{y}_{ij} = \hat{x}_k$ if $i = i_0$, $j = i_k$, and $\hat{y}_{ij} = 0$ otherwise. Then $Z^{\mathrm{T}} \mathscr{E}(\hat{y}) Z = \sum_{k=1}^{\bar{r}} \hat{x}_k (z^{i_0} {z^{i_k}}^{\mathrm{T}} + z^{i_k} {z^{i_0}}^{\mathrm{T}}) = 2 z^{i_0} {z^{i_0}}^{\mathrm{T}}$ is a nonzero positive semidefinite matrix. Furthermore, the null space of $Z^{\mathrm{T}} \mathscr{E}(\hat{y}) Z =$ null space of ${z^{i_0}}^{\mathrm{T}} \subseteq$ null space of $P^{\mathrm{T}} \mathscr{E}(\hat{y}) Z = p^{i_0} {z^{i_0}}^{\mathrm{T}} + \sum_{k=1}^{\bar{r}} \hat{x}_k p^{i_k} {z^{i_0}}^{\mathrm{T}}$. Hence by Corollary 3.2, $G(P)$ is dimensionally flexible. $\square$

The following is an immediate corollary of Theorem 3.2.

Corollary 3.4. *Let $G(P)$ be a given framework in $\Re^{n-2}$. Assume that G is not a complete graph, and that P is in general position. Then P is dimensionally flexible.*

Note that Theorem 3.2 and Corollary 3.4 are false if the r-configuration P is not in general position as shown by the following example.

Example 3.2. Consider the following two frameworks $G_1(P_1)$ and $G_2(P_2)$ in $\Re^2$ (see Fig. 3), where $\bar{G}_1 = (V1 = \{1, 2, 3, 4\}, \bar{E}1 = \{(1, 4)\})$, $\bar{G}_2 = (V2 = \{1, 2, 3, 4, 5\}, \bar{E}2 = \{(1, 4), (1, 5)\})$, and

$$P_1 = \begin{bmatrix} -1 & -1/4 \\ 0 & -1/4 \\ 1 & -1/4 \\ 0 & 3/4 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} -1 & -2/3 \\ 0 & -2/3 \\ 1 & -2/3 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Both $G_1(P_1)$ and $G_2(P_2)$ are dimensionally rigid and both P_1 and P_2 are not in general position. Note that $\delta(G_2) = 2 = r$.

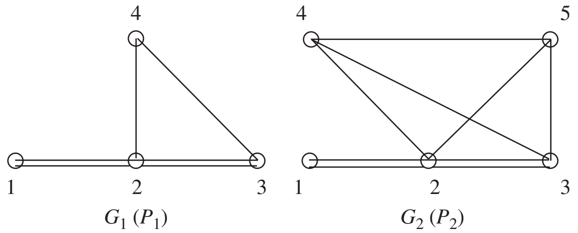

Fig. 3. frameworks $G_1(P_1)$ and $G_2(P_2)$ of Example 3.2. Both frameworks are dimensionally rigid, and both P_1 and P_2 are not in general position.

4. Checking the validity of Condition (14)

In this section we formulate the problem of checking whether Condition (14) in Corollary 3.2 holds or not, as a semidefinite programming (SDP) problem. Given a framework $G(P)$ in $\Re^r$, let Z be the Gale matrix corresponding to P. Let $\mathscr{L}$ be the subspace of $\mathscr{S}_{\bar{r}}$ spanned by the matrices $(z^i {z^j}^{\mathrm{T}} + z^j {z^i}^{\mathrm{T}})$ for all $(i, j) \in \bar{E}$, where $\bar{E}$ is the edge set of the complement graph $\bar{G}$. Note that these matrices need not be linearly independent. If $\mathscr{L} = \mathscr{S}_{\bar{r}}$, then we have the following result.

Lemma 4.1. *If $\mathscr{L} = \mathscr{S}_{\bar{r}}$, i.e., if the dimension of $\mathscr{L} = (\bar{r}(\bar{r}+1))/2$, then $G(P)$ is dimensionally flexible.*

Proof. If $\mathscr{L}=\mathscr{S}_{\bar{r}}$, then there exist $\hat{y}_{ij}$'s, not all of which are zero, such that $I_{\bar{r}}=\sum_{(i,j)\in\bar{E}} \hat{y}_{ij}(z^i {z^j}^{\mathrm{T}}+z^j {z^i}^{\mathrm{T}})=Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z$. The result follows trivially from Corollary 3.2 since the null space of $Z^{\mathrm{T}}\mathscr{E}(\hat{y})Z=\emptyset$. □

Now if $\mathscr{L} \neq \mathscr{S}_{\bar{r}}$, then let N^i for $i = 1, \ldots, \bar{n}$ be a basis of $\mathscr{L}^{\perp}$, the orthogonal complement of $\mathscr{L}$. In the sequel, all matrices are of order $\bar{r}$. Thus, we drop the subscript from the identity matrix $I_{\bar{r}}$. Consider the following SDP problem:

$$
\text{(P)} \quad \begin{array}{lll} p^* = \max & t & \\ \text{subject to} & -tI + \sum_i^{\bar{n}} x_i N^i & \succeq 0, \\ & \sum_i^{\bar{n}} x_i N^i & \preceq I, \end{array} \tag{16}
$$

and its dual

$$
\text{(D)} \quad \begin{array}{llll} d^* = \min & \operatorname{tr} Y_2 & & \\ \text{subject to} & \langle N^i, Y_2\rangle - \langle N^i, Y_1\rangle & = 0, & i = 1, \ldots, \bar{n} \\ & \operatorname{tr} Y_1 & = 1, & \\ & Y_1 \succeq 0,\ Y_2 \succeq 0. & & \end{array} \tag{17}
$$

Since there exists $(\hat{t}, \hat{x})$, namely $(-1, 0)$, such that $-\hat{t}I + \sum_i^{\bar{n}} \hat{x}_i N^i \succ 0$, $\sum_i^{\bar{n}} \hat{x}_i N^i \prec I$; and since $Y_1 = I/\bar{r} \succ 0$ and $Y_2 = I/\bar{r} \succ 0$ are dual feasible, Slater constraint qualification condition holds for both problems. Hence, by the SDP strong duality theorem [17], $p^* = d^*$. In addition, we have the following lemma.

Lemma 4.2. *In problem* (16), *p^* is finite and nonnegative. Furthermore, $p^* > 0$ if and only Condition* (14) *in Corollary* 3.2 *holds, i.e., there exists a positive definite matrix Ψ such that ${z^i}^{\mathrm{T}} \Psi z^j = 0$ for all $(i, j) \in \bar{E}$.*

Proof. The nonnegativeness of p^* follows from the fact that $p^* = d^*$, and the fact that $\operatorname{tr} Y_2 \geqslant 0$ since Y_2 is positive semidefinite. The finiteness of p^* follows from the second constraint in (16), which is added solely for this purpose.

Now it is clear from the constraint $-tI + \sum_i x_i N^i \succeq 0$ that $p^* = \lambda_{\min}(\sum_i x_i N^i)$, where $\lambda_{\min}(B)$ denotes the minimum eigenvalue of B. Thus, the result follows from the definition of N^i's by setting $\Psi = \sum_i x_i N^i$. □

Semidefinite programs can be solved efficiently using interior-point methods [17]. SeDuMi by Sturm [15] is a widely available SDP solver.

5. Summary and concluding remarks

Given a joint-and-bar framework $G(P)$ in $\Re^r$, $G(P)$ is said to be dimensionally rigid iff there does not exist a framework $G(q)$ in $\Re^s$, equivalent to $G(P)$, for some $s \geqslant r + 1$. In this paper, we presented necessary and sufficient conditions for $G(P)$ to be dimensionally rigid in terms of Z (Theorem 3.1, Corollary 3.2), the Gale matrix corresponding to P. We showed that these conditions can be strengthened in the case where $r = n - 2$ (Corollary 3.3), and in case where P is in general position (Theorem 3.2). We also showed that if a given framework $G(P)$ is both rigid and dimensionally rigid, then $G(P)$ is unique (Corollary 3.1). Finally, we formulated the problem of checking the validity of Condition (14) as a SDP.

The following problem, not discussed in this paper, is of great interest. Given a framework $G(p)$ in $\Re^r$, determine whether or not there exists a framework $G(q)$ in $\Re^s$, equivalent to $G(p)$, for some $s \leqslant r - 1$. This problem seems to be quite difficult in general, especially if a constructive proof is desired. Finally, the following two problems are also of interest and merit further investigation. The first problem is that of obtaining a complete characterization of the cases where Condition (15) holds. And the second problem is that of devising a combinatorial algorithm for checking the validity of Condition (14).

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