Oscillation theorems for certain delay partial difference equations✩

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Abstract

In this work we obtain some new oscillation criteria for certain delay partial difference equations with continuous arguments by some new techniques.

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1. Introduction

The qualitative analysis of partial difference equations has received much attention in the past few years (see the survey paper [1]). In particular, the oscillation of partial difference equations with continuous arguments has been investigated in some papers [2–5]. In this work, we consider the partial difference equations of the form

\[ d_1A(x + a, y + b) + d_2A(x + a, y) + d_3A(x, y + b) - d_4A(x, y) + \sum_{i=1}^{u} p_i(x, y) A(x - \tau_i, y - \sigma_i) = 0, \]

(1)

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where \( p_i \in C(R^+ \times R^+, R^+) \), \( a, b, \tau_i \) and \( \sigma_i \) are positive, \( d_i, i = 1, 2, 3 \), are nonnegative and \( d_4 \) is positive. Eq. (1) has been investigated by Agarwal and Zhou [5].

By a solution of (1), we mean a continuous function \( A(x, y) \) which satisfies (1) for \( x \geq x_0 \geq 0 \), \( y \geq y_0 \geq 0 \). A solution \( A(x, y) \) of (1) is said to be eventually positive if \( A(x, y) > 0 \) for all large \( x \) and \( y \), and eventually negative if \( A(x, y) < 0 \) for all large \( x \) and \( y \). It is said to be oscillatory if it is neither eventually positive nor eventually negative.

Our purpose is to obtain new oscillation criteria for the oscillation of (1) by various new techniques.

2. Main results

Throughout this section, we assume the following for \( i = 1, 2, \ldots, u \):

(i) \( \tau_i = k_i a + \theta_i, \sigma_i = l_i b + \eta_i \), where \( k_i, l_i \) are nonnegative integers, \( \theta_i \in [0, a) \), \( \eta_i \in [0, b) \);

(ii) \[ Q_i(x, y) = \min\{ p_i(z, w) \mid x \leq z \leq x + a, y \leq w \leq y + b, x \geq x_0, y \geq y_0 \} \]

and \( \inf_{x \geq x_0, y \geq y_0} Q_i(x, y) = q_i \geq 0, \quad i = 1, 2, \ldots, u. \)

From [5] we have the following result:

**Lemma 2.1.** Let \( A(x, y) \) be an eventually positive solution of (1). Set

\[ \omega(x, y) = \int_{x}^{x+a} \int_{y}^{y+b} A(u, v) dv du. \]  

Then \( \omega(x, y) \) is an eventually positive solution of the difference inequality

\[ d_1 \omega(x + a, y + b) + d_2 \omega(x + a, y) + d_3 \omega(x, y + b) - d_4 \omega(x, y) \]

\[ + \sum_{i=1}^{u} Q_i(x, y) \omega(x - k_i a, y - l_i b) \leq 0, \]

and \( \frac{\partial \omega}{\partial x} < 0, \frac{\partial \omega}{\partial y} < 0. \)

From (4), \( d_2 \omega(x + a, y) \leq d_4 \omega(x, y) \) and \( d_3 \omega(x, y + b) \leq d_4 \omega(x, y) \) for sufficiently large \( x \) and \( y \).

Let \( \lambda_1 = 0 \). Then for sufficiently large \( x \) and \( y \), we have

\[ \omega(x - a, y) \geq e^{-\lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \omega(x, y), \]

\[ \omega(x - k_i a, y) \geq e^{-\lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \omega(x, y) \]

and

\[ \omega(x, y - b) \geq e^{-\lambda_1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y), \]

\[ \omega(x, y - l_i b) \geq e^{-\lambda_1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y). \]
Hence,
\[ \omega(x - k_i a, y - l_i b) \geq e^{-k_i \lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \omega(x, y) \]
\[ \geq e^{-(k_i + l_i) \lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y). \]

From (4) and (ii), we have
\[ d_1 \omega(x + a, y + b) + d_2 \omega(x + a, y) + d_3 \omega(x, y + b) - d_4 \omega(x, y) \]
\[ + \sum_{i=1}^{u} q_i \omega(x - k_i a, y - l_i b) \leq 0. \] (5)

Hence, for sufficiently large \( x \) and \( y \),
\[ \frac{d_2}{d_4} \omega(x + a, y) \leq \omega(x, y) \left( 1 - \frac{1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i) \lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} \right) \]
and
\[ \frac{d_3}{d_4} \omega(x, y + b) \leq \omega(x, y) \left( 1 - \frac{1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i) \lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} \right). \]

Let
\[ e^{\lambda_2} = 1 - \frac{1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i) \lambda_1} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}. \]

Obviously, \( \lambda_2 \leq \lambda_1 = 0 \). For sufficiently large \( x \) and \( y \),
\[ \frac{d_2}{d_4} \omega(x + a, y) \leq e^{\lambda_2} \omega(x, y), \quad \frac{d_3}{d_4} \omega(x, y + b) \leq e^{\lambda_2} \omega(x, y). \]

Thus there exists \( \lambda_2 \leq 0 \) with
\[ \omega(x - k_i a, y) \geq e^{-\lambda_2 k_i} \left( \frac{d_2}{d_4} \right)^{k_i} \omega(x, y) \]
and
\[ \omega(x, y - b) \geq e^{-\lambda_2} \frac{d_3}{d_4} \omega(x, y), \quad \omega(x, y - l_i b) \geq e^{-\lambda_2 l_i} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y). \]

By induction, for \( n \geq 1 \),
\[ \omega(x - a, y) \geq e^{-\lambda_n \lambda_2} \frac{d_2}{d_4} \omega(x, y), \quad \omega(x - k_i a, y) \geq e^{-\lambda_n k_i} \left( \frac{d_2}{d_4} \right)^{k_i} \omega(x, y), \]
\[ \omega(x, y - b) \geq e^{-\lambda_n \lambda_2} \frac{d_3}{d_4} \omega(x, y), \quad \omega(x, y - l_i b) \geq e^{-\lambda_n l_i} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y), \]
where
\[ e^{\lambda_n} = 1 - \frac{1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda_{n-1}} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}. \]

Observe that \(-\infty < \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 = 0\). Therefore, \(\lim_{n \to \infty} \lambda_n = \lambda^* < 0\) exists and
\[ e^{\lambda^*} = 1 - \frac{1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}. \] (6)

Hence, we have
\[ \omega(x - k_i a, y - l_i b) \geq e^{-(k_i+l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y). \] (7)

From (4),
\[ d_4 \omega(x, y) \geq d_1 \omega(x + a, y + b) + d_2 \omega(x + a, y) + d_3 \omega(x, y + b) \]
\[ + \sum_{i=1}^{u} Q_i(x, y) \omega(x - k_i a, y - l_i b). \] (8)

By (7), we have
\[ \omega(x - k_i a, y - l_i b) \geq e^{-(k_i+l_i)(1-1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x - a, y), \]
and we have
\[ \omega(x, y) \geq \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i+l_i-1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x - a, y). \]
Hence
\[ \omega(x + a, y) \geq \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x + a, y) e^{-(k_i+l_i-1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y). \]

Similarly, we have
\[ \omega(x, y + b) \geq \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y + b) e^{-(k_i+l_i-1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1} \omega(x, y). \]

Substituting the above inequalities into (8), we obtain
\[ d_4 \omega(x, y) \geq d_1 \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x + a, y + b) e^{-(k_i+l_i-2)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1} \omega(x, y) \]
\[ + d_2 \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x + a, y) e^{-(k_i+l_i-1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x, y) \]
Similarly, we have

\[ Set \]

\[
R(x, y) = d_4 - \frac{d_1}{d_4} \sum_{i=1}^{u} Q_i(x + a, y + b) e^{-(k_i + l_i - 2)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1} - \frac{d_2}{d_4} \sum_{i=1}^{u} Q_i(x + a, y) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1} - \frac{d_3}{d_4} \sum_{i=1}^{u} Q_i(x, y + b) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1}. \]

Then (9) leads to

\[ \omega(x, y) \geq \frac{1}{R(x, y)} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} \omega(x - a, y). \] (11)

Similarly, we have

\[ \omega(x, y) \geq \frac{1}{R(x, y)} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1} \omega(x, y - b), \] (12)

and

\[ \omega(x, y) \geq \frac{1}{R(x, y)} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i - 2)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1} \omega(x - a, y - b). \] (13)

From (8), we have, for all sufficiently large \( x \) and \( y \),

\[ 1 \geq \frac{d_1}{d_4} \frac{\omega(x + a, y + b)}{\omega(x, y)} + \frac{d_2}{d_4} \frac{\omega(x + a, y)}{\omega(x, y)} + \frac{d_3}{d_4} \frac{\omega(x, y + b)}{\omega(x, y)} + \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) \frac{\omega(x - k_i a, y - l_i b)}{\omega(x, y)} \]

\[ \geq \frac{d_1}{d_4} R(x + a, y + b) \sum_{i=1}^{u} Q_i(x + a, y + b) e^{-(k_i + l_i - 2)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1} + \frac{d_2}{d_4} R(x + a, y) \sum_{i=1}^{u} Q_i(x + a, y) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i} + \frac{d_3}{d_4} R(x, y + b) \sum_{i=1}^{u} Q_i(x, y + b) e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1} + \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} \triangleq H(x, y). \] (14)
From (14), we obtain the main result in this work.

**Theorem 2.1.** Assume that

\[
\limsup_{x,y \to \infty} H(x, y) > 1. \quad (15)
\]

Then every solution of (1) is oscillatory.

From (6), we have

\[
d_4(1 - e^{\lambda x}) = \sum_{i=1}^{u} q_i e^{-(k_i + l_i)x} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}. \quad (16)
\]

Using (16) we can obtain a simpler condition from (15).

In view of (10), we have, for all \( x \) and \( y \) sufficiently large,

\[
R(x, y) \leq d_4 - \frac{d_1}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i - 2)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i-1} \left( \frac{d_3}{d_4} \right)^{l_i-1}
\]

\[
- \frac{d_2}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1}
\]

\[
- \frac{d_3}{d_4} \sum_{i=1}^{u} q_i e^{-(k_i + l_i - 1)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i-1}
\]

\[
= d_4 - \frac{d_1}{d_4} d_4(1 - e^{\lambda^*})e^{2\lambda^*} \frac{d_2^2}{d_2 d_3} - \frac{d_2}{d_4} d_4(1 - e^{\lambda^*})e^{\lambda^*} \frac{d_4}{d_2} - \frac{d_3}{d_4} d_4(1 - e^{\lambda^*})e^{\lambda^*} \frac{d_4}{d_3}
\]

\[
= d_4 \left[ 1 - (1 - e^{\lambda^*}) \left( \frac{d_1 d_4}{d_2 d_3} e^{2\lambda^*} + 2e^{\lambda^*} \right) \right]
\]

\[
= d_4 \left[ 1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right) \right].
\]

Therefore

\[
H(x, y) \geq \frac{1}{d_4 \left[ 1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right) \right]}
\]

\[
\times \left\{ \frac{d_1}{d_4} d_4(1 - e^{\lambda^*})e^{2\lambda^*} \frac{d_2^2}{d_2 d_3} + \frac{d_2}{d_4} d_4(1 - e^{\lambda^*})e^{\lambda^*} \frac{d_4}{d_2} + \frac{d_3}{d_4} d_4(1 - e^{\lambda^*})e^{\lambda^*} \frac{d_4}{d_3} \right\}
\]

\[
+ \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}
\]

\[
= \frac{(1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right) + \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}}{1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right) + \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x, y) e^{-(k_i + l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i}}. \quad (17)
\]

Then we have the following simpler result.
Corollary 2.1. Assume that

\[
\limsup_{x,y \to \infty} \frac{1}{d_4} \sum_{i=1}^{u} Q_i(x,y) e^{-(k_i+l_i)\lambda^*} \left( \frac{d_2}{d_4} \right)^{k_i} \left( \frac{d_3}{d_4} \right)^{l_i} > \frac{1 - 2(1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)}{1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)}.
\]

Then every solution of (1) oscillates.

From (17), we have

\[
H(x, y) \geq \frac{(1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)}{1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)} + (1 - e^{\lambda^*}).
\]

Corollary 2.2. If

\[
\frac{(1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)}{1 - (1 - e^{\lambda^*})e^{\lambda^*} \left( \frac{d_1 d_4}{d_2 d_3} e^{\lambda^*} + 2 \right)} + (1 - e^{\lambda^*}) > 1,
\]

then every solution of (1) oscillates.

Consider the special case of (1): \( u = 1, k = l = 0, d_1 = d_2 = d_3 = d_4 = 1 \). From Theorem 2.9 of [5], if

\[
\limsup_{x,y \to \infty} Q(x,y) > 1,
\]

then every solution of (1) oscillates.

In this case, \( e^{\lambda^*} = 1 - q \). By Corollary 2.1, if

\[
\limsup_{x,y \to \infty} Q(x,y) > \frac{1 - 2q(1 - q)(3 - q)}{1 - q(1 - q)(3 - q)},
\]

then every solution of (1) oscillates.

Obviously, the right-hand side of (22) is less than 1. So our result is sharper than that in [5] in this case.

Example 2.1. Consider the partial difference equation

\[
A(x + 2\pi, y + 2\pi) + A(x + 2\pi, y) + A(x, y + 2\pi) - A(x, y) + p(x, y)A(x - \pi, y - 3\pi) = 0,
\]

where \( p(x, y) = \frac{11}{5} + \sin x + \sin y \). Then

\[
Q(x, y) = \min_{x \leq z \leq x + 2\pi, y \leq w \leq y + 2\pi} p(z, w) = \frac{1}{5}
\]

i.e., \( q = \frac{1}{5} \). By (16), \( e^{\lambda^*} = 1 - \frac{1}{5}e^{-\lambda^*} \).
Hence $e^{\lambda^*} = \frac{1 - \sqrt{15}}{2} \approx 0.276393$, and
\[
\frac{(1 - e^{\lambda^*})e^{\lambda^*}(e^{\lambda^*} + 2)}{1 - (1 - e^{\lambda^*})e^{\lambda^*}(e^{\lambda^*} + 2)} + (1 - e^{\lambda^*}) = 1.5594133 > 1.
\]
By Corollary 2.2, every solution of (23) oscillates. But condition (48) of [5] does not hold. Our result is again sharper than the corresponding result in [5].

References