



Integrality of the Chern character in small codimension

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Abstract

We prove an integrality property of the Chern character with values in Chow groups. As a consequence, we obtain a construction of the $p - 1$ first homological Steenrod operations on Chow groups modulo p and p -primary torsion, over an arbitrary field. We provide applications to the study of correspondences between algebraic varieties.

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0. Introduction

The Grothendieck–Riemann–Roch theorem of [5] can be expressed as the existence of the homological Chern character, a collection of group morphisms

$$\text{ch}_i : K'_0(X) \rightarrow \text{CH}_i(X)_{\mathbb{Q}},$$

which are compatible with projective push-forwards. Here X is a (possibly singular) algebraic variety, $K'_0(X)$ the Grothendieck group of coherent sheaves, and $\text{CH}_i(X)_{\mathbb{Q}}$ the Chow group of X of algebraic cycles of dimension i , with \mathbb{Q} -coefficients. Compatibility with pull-backs is expressed by a Riemann–Roch type formula involving the Todd class.

Given a prime number p , one can formulate the following statement describing a p -integrality property of the homological Chern character.

Conjecture. *Let X be a variety, and n a positive integer. Then*

$$p^{\lfloor n/(p-1) \rfloor} \cdot \text{ch}_{\dim X - n}[\mathcal{O}_X] \in \text{im}(\text{CH}_{\dim X - n}(X)_{\mathbb{Z}(p)} \rightarrow \text{CH}_{\dim X - n}(X)_{\mathbb{Q}}).$$

The topological analog of this conjecture was stated and proved by Adams in [1, Theorem 1]. A more elementary proof of this topological result was later given by Atiyah [4, Theorem 7.1], under the quite restrictive assumption that X is a CW-complex whose homology has no torsion. Although our approach is more akin to Atiyah’s than Adams’ (where in particular Steenrod operations were used), no condition of such nature will appear.

In [14, Theorem 4.4], we proved the conjecture for $p = 2$ and $n = 1$. In the present article, we prove it for $n < p(p - 1)$, and any p (Section 3). In particular we recover the previously obtained result, using an argument of different nature. We discuss in Appendix how this result can be extended to the more general setting of a scheme of finite type over a regular base.

Let us mention that the conjecture is an immediate consequence of the formula expressing compatibility of the homological Chern character with pull-backs when X is smooth, or more generally a local complete intersection variety. This remark can be used to prove the conjecture in the presence of some form of resolution of singularities. Using a result of Gabber on existence of regular alterations, we prove the conjecture for all n , when the characteristic of the base field is not p .

The n -th Steenrod operation on Chow groups modulo p is an endomorphism of the Chow group with \mathbb{Z}/p -coefficients which lowers dimension of cycles by $n(p - 1)$. The Steenrod operations constitute an efficient tool for studying rationality properties of projective homogeneous varieties.

The classical constructions of the operations modulo p [6–8,21,30] do not work over a field of characteristic p . In [14], we constructed a weak form of the operation for $p = 2$ and $n = 1$, over any field. We later obtained the full version of this operation in [16]. In the present paper,

we construct a weak form of the n -th Steenrod operation modulo p , for p an arbitrary prime and $n = 1, \dots, p - 1$, over any field.

More precisely, assuming that the conjecture above holds true for $n \leq m(p - 1)$, and given any (possibly singular) variety X over an arbitrary field, we construct for each integer i such that $0 < i \leq m$, a morphism of graded abelian groups

$$T_i: \text{CH}_\bullet(X) \otimes \mathbb{Z}/p \rightarrow \widetilde{\text{CH}}_{\bullet-i(p-1)}(X) \otimes \mathbb{Z}/p.$$

Here $\widetilde{\text{CH}}_d$ is the quotient of the Chow group CH_d by its torsion subgroup. These morphisms are compatible with proper push-forwards, external products, and extension of the base field. We also provide a Riemann–Roch type formula expressing compatibility with pull-backs along local complete intersection morphisms.

Thus we construct operations T_1, \dots, T_{p-1} unconditionally, and operations T_i for all $i > 0$ when the characteristic of the base field is not p . The operation T_i corresponds – up to a sign – to the i -th Steenrod operation modulo p when $i \leq p$. More generally, the total operation $\sum_i T_i$ corresponds to the inverse of the total Steenrod operation $\sum_j S_j$. We explain how to recover the operation induced by S_j on reduced Chow groups (the *reduced Steenrod operation*) from the operations T_i .

In the last part of this paper, we prove the degree formula corresponding to the operations T_i . Since we only proved the conjecture in general for small values of n , we need to work with varieties of dimension $\leq p(p - 1)$ (note that the dimension $p(p - 1)$ is allowed). We first give a bound on the p -adic valuation of the index of a projective variety in terms of its dimension and of the Euler characteristic of its structure sheaf. Then we deduce from the degree formula that the Severi–Brauer variety of a central division algebra of prime degree p is strongly p -incompressible. This is a new statement when the base field has characteristic p . Finally, we show that the Euler characteristic of the structure sheaf of a smooth projective variety of dimension $p - 1$, possessing a special correspondence but no zero-cycle of degree prime to p , is prime to p . This extends a statement of Rost to arbitrary base fields.

1. Notations

Varieties. We fix a base field k . A *variety* is a finite type, separated, quasi-projective scheme over k . A morphism of varieties is a morphism of schemes over k . We will denote by Var/k this category. We also denote by Proj/k the category with the same objects, but with morphisms only those which are projective. The function field of an integral variety X shall be denoted by $k(X)$.

Integral elements. For a number $\alpha \in \mathbb{Q}$, we write $[\alpha]$ for the greatest integer $\leq \alpha$.

Let A be an abelian group. For any commutative ring R we write A_R for $A \otimes_{\mathbb{Z}} R$, and, if S is another commutative ring containing R , we write

$$A_{R \subset S} = \text{im}(A_R \rightarrow A_S).$$

Note that if A is a ring, then $A_{R \subset S}$ is a subring of A_S .

Let p be a prime number. When $R = \mathbb{Z}_{(p)}$ (the subring of \mathbb{Q} consisting of fractions with denominator prime to p), $S = \mathbb{Q}$, and $x \in A_{\mathbb{Q}} - \{0\}$, we denote by $v_p(x)$ the greatest integer r such that $p^{-r} \cdot x \in A_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}$. We set $v_p(0) = \infty$.

When $x, y \in A_{\mathbb{Q}}$, we have

$$v_p(x + y) \geq \min(v_p(x), v_p(y)).$$

If A is a commutative ring, M an A -module, $a \in A_{\mathbb{Q}}$, and $m \in M_{\mathbb{Q}}$, we have

$$v_p(a \cdot m) \geq v_p(a) + v_p(m). \tag{1}$$

Grothendieck groups of schemes. We denote by $K_0(-)$ the presheaf of rings on \mathbf{Var}/k which associates to a variety X the Grothendieck ring of locally-free coherent \mathcal{O}_X -modules. We denote by $K'_0(-)$ the functor $\mathbf{Proj}/k \rightarrow \mathbf{Ab}$ which associates to a variety X the Grothendieck group of coherent \mathcal{O}_X -modules. Here, and in the rest of the paper, the notation \mathbf{Ab} stands for the category of abelian groups.

Local complete intersection morphisms. (See [9, Appendix B.7.6].) These are the morphisms of varieties $f: Y \rightarrow X$ which can be factored as $p \circ i$, with i a regular closed embedding, and p a smooth morphism. Such a morphism admits a *virtual tangent bundle* $T_f = i^*[T_p] - [N_i] \in K_0(Y)$, where T_p is the tangent bundle of p , and N_i the normal bundle of i . This element does not depend on the choice of the factorization. There are pull-backs $f^*: \mathbf{CH}(X) \rightarrow \mathbf{CH}(Y)$ and $f^*: K'_0(X) \rightarrow K'_0(Y)$. We call a variety X a *local complete intersection variety* if its structural morphism $x: X \rightarrow \mathbf{Spec}(k)$ is a local complete intersection morphism. We denote by $T_X \in K_0(X)$ the virtual tangent bundle T_x of x .

Topological filtration. Let X be a variety, and d an integer. We denote by $K'_0(X)_{(d)}$ the subgroup of $K'_0(X)$ generated by those \mathcal{O}_X -modules whose support has dimension $\leq d$. It coincides with the subgroup generated by elements of type $i_*[\mathcal{O}_Z]$, where $i: Z \hookrightarrow X$ is a closed embedding, and $\dim Z \leq d$.

When $f: Y \rightarrow X$ is a projective morphism, we have

$$f_* K'_0(Y)_{(d)} \subset K'_0(X)_{(d)}.$$

When $f: Y \rightarrow X$ is a local complete intersection morphism of relative dimension n , we have [11, Theorem 83 and Lemma 84]

$$f^* K'_0(X)_{(d)} \subset K'_0(Y)_{(d+n)}.$$

The associated graded group is denoted by

$$\mathrm{gr}_d K'_0(X) = K'_0(X)_{(d)} / K'_0(X)_{(d-1)}.$$

There is a natural transformation of functors $\mathbf{Proj}/k \rightarrow \mathbf{Ab}$

$$\varphi_-: \mathbf{CH}(-) \rightarrow \mathrm{gr} K'_0(-) \tag{2}$$

which respects the gradings [9, Example 15.5]. It is moreover compatible with pull-backs along local complete intersection morphisms, external products, and extension of the base field (see [15, Section 3]).

Chow cohomology ring. Let X be a possibly singular variety, and consider the operational Chow ring $\mathbf{CH}(X \xrightarrow{\mathrm{id}} X)$ (see [9, Definition 17.3]). When resolution of singularities is available, this ring is commutative [9, Example 17.4.4]. We denote by $\mathbf{CH}(X)$ the subgroup of elements $x \in \mathbf{CH}(X \xrightarrow{\mathrm{id}} X)$ such that for any morphism $f: Y \rightarrow X$ the element f^*x belongs to the center of $\mathbf{CH}(Y \xrightarrow{\mathrm{id}} Y)$. The group $\mathbf{CH}(X)$ has a natural structure of a $\mathbf{CH}(X)$ -module. We obtain a presheaf of commutative rings \mathbf{CH} on \mathbf{Var}/k , in which Chern classes of vector bundles take their values [9, Proposition 17.3.2].

When E is a vector bundle over a variety X , we write

$$\text{Td}(E) \in \text{CH}(X)_{\mathbb{Q}}$$

for the total Todd class [9, Example 3.2.4]. The individual components are

$$\text{Td}^i(E) \in \text{CH}^i(X)_{\mathbb{Q}}.$$

Chern character. The homological Chern character is a natural transformation of functors $\text{Proj} \rightarrow \text{Ab}$

$$\text{ch}: K'_0(-) \rightarrow \text{CH}(-)_{\mathbb{Q}}.$$

This is the map τ of [9, Theorem 18.3]. Its component of dimension i will be

$$\text{ch}_i: K'_0(-) \rightarrow \text{CH}_i(-)_{\mathbb{Q}}.$$

Adams operations. (See [28]) Let $l \in \mathbb{Z} - \{0\}$. The l -th homological Adams operation is a natural transformation of functors $\text{Proj} \rightarrow \text{Ab}$

$$\psi_l: K'_0(-) \rightarrow K'_0(-) \otimes \mathbb{Z}[1/l].$$

Part 1. Integrality of the Chern character

We fix a prime number p . We say that p -integrality holds in codimension $\leq c$ if the following statement is true.

Statement 1.1. *Let X be a variety and $d \geq 0$ an integer. Then for all integers n such that $0 \leq n \leq c$, and all elements $x \in K'_0(X)_{(d)}$, we have*

$$v_p(\text{ch}_{d-n}(x)) \geq -\left\lceil \frac{n}{p-1} \right\rceil.$$

We prove in Section 3 that p -integrality holds in codimension $\leq p(p-1) - 1$. Moreover, we prove in Section 4 that p -integrality holds in codimension $\leq m$ for all m when the characteristic of the base field is not p . We expect that this assumption is superfluous.

2. Adams operations and Chern character

We fix an element $l \in \mathbb{Z} - \{0\}$. The sole purpose of this section is to prove, in Proposition 2.4, the relation for $n \geq 0$,

$$\text{ch}_n \circ \psi_l = l^{-n} \cdot \text{ch}_n. \tag{3}$$

This is certainly very classical, at least so is the cohomological counterpart of this statement. Due to the lack of suitable reference, we give some details in this section. We shall need to introduce some notations, but except in Appendix, the only thing used in the sequel will be the relation (3).

Cohomological variants. The cohomological (i.e. usual) Chern character [9, Example 3.2.3] is a morphism of presheaves of rings on Var/k

$$\text{ch}: K_0(-) \rightarrow \text{CH}(-)_{\mathbb{Q}}.$$

The cohomological (i.e. usual) l -th Adams operation is a morphism of presheaves of rings on Var/k

$$\psi^l : K_0(-) \rightarrow K_0(-),$$

sending the class of a line bundle to its l -th tensor power.

When X is connected, the ring $K_0(X)$ has an augmentation

$$\text{rank} : K_0(X) \rightarrow \mathbb{Z}$$

sending the class of a vector bundle to its rank.

Bott's class. Consider the Laurent polynomial in the variable x

$$t^l(x) = \frac{1 - x^l}{1 - x}.$$

There is a unique morphism of presheaves of abelian groups on Var/k

$$\theta^l : K_0(-) \rightarrow (\mathbb{Z}[1/l] \otimes K_0(-))^{\times}$$

such that for all line bundles L ,

$$\theta^l(L) = t^l[L^{\vee}].$$

Note that this does not follow [28], where $\theta^l(L) = t^l[L]$.

K-theory with supports. When X is a closed subvariety of a variety W , the notation $K_0^X(W)$ stands for the Grothendieck group of complexes of locally free \mathcal{O}_W -modules, which are acyclic off X , modulo the classes of acyclic complexes. It has a natural structure of a $K_0(W)$ -module. We denote by $\text{CH}(X \rightarrow W)$ the bivariant Chow group (see [9, Definition 17.1]). There are maps

$$- \cap [\mathcal{O}_W] : K_0^X(W) \rightarrow K_0^X(W), \quad - \cap [W] : \text{CH}(X \rightarrow W) \rightarrow \text{CH}(X)$$

which are isomorphisms when W is smooth ([12, Lemma 1.9] and [9, Proposition 17.4.2]). We write

$$\text{ch}_X^W : K_0^X(W) \rightarrow \text{CH}(X \rightarrow W)_{\mathbb{Q}}$$

for the local Chern character [9, Section 18.1], and

$$\psi_X^l : K_0^X(W) \rightarrow K_0^X(W)$$

for the Adams operation with supports [12, Section 4].

Assume that W is smooth, and that $x \in K_0^X(W)$. Then the homological Chern character is given by the formula [9, Theorem 18.3, (3)],

$$\text{ch}(x \cap [\mathcal{O}_W]) = (\text{Td}(T_W) \cdot \text{ch}_X^W(x)) \cap [W], \tag{4}$$

while homological Adams operations are defined by the formula [28, Théorème 7]

$$\psi_l(x \cap [\mathcal{O}_W]) = (\theta^l(-T_W) \cdot \psi_X^l(x)) \cap [\mathcal{O}_W]. \tag{5}$$

Composing the maps ch_X^W with projections $\text{CH}(X \rightarrow W)_{\mathbb{Q}} \rightarrow \text{CH}^j(X \rightarrow W)_{\mathbb{Q}}$, we obtain individual components $(\text{ch}_X^W)^j$.

We now prove a few lemmas that will be used to obtain (3).

Lemma 2.1. *Let X be a connected variety. We have in $\text{CH}(X)_{\mathbb{Q}}$, for all $u \in K_0(X)$,*

$$\text{Td}(-u) \cdot \text{ch} \circ \theta^l(u) = l^{\text{rank}(u)} \cdot \text{Td}(-\psi^l(u)).$$

Proof. Assume that L is a line bundle. Then

$$\text{ch} \circ \theta^l(L) = \text{ch}(t^l[L^\vee]) = t^l(\text{ch}[L^\vee]) = t^l(x^{-1}) = \frac{1 - x^{-l}}{1 - x^{-1}},$$

where $x = e^{c_1(L)}$. Then

$$\begin{aligned} \text{Td}(-L) \cdot \text{ch} \circ \theta^l(L) &= \frac{1 - x^{-1}}{c_1(L)} \cdot \frac{1 - x^{-l}}{1 - x^{-1}} \\ &= \frac{1 - x^{-l}}{c_1(L)} \\ &= l \cdot \frac{1 - e^{-c_1(L^{\otimes l})}}{c_1(L^{\otimes l})} \\ &= l \cdot \text{Td}(-L^{\otimes l}). \end{aligned}$$

Thus the claimed formula holds when u is the class of a line bundle. Both sides of this formula define morphisms of presheaves of abelian groups on Var/k

$$K_0(-) \rightarrow (\text{CH}(-)_{\mathbb{Q}})^{\times}.$$

Therefore, we can conclude with the splitting principle. \square

Lemma 2.2. *Let X be a variety. We have in $\text{CH}(X)_{\mathbb{Q}}$, for all $u \in K_0(X)$,*

$$\text{Td}(\psi^l(-u)) = \sum_{i \geq 0} l^i \cdot \text{Td}^i(-u).$$

Proof. For a line bundle L , we have

$$\text{Td}(-L^{\otimes l}) = \sum_{i \geq 0} \frac{(-c_1(L^{\otimes l}))^i}{(i+1)!} = \sum_{i \geq 0} l^i \cdot \frac{(-c_1(L))^i}{(i+1)!} = \sum_{i \geq 0} l^i \cdot \text{Td}^i(-L),$$

proving the formula for u the class of a line bundle. The lemma follows as above from the splitting principle. \square

Lemma 2.3 ([19, Theorem 3.1]). *Let X be a closed subvariety of a smooth variety W . We have in $\text{CH}(X \rightarrow W)_{\mathbb{Q}}$, for $x \in K_0^X(W)$,*

$$\text{ch}_X^W(\psi_X^l(x)) = \sum_{j \geq 0} l^j \cdot (\text{ch}_X^W)^j(x).$$

Proposition 2.4. *Let $n \geq 0$ be an integer. We have $\text{ch}_n \circ \psi_l = l^{-n} \cdot \text{ch}_n$.*

Proof. It will be enough to prove that the two sides of the formula have the same effect on any element of $K'_0(X)$, for X a connected variety. Choose a closed embedding $X \hookrightarrow W$ into a smooth connected variety W of dimension d . Then, we have in $\text{CH}(X)_{\mathbb{Q}}$, for all $x \in K'_0(X)$,

$$\begin{aligned} \text{ch} \circ \psi_l(x \cap [\mathcal{O}_W]) &= \text{Td}(T_W|_X) \cdot (\text{ch}_X^W(\theta^l(-T_W) \cdot \psi_X^l(x)) \cap [W]) \\ &= (\text{Td}(T_W|_X) \cdot \text{ch} \circ \theta^l(-T_W|_X)) \cdot (\text{ch}_X^W(\psi_X^l(x)) \cap [W]) \\ &= l^{-d} \cdot \text{Td}(\psi^l(T_W|_X)) \cdot (\text{ch}_X^W(\psi_X^l(x)) \cap [W]) \\ &= \sum_{i=0}^d l^{i-d} \cdot \text{Td}^i(T_W|_X) \cdot \sum_{j \geq 0} l^j \cdot ((\text{ch}_X^W)^j(x) \cap [W]) \\ &= \sum_{n=0}^d l^{-n} \cdot \sum_{i+j=d-n} \text{Td}^i(T_W|_X) \cdot ((\text{ch}_X^W)^j(x) \cap [W]) \\ &= \sum_{n=0}^d l^{-n} \cdot \text{ch}_n(x \cap [\mathcal{O}_W]). \end{aligned}$$

The first equality is (4) and (5). The second follows from [9, Proposition 18.1, (c)]. For the third one, we use Lemma 2.1. The fourth one uses Lemmas 2.2 and 2.3. The last one is again (4). \square

3. Integrality in arbitrary characteristic

Proposition 3.1. *Let $l \in \mathbb{Z} - \{0\}$, X a variety, and $x \in K'_0(X)_{(d)}$. Then*

$$\psi_l(x) - l^{-d} \cdot x \in K'_0(X)_{(d-1)} \otimes \mathbb{Z}[1/l].$$

Proof. When $l = p$ is a prime number, this is [15, Lemma 6.1]. Actually we obtained this result using the formula

$$\theta^p(-d) = p^{-d}, \quad (d \in \mathbb{Z}),$$

which is true when p is an arbitrary element of $\mathbb{Z} - \{0\}$. Therefore, we can obtain the proposition using the proof of [15, Lemma 6.1].

Alternatively, one can assume that X is integral and $d = \dim X$. Then we have an exact sequence

$$0 \rightarrow K'_0(X)_{(d-1)} \otimes \mathbb{Z}[1/l] \rightarrow K'_0(X)_{(d)} \otimes \mathbb{Z}[1/l] \rightarrow K'_0(\text{Spec}(k(X))) \otimes \mathbb{Z}[1/l] \rightarrow 0,$$

and the statement is a consequence of [16, Lemma 4.2]. \square

Theorem 3.2. *p -integrality holds in codimension $\leq p(p - 1) - 1$.*

Proof. We must prove Statement 1.1 with $c = p(p - 1) - 1$. For all d , the case $n = 0$ immediately follows from [9, Theorem 18.3, (5)]. In particular, the theorem is proved when $d = 0$. We can therefore proceed by induction on d , and assume that $n > 0$.

For all $i \geq 1$, define an integer

$$a_p(i) = \begin{cases} 1 & \text{if } p - 1 | i \\ 0 & \text{otherwise} \end{cases}$$

so that we have

$$\left[\frac{n}{p-1} \right] = \left[\frac{n-1}{p-1} \right] + a_p(n). \tag{6}$$

Since $0 < n < p(p-1)$, we can use Lemma 3.3 below to choose an integer l , prime to p , such that

$$v_p(l^n - 1) = a_p(n). \tag{7}$$

Write, in $K'_0(X) \otimes \mathbb{Z}[1/l]$, using Proposition 3.1,

$$\psi_l(x) = l^{-d} \cdot x + l^{-d-m} \cdot \alpha$$

with $\alpha \in K'_0(X)_{(d-1)}$, and $m \geq 0$. Then applying $l^{d+m} \cdot \text{ch}_{d-n}$ to the relation above, and using Proposition 2.4, we obtain in $\text{CH}_{d-n}(X)_{\mathbb{Q}}$

$$l^m \cdot (l^n - 1) \cdot \text{ch}_{d-n}(x) = \text{ch}_{d-n}(\alpha).$$

Because of (7), we get

$$v_p(\text{ch}_{d-n}(x)) = v_p(\text{ch}_{d-n}(\alpha)) + a_p(n).$$

We conclude using (6) and the induction hypothesis for $\alpha \in K'_0(X)_{(d-1)}$. \square

Lemma 3.3. *Let p be a prime number. Then there exists an integer l , prime to p , with the following properties*

- (a) *If n is not divisible by $p-1$, then $l^n - 1$ is not divisible by p .*
- (b) *If n is not divisible by $p(p-1)$, then $l^n - 1$ is not divisible by p^2 .*

Proof. Take for l an integer whose class modulo p^2 generates the cyclic group $(\mathbb{Z}/p^2)^\times$. Then one checks easily that the class of l modulo p generates the cyclic group $(\mathbb{Z}/p)^\times$. \square

Remark 3.4. In contrast with [14] where the normalization procedure was used, we do not use geometric arguments here. We only use basic relations between Adams operations, Chern character, and filtration on K -theory. Using [3, (1.5)] in place of Proposition 3.1, and [2, Theorem 5.1, (vi)] in place of Proposition 2.4, one could proceed as above to replace the assumption “ X is without torsion” by “ $n < p(p-1)$ ” in [4, Theorem 7.1], thus giving an elementary proof of the p -primary part of [1, Theorem 1], for small n .

Note also that the approach of [14] is limited by nature to schemes whose normalization morphism is of finite type, while Theorem 3.2 is valid for more general schemes (see Appendix).

4. Consequences of Gabber’s Theorem

In this section, we use the theorem of Gabber asserting that, over a field of characteristic different from p , any integral variety X admits a regular alteration of degree prime to p . This means that there exists a projective morphism $Z \rightarrow X$, generically finite of degree prime to p , with Z regular. Some details can be found in [10,17].

If $d \geq 0$ is an integer, the d -th Todd number is

$$\tau_d = \prod_{p \text{ prime}} p^{\lfloor d/(p-1) \rfloor}. \tag{8}$$

Proposition 4.1. *Let X be a local complete intersection variety. Then, for all n*

$$\tau_n \cdot \text{ch}_{\dim X - n}[\mathcal{O}_X] \in \text{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}}.$$

Proof. Let $x : X \rightarrow \text{Spec}(k)$ be the structural morphism of X . Then by [9, Theorem 18.3, (4) and (5)],

$$\text{ch}[\mathcal{O}_X] = \text{ch} \circ x^*[\mathcal{O}_{\text{Spec}(k)}] = \text{Td}(T_X) \cdot x^* \circ \text{ch}[\mathcal{O}_{\text{Spec}(k)}] = \text{Td}(T_X) \cdot [X].$$

Then the proposition follows from Lemma 6.3 below. \square

Theorem 4.2. *Let p be a prime different from the characteristic of the base field. Then p -integrality holds in codimension $\leq c$, for any c .*

Proof. We must prove Statement 1.1 with c arbitrary large. We proceed by induction on d , the case $d = 0$ being obvious. It is enough to consider the case X integral of dimension d , and $x = [\mathcal{O}_X]$. Using Gabber’s theorem, we choose $f : Z \rightarrow X$ a projective morphism, with Z a regular connected variety of dimension d , and such that $[k(Z) : k(X)] = \lambda$ is prime to p . Then there is an element $\delta \in K'_0(X)_{(d-1)}$ such that

$$f_*[\mathcal{O}_Z] = \lambda \cdot x + \delta.$$

Thus

$$\begin{aligned} v_p(\text{ch}_{d-n}(x)) &\geq \min(v_p(f_* \text{ch}_{d-n}[\mathcal{O}_Z]), v_p(\text{ch}_{d-n}(\delta))) \\ &\geq \min(v_p(\text{ch}_{d-n}[\mathcal{O}_Z]), v_p(\text{ch}_{(d-1)-(n-1)}(\delta))). \end{aligned}$$

We conclude using Proposition 4.1 for Z , and the induction hypothesis on δ . \square

The combination of Theorems 3.2 and 4.2 implies the following.

Corollary 4.3. *Let X be a variety over a field of characteristic p , and d be an integer. Then for all integers n such that $0 \leq n < p(p - 1)$, and all elements $x \in K'_0(X)_{(d)}$, we have*

$$\tau_n \cdot \text{ch}_{d-n}(x) \in \text{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}}.$$

Part 2. Operations on Chow groups

5. Construction

We proceed as in [14] to derive operations from the integrality property of the Chern character stated above. This follows [4, Section 7].

Let A be an abelian group. Then the natural map

$$A_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p \rightarrow A_{\mathbb{Z}(p) \subset \mathbb{Q}} \otimes \mathbb{Z}/p, \tag{9}$$

is an isomorphism. This group can also be described as A modulo p and p -primary torsion. By this we mean the quotient of A by $p \cdot A + T$, where T is the subgroup of elements $a \in A$ such that $p^n \cdot a = 0$ for some integer $n > 0$.

We assume that p -integrality holds in codimension $\leq m(p - 1)$, for some integer m . Let $q > 0$ be an integer. For every integer i such that $0 < i \leq m$, we see using Statement 1.1, for $d = q$ and $n = i(p - 1)$, that the association $x \mapsto p^i \cdot \text{ch}_{q-i(p-1)}(x)$ induces a natural transformation

of functors $\text{Proj} \rightarrow \text{Ab}$

$$K'_0(-)_{(q)} \rightarrow \text{CH}_{q-i(p-1)}(-)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}.$$

Let X be a variety, and $x \in K'_0(X)_{(q-1)} \subset K'_0(X)_{(q)}$. Then

$$p^i \cdot \text{ch}_{q-i(p-1)}(x) = p \cdot (p^{i-1} \cdot \text{ch}_{(q-1)-(i(p-1)-1)}(x)).$$

Using **Statement 1.1** for $d = q - 1$ and $n = i(p - 1) - 1$, we see that this element belongs to $p \cdot \text{CH}_{q-i(p-1)}(X)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}$. This gives a natural transformation of functors $\text{Proj} \rightarrow \text{Ab}$

$$\text{gr}_q K'_0(-) \otimes \mathbb{Z}/p \rightarrow \text{CH}_{q-i(p-1)}(-)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

Composing on the left with the map (2), and on the right with the inverse of the map (9) with $A = \text{CH}_{q-i(p-1)}(-)$, we obtain a natural transformation of functors $\text{Proj} \rightarrow \text{Ab}$

$$T_i: \text{CH}_q(-) \otimes \mathbb{Z}/p \rightarrow \text{CH}_{q-i(p-1)}(-)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

It will be convenient to agree that

$$T_0: \text{CH}_q(-) \otimes \mathbb{Z}/p \rightarrow \text{CH}_q(-)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$$

is the quotient map modulo the image modulo p of p -primary torsion.

6. The inverse Todd genus

We consider the power series in the variable x

$$R^{(p)}(x) = \sum_{i=0}^{\infty} (-1)^i \cdot x^{p^i-1}. \tag{10}$$

There is a unique morphism of presheaves of abelian groups on Var/k

$$r^{(p)}: K_0(-) \rightarrow (\text{CH}(-) \otimes \mathbb{Z}/p)^\times$$

such that for L a line bundle,

$$r^{(p)}(L) = R^{(p)}(c_1(L)).$$

Individual components are denoted

$$r_i^{(p)}: K_0(-) \rightarrow \text{CH}^{i(p-1)}(-) \otimes \mathbb{Z}/p.$$

Using the terminology of [24], this class will be the *inverse Todd genus* of the operation $T = \sum_i T_i$ constructed in the previous section. In this section we describe a relation between the characteristic class $r^{(p)}$ and the Todd class.

Remark 6.1. The inverse Todd genus of the usual Steenrod operation $\sum_i S_i$ is defined by the power series $W^{(p)}(x) = 1 + x^{p-1}$. One easily checks that $x \cdot R^{(p)}(x)$ is the left inverse of $x \cdot W^{(p)}(x)$ for the law given by composition of power series with coefficients in \mathbb{Z}/p .

We begin by stating an easy argument that we will use repeatedly below.

Lemma 6.2. Let A be a commutative ring, M an A -module, $a_u \in A_{\mathbb{Q}}$ and $m_u \in M_{\mathbb{Q}}$ for $u = 0, \dots, i(p - 1)$ satisfying

$$v_p(a_u) \geq -\left\lfloor \frac{u}{p-1} \right\rfloor \quad \text{and} \quad v_p(m_u) \geq -\left\lfloor \frac{u}{p-1} \right\rfloor.$$

Then we have an equality in $M_{\mathbb{Z}(p) \subset \mathbb{Q}} \otimes \mathbb{Z}/p$,

$$\sum_{u=0}^{i(p-1)} (p^i \cdot a_u \cdot m_{i(p-1)-u}) = \sum_{j=0}^i (p^j \cdot a_{j(p-1)}) \cdot (p^{i-j} \cdot m_{(i-j)(p-1)}).$$

Proof. We compute, for all u , using (1),

$$\begin{aligned} v_p(a_u \cdot m_{i(p-1)-u}) &\geq v_p(a_u) + v_p(m_{i(p-1)-u}) \\ &\geq -\left[\frac{u}{p-1}\right] - \left[\frac{i(p-1)-u}{p-1}\right] \\ &= -\left[\frac{u}{p-1}\right] - i - \left[\frac{-u}{p-1}\right]. \end{aligned}$$

This integer is always $\geq -i$; hence the elements delimited by the parentheses in the formula of the statement belong to the submodule $M_{\mathbb{Z}(p) \subset \mathbb{Q}} \subset M_{\mathbb{Q}}$. Moreover, when u is not divisible by $p-1$, this integer is $> -i$. This gives the requested formula modulo p . \square

Lemma 6.3. We have, for all integers n , and all $x \in K_0(X)$

$$v_p(\text{Td}^n(x)) \geq -\left[\frac{n}{p-1}\right].$$

Proof. Let X be a variety. Then the set of elements $1 + y$ with

$$y \in \bigoplus_{n>0} p^{-[n/(p-1)]} \cdot \text{CH}^n(X)_{\mathbb{Z}(p) \subset \mathbb{Q}}$$

is a subgroup $B(X)$ of $(\text{CH}(X)_{\mathbb{Q}})^{\times}$. If $f: Y \rightarrow X$ is a morphism of varieties, then $f^*B(X) \subset B(Y)$. Thus we can define a presheaf of abelian groups C on Var/k by

$$C(-) = (\text{CH}(-)_{\mathbb{Q}})^{\times} / B(-).$$

We have a morphism of presheaves of abelian groups on Var/k

$$v: K_0(-) \rightarrow C(-), \quad x \mapsto \text{Td}(-x).$$

When L is a line bundle, we have by definition of the Todd class,

$$v[L] = \sum_{n \geq 0} \frac{(-c_1(L))^n}{(n+1)!}.$$

Then v vanishes on the classes of line bundles, because of Lemma 6.5 below. The splitting principle allows us to conclude.¹ \square

Proposition 6.4. Let $j \geq 0$ be an integer, X a variety, and $x \in K_0(X)$. Then the elements $p^j \cdot \text{Td}^{j(p-1)}(-x)$ and $r_j^{(p)}(x)$ have same image in

$$\text{CH}^{j(p-1)}(X)_{\mathbb{Z}(p) \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

¹ If $f: Y \rightarrow X$ is a projective bundle, one can check that the pull-back $C(X) \rightarrow C(Y)$ is injective using the fact that $f^*: \text{CH}(X) \rightarrow \text{CH}(Y)$ is a split monomorphism.

Proof. We consider the presheaf of commutative rings on Var/k

$$\widetilde{\text{CH}}(-) = \text{CH}(-)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

The characteristic class $r^{(p)}$ induces a morphism of presheaves of abelian groups on Var/k

$$\widetilde{r}^{(p)} : K_0(-) \rightarrow \widetilde{\text{CH}}(-)^\times.$$

By Lemma 6.3, we can consider the association

$$\rho : K_0(-) \rightarrow \widetilde{\text{CH}}(-), \quad x \mapsto \sum_{j \geq 0} p^j \cdot \text{Td}^{j(p-1)}(-x).$$

It is clearly compatible with arbitrary pull-backs. It sends sums to products because the total Todd class has the same property, and because of Lemmas 6.2 and 6.3.

Let L be a line bundle. Then by definition of the Todd class, we have

$$\rho[L] = \sum_{j \geq 0} \frac{p^j}{(j(p-1) + 1)!} \cdot (-c_1(L))^{j(p-1)}.$$

Using Lemma 6.5 below, we see that the j -th coefficient is an element of $\mathbb{Z}_{(p)}$. Moreover, we see that it belongs to $p\mathbb{Z}_{(p)}$ if and only if $j(p-1) + 1$ is not a power of p . Otherwise, when say $j(p-1) + 1 = p^i$, we see using Lemma 6.6 below, that the j -th coefficient is $(-1)^i$ modulo $p\mathbb{Z}_{(p)}$. In view of (10), the characteristic class ρ coincides with $\widetilde{r}^{(p)}$ on the classes of line bundles, and the claim follows from the splitting principle. \square

Lemma 6.5 ([20, p.10]). *Let p be a prime number, and n a positive integer. Let s be the sum of the digits appearing in the expression of n in base p . Then*

$$v_p(n!) = \frac{n - s}{p - 1}.$$

Lemma 6.6. *Let $i \geq 0$ be an integer, and p a prime number. We have*

$$(p^i)! \cdot p^{-\frac{p^i-1}{p-1}} = (-1)^i \pmod p.$$

Proof. We proceed by induction on i . For $i = 0$ this is immediate. Let $i \geq 1$ and consider the set $U_i = \{1, 2, \dots, p^i - 1, p^i\}$. The product of all the integers sitting between two consecutive multiples of p is congruent to -1 modulo p , by Wilson’s Theorem. Thus the product of all elements of U_i which are non-divisible by p is congruent modulo p to $(-1)^{p^{i-1}} = -1$. The product of the p^{i-1} elements of U_i which are divisible by p is $p^{p^{i-1}} \cdot (p^{i-1})!$. Hence we have

$$(p^i)! \cdot p^{-\frac{p^i-1}{p-1}} = (-1) \cdot p^{p^{i-1}} \cdot (p^{i-1})! \cdot p^{-\frac{p^i-1}{p-1}} = -p^{-\frac{p^{i-1}-1}{p-1}} \cdot (p^{i-1})!.$$

By the induction hypothesis, this is $(-1)^i$, as requested. \square

7. Properties

Let p be a prime number. In this section, we assume that integrality holds in codimension $\leq m(p-1)$, and describe some functorial properties of the operations T_0, \dots, T_m constructed in Section 5.

Proposition 7.1. *Let $f: Y \rightarrow X$ be a local complete intersection morphism. Let i be an integer such that $0 \leq i \leq m$. Then we have*

$$T_i \circ f^* = \sum_{j=0}^i r_j^{(p)}(-T_f) \cdot f^* \circ T_{i-j}.$$

Proof. We have, by [9, Theorem 18.3, (4)],

$$\text{ch}_{n-i(p-1)} \circ f^* = \sum_{u=0}^{i(p-1)} \text{Td}^u(T_f) \cdot f^* \circ \text{ch}_{n-i(p-1)+u}.$$

In view of Lemmas 6.3 and 6.2 and Theorem 3.2, we have, in the group $\text{CH}_{n-i(p-1)}(Y)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$,

$$p^i \cdot \text{ch}_{n-i(p-1)} \circ f^* = \sum_{j=0}^i (p^j \cdot \text{Td}^{j(p-1)}(T_f)) \cdot f^* \circ (p^{i-j} \cdot \text{ch}_{n-(i-j)(p-1)}).$$

Since the map (2) is compatible with f^* , Proposition 6.4 allows us to conclude. \square

Applying Proposition 7.1 to the structural morphism of a local complete intersection variety, we obtain the formula computing the effect of the operation T_i on regular cycles.

Corollary 7.2. *Let X be a local complete intersection variety (e.g. a regular variety). Let i be an integer such that $0 \leq i \leq m$. Then*

$$T_i[X] = r_i^{(p)}(-T_X) \cdot [X].$$

Proposition 7.3. *Let X and Y be two varieties over the same field. Then for $x \in \text{CH}_d(X)$, and $y \in \text{CH}_e(Y)$, and $0 \leq i \leq m$, we have*

$$T_i(x \times y) = \sum_{j=0}^i T_j(x) \times T_{i-j}(y).$$

Proof. Let $a \in K'_0(X)_{(d)}$ (resp. $b \in K'_0(X)_{(e)}$) be an element such that $\varphi_X(x) = a \bmod K'_0(X)_{(d-1)}$ (resp. $\varphi_Y(y) = b \bmod K'_0(Y)_{(e-1)}$), where φ_- is the map (2). We have, by [9, Example 18.3.1] (and [9, Theorem 18.3, (5)])

$$\text{ch}_{d+e-i(p-1)}(a \times b) = \sum_{u=0}^{i(p-1)} \text{ch}_{d-u}(a) \times \text{ch}_{e+u-i(p-1)}(b).$$

By Lemma 6.2, this gives in $\text{CH}_{d+e-i(p-1)}(X \times Y)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$,

$$p^i \cdot \text{ch}_{d+e-i(p-1)}(a \times b) = \sum_{j=0}^i (p^j \cdot \text{ch}_{d-j(p-1)}(a)) \times (p^{i-j} \cdot \text{ch}_{e-(i-j)(p-1)}(b)).$$

We can conclude using the fact that φ is compatible with external products. \square

Assume that X is a local complete intersection variety. Define, for $i \leq m$, the i -th cohomological operation T^i by the formula

$$T^i = \sum_{j=0}^i r_j^{(p)}(T_X) \cdot T_{i-j} \quad : \quad \text{CH}_\bullet(X) \rightarrow \text{CH}_{\bullet-i(p-1)}(X)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

Proposition 7.4. *Let X and Y be local complete intersection varieties over a common field, and i an integer such that $0 < i \leq m$.*

(i) *For $x \in \text{CH}(X)$, $y \in \text{CH}(Y)$, we have*

$$T^i(x \times y) = \sum_{j=0}^i T^j(x) \times T^{i-j}(y).$$

(ii) *When X is smooth, $x, y \in \text{CH}(X)$, we have*

$$T^i(x \cdot y) = \sum_{j=0}^i T^j(x) \cdot T^{i-j}(y).$$

(iii) *The operation T^0 is the identity.*

(iv) *$T^i[X] = 0$.*

(v) *Assume that X is smooth, and let $x \in \text{CH}^1(X)$. Then $T^i(x) = -x^p$ if $i = 1$, and $T^i(x) = 0$ otherwise.*

(vi) *If $f : Y \rightarrow X$ is a local complete intersection morphism, then $T^i \circ f_* = f_* \circ T^i$.*

(vii) *Let $f : Y \rightarrow X$ be a projective morphism. Let $T_f = T_Y - f_* T_X \in K_0(Y)$. Then*

$$T^i \circ f_* = f_* \circ \sum_{j=0}^i r_j^{(p)}(-T_f) \cdot T^{i-j}.$$

Proof. Statement (i) follows from Proposition 7.3, statement (vi) from Proposition 7.1, and (vii) from the fact that the operations T_i are compatible with projective push-forwards. Then (ii) follows from (i) and (vi), statement (iv) from (vi) and the obvious computation for $X = \text{Spec}(k)$. Statement (iii) is a consequence of the definition of T_0 .

We now prove (v). We can assume that $x = [Z]$ for some integral closed subvariety Z of X . Since X is locally factorial, Z is a locally principal divisor. In particular, the closed embedding $f : Z \hookrightarrow X$ is regular, and Z is a local complete intersection variety. Then, using (iv) and (vii), where $T_f = -f_*[\mathcal{O}_X(Z)]$, we obtain for $i > 0$,

$$\begin{aligned} T^i(x) &= T^i \circ f_*[Z] = f_* \circ T^i[Z] - f_*(c_1(f_*[\mathcal{O}_X(Z)]))^{p-1} \cdot T^{i-1}[Z] \\ &= -c_1(\mathcal{O}_X(Z))^{p-1} \cdot f_* \circ T^{i-1}[Z]. \end{aligned}$$

This is zero when $i > 1$ by (iv). When $i = 1$, this is, by (iii),

$$-c_1(\mathcal{O}_X(Z))^{p-1} \cdot [Z] = -c_1(\mathcal{O}_X(Z))^p = -x^p,$$

as requested. \square

Finally, we mention the following.

Proposition 7.5. *Let L/k be a field extension. Then, for all varieties X , all $x \in \text{CH}(X)$, and i such that $0 \leq i \leq m$, we have*

$$T_i(x_L) = T_i(x)_L.$$

Proof. This follows from [14, (2) and Theorem 3.1, (d)]. \square

8. Relation with Steenrod operations

In this section, we compare the operations T_i with the Steenrod operations constructed by other methods. We therefore make the assumption that the characteristic of the base field is different from p .

Consider the i -th homological Steenrod operation of, say, [7]

$$S_i : \text{CH}_\bullet(-) \otimes \mathbb{Z}/p \rightarrow \text{CH}_{\bullet-i(p-1)}(-) \otimes \mathbb{Z}/p,$$

and write $S = \sum_i S_i$. Since $S_0 = \text{id}$, we have $S = \text{id} + f$, where for every variety X the induced endomorphism f^X of $\text{CH}(X) \otimes \mathbb{Z}/p$ is nilpotent. We write T'^X for the left inverse of S^X ; this defines a natural transformation of functors $\text{Var}/k \rightarrow \text{Ab}$ with components

$$T'_i : \text{CH}_\bullet(-) \otimes \mathbb{Z}/p \rightarrow \text{CH}_{\bullet-i(p-1)}(-) \otimes \mathbb{Z}/p.$$

We have $T'_0 = \text{id}$ and, for $i > 0$, one can express T'_i inductively as

$$T'_i = - \sum_{j=1}^i T'_{i-j} \circ S_j. \tag{11}$$

Proposition 8.1. *We have for $i \leq p$*

$$T'_i = (-1)^i \cdot S_i.$$

Proposition 8.2. *Assume that X is a smooth variety. Then*

$$T'[X] = r^{(p)}(-T_X)[X].$$

Proof (*Proof of Propositions 8.1 and 8.2.*) Let Z be a variety. It is proven in [22, Lemma 5.2 and Proposition 3.1] that the homological Steenrod operations S_\bullet induce an action of the Steenrod \mathbb{Z}/p -algebra \mathcal{S} on $\text{CH}(Z) \otimes \mathbb{Z}/p$. The operation S_j acts as the element $\mathcal{P}^j \in \mathcal{S}$ (in case $p = 2$ we write \mathcal{P}^j for Sq^{2j}). Using (11), we see that the action of T'_j on $\text{CH}(Z) \otimes \mathbb{Z}/p$ coincides with the action of the conjugate $c(\mathcal{P}^j)$ of \mathcal{P}^j , in the sense of [23, Section 7]. Let b_i for $i \in \mathbb{N} - \{0\}$ be indeterminates. If $R = (r_1, r_2, \dots)$ is a sequence of non-negative integers, we write b^R for the product $b_1^{r_1} b_2^{r_2} \dots$. By [23, Corollary 6], the morphism

$$\mathcal{S}[[b_1, b_2, \dots]] \rightarrow \mathcal{S}[[x]], \quad b_i \mapsto (-1)^{\frac{p^i-1}{p-1}} \cdot x^{p^i-1} = (-1)^i \cdot x^{p^i-1} \pmod p$$

sends the total Steenrod operation $\sum_R \mathcal{P}^R \cdot b^R$ to $\sum_j c(\mathcal{P}^j) \cdot x^{j(p-1)}$.

We obtain Proposition 8.1 by noticing that for $i \leq p$ the element \mathcal{P}^i is the only basis element of type \mathcal{P}^R of the same dimension as $c(\mathcal{P}^i)$.

Moreover, by [22, Proposition 5.3, (2)], we obtain

$$T'[X] = d(-T_X)[X],$$

where d is the total Chern class $\sum_R c_R \cdot b^R$ evaluated at $b_i = (-1)^i$. Inspection of the value of d on a line bundle reveals that $d = r^{(p)}$, establishing Proposition 8.2. \square

Proposition 8.3. *Assume that the base field is of characteristic different from p . Let i be any integer ≥ 0 . Then we have a commutative diagram*

$$\begin{array}{ccc}
 \mathrm{CH}(-) \otimes \mathbb{Z}/p & \xrightarrow{T'_i} & \mathrm{CH}(-) \otimes \mathbb{Z}/p \\
 & \searrow T_i & \downarrow \\
 & & \mathrm{CH}(-)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p
 \end{array}$$

Proof. By Lemma 8.4 below and Proposition 7.5, we can assume that the base field is perfect.

Since both operations are compatible with proper push-forwards, it will be enough to compare their values in $\mathrm{CH}_{d-i(p-1)}(X)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$ on the class $[X]$, when X is an integral variety of dimension d . Using Gabber’s theorem, we find a projective morphism $f: Z \rightarrow X$, with Z smooth of dimension d , and f dominant with $[k(Z) : k(X)] = \lambda$ an integer prime to p . Let $v \in (\mathbb{Z}/p)^\times$ be the inverse modulo p of λ . Then, using Corollary 7.2,

$$T_i[X] = v \cdot T_i(\lambda \cdot [X]) = v \cdot T_i \circ f_*[Z] = v \cdot f_* \circ T_i[Z] = v \cdot f_*(r_i^{(p)}(-T_Z) \cdot [Z]).$$

A similar computation, and Proposition 8.2, show that $T'_i[X]$ has the same value, as requested. \square

Lemma 8.4. *Let X be a variety over a field k of characteristic $\neq p$. Let F/k be a perfect closure. Then scalars extension induces an injective morphism*

$$\mathrm{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p \rightarrow \mathrm{CH}(X_F)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p.$$

Proof. Let L/k be a finite field extension. A transfer argument shows that $\mathrm{CH}(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}(X_L)_{\mathbb{Q}}$ is injective, hence so is $\mathrm{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}} \rightarrow \mathrm{CH}(X_L)_{\mathbb{Z} \subset \mathbb{Q}}$, and therefore so is

$$\mathrm{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}_{(p)} \rightarrow \mathrm{CH}(X_L)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}_{(p)}.$$

When the degree of the extension L/k is prime to p , the transfer induces a splitting of this monomorphism, so that

$$\mathrm{CH}(X)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p \rightarrow \mathrm{CH}(X_L)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$$

is injective.

By [29, Theorem 5.20 and Proposition 5.26], the group $\mathrm{CH}(X_F)$ is the direct limit of the groups $\mathrm{CH}(X_L)$ with L/k a finite subextension of F/k . Such an extension L/k is of degree prime to p because it is a purely inseparable extension of a field of characteristic $\neq p$. Using the fact that the functor $A \mapsto A_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$ commutes with any exact functor commuting with tensor products, we see that the natural map

$$\mathrm{colim}_{\substack{k \subset L \subset F \\ [L:k] < \infty}} (\mathrm{CH}(X_L)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p) \rightarrow \mathrm{CH}(X_F)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$$

is an isomorphism. The lemma follows. \square

Remark 8.5. Proposition 8.3 actually gives some new information about Steenrod operations modulo p in characteristic not p . Denote by $\mu: \mathrm{CH}(-) \otimes \mathbb{Z}/p \rightarrow \mathrm{CH}(-)_{\mathbb{Z} \subset \mathbb{Q}} \otimes \mathbb{Z}/p$ the natural quotient map. Let X be a variety, and

$$\mathcal{K}(X) = \ker(\mathrm{CH}(X) \rightarrow \mathrm{gr} K'_0(X)). \tag{12}$$

Then $\mathcal{K}(X)$ consists of torsion elements. Let $\mathcal{K}_p(X)$ be its image in $\text{CH}(X) \otimes \mathbb{Z}/p$. One can ask the following question: “What is the set \mathcal{E} of operations Q on Chow groups modulo p such that $\mu \circ Q$ vanishes on $\mathcal{K}_p(X)$, for all X ?”

By construction, we have $T_i(\mathcal{K}_p(X)) = 0$; hence a consequence of Proposition 8.3 is that $T'_i \in \mathcal{E}$, for all i . On the other hand when $p = 2$ and $j > 1$, then the cohomological Steenrod square S^{2^j-1} does not belong to \mathcal{E} (see [16, Remark 8.4]).

This can also be used to obtain information on $\text{gr } K'_0(X)$. For example, a direct consequence of Proposition 8.3 is that $\text{gr}_2 K'_0(D)$ contains non-zero 3-torsion, where D is the variety of [27] (over a field of characteristic zero).

Remark 8.6. One can use the operations T_i to recover some of the results of [15]. Assume that X is a variety and L/k an extension of the base field such that $\text{CH}(X_L)$ is torsion-free. Then T_i is an endomorphism of $\text{CH}(X_L) \otimes \mathbb{Z}/p$. We set $S_0 = \text{id}$, and define inductively, for $n > 0$ and $x \in \text{CH}(X_L) \otimes \mathbb{Z}/p$,

$$S_n(x) = - \sum_{i=1}^n T_i \circ S_{n-i}(x).$$

Let $\overline{\text{Ch}}(X) = \text{im}(\text{CH}(X) \otimes \mathbb{Z}/p \rightarrow \text{CH}(X_L) \otimes \mathbb{Z}/p)$ be the reduced Chow group modulo p . Since the operations T_i commute with extension of the base field, they send the subgroup $\overline{\text{Ch}}(X) \subset \text{CH}(X_L) \otimes \mathbb{Z}/p$ to itself. Therefore, the association $x \mapsto S_n(x)$ induces an operation of this subgroup, the n -th reduced Steenrod operation.

Part 3. Degree formula and applications.

We now describe some consequences of the results obtained above. In order to make unconditional statements, we restrict ourselves to applications of the fact that p -integrality holds in codimension $\leq p(p - 1) - 1$ (Theorem 3.2).

We fix prime number p . When X is a projective variety, we let n_X be the positive generator of the image of the degree map $\text{CH}(X) \rightarrow \mathbb{Z}$. We also set

$$n_X(p) = p^{v_p(n_X)}.$$

Let $x : X \rightarrow \text{Spec}(k)$ be the structural morphism of X . We denote by $\chi(\mathcal{O}_X)$ the integer corresponding to $x_*[\mathcal{O}_X]$ under the identification $K'_0(\text{Spec}(k)) = \mathbb{Z}$. In other words,

$$\chi(\mathcal{O}_X) = \sum_{j \geq 0} (-1)^j \dim_k H^j(X, \mathcal{O}_X).$$

Note that $\text{ch}[\mathcal{O}_{\text{Spec}(k)}] = [\text{Spec}(k)]$, so that

$$\chi(\mathcal{O}_X) = \text{deg} \circ \text{ch}_0[\mathcal{O}_X].$$

9. Index and Euler characteristic

We begin by giving a restriction on the possible values of the index n_X of a projective variety X , in terms of two integers which do not change under field extensions: the dimension of X and the Euler characteristic $\chi(\mathcal{O}_X)$.

Proposition 9.1. *Let X be a projective variety of dimension $< p(p - 1)$. Then*

$$v_p(n_X) \leq \left\lceil \frac{\dim X}{p - 1} \right\rceil + v_p(\chi(\mathcal{O}_X)).$$

Proof. We have, by [Theorem 3.2](#),

$$v_p(\text{ch}_0[\mathcal{O}_X]) \geq -\left\lceil \frac{\dim X}{p-1} \right\rceil,$$

which means that there is an integral zero-cycle $x \in \text{CH}_0(X)$, and an integer λ prime to p such that

$$x = \lambda \cdot p^{\lceil \dim X / (p-1) \rceil} \cdot \text{ch}_0[\mathcal{O}_X].$$

Taking degrees, we obtain

$$n_X |\lambda \cdot p^{\lceil \dim X / (p-1) \rceil} \cdot \chi(\mathcal{O}_X),$$

whence the result. \square

Let X be an integral projective variety of dimension $i(p-1)$, with $i > 0$. We define in $\mathbb{Z}/n_X(p)$

$$t_p(X) = p^{i-1} \cdot \chi(\mathcal{O}_X) \text{ mod } n_X(p).$$

Then [Proposition 9.1](#) amounts to saying that we have, when $i < p$,

$$p \cdot t_p(X) = 0 \text{ mod } n_X(p).$$

The same conclusion holds for any $i > 0$ when X is a local complete intersection variety by [Proposition 4.1](#), or if p -integrality holds in codimension $\leq i(p-1)$.

Remark 9.2. By [[26](#), Corollary 1], the class $t_p(X)$ vanishes when the base field is perfect of characteristic p .

Remark 9.3. Assume that $i < p$. Let $u_p(X) \in \text{CH}_0(X)$ be a cycle whose class modulo p and p -primary torsion is $T_i[X]$. Then its image $\overline{u}_p(X)$ in $\text{CH}_0(X)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}$ satisfies

$$\overline{u}_p(X) = p^i \cdot \text{ch}_0[\mathcal{O}_X] \text{ mod } p \text{ CH}_0(X)_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}.$$

Hence

$$\text{deg } \overline{u}_p(X) = p^i \cdot \chi(\mathcal{O}_X) \text{ mod } pn_X \mathbb{Z}_{(p)}$$

and

$$\frac{\text{deg } u_p(X)}{p} = p^{i-1} \cdot \chi(\mathcal{O}_X) \text{ mod } n_X(p).$$

Thus the class $t_p(X)$ corresponds to the Rost numbers for a certain combination of Steenrod operations (see [[22](#), p. 20]) under the map $\mathbb{Z}/n_X \rightarrow \mathbb{Z}/n_X(p)$.

When $f: Y \rightarrow X$ is a morphism between varieties of the same dimension, then the integer $\text{deg } f$ is defined as the degree of the extension of function fields when f is dominant, and as zero otherwise.

Theorem 9.4 (Degree Formula). *Let $f: Y \rightarrow X$ be a morphism of integral projective varieties of the same dimension $i(p-1)$ with $0 < i \leq p$. Then*

$$t_p(Y) = \text{deg } f \cdot t_p(X) \text{ mod } n_X(p).$$

Proof. We have in $K'_0(X)$

$$f_*[\mathcal{O}_Y] = \text{deg } f \cdot [\mathcal{O}_X] + \delta \tag{13}$$

with $\delta \in K'_0(X)_{(i(p-1)-1)}$. This element δ can be written as a linear combination of elements of type $[\mathcal{O}_Z]$, for Z a closed subvariety of X , with $\dim Z \leq i(p-1)-1$. Note that $n_X(p) | n_Z(p)$ for such Z . The result is obtained by projecting (13) to $K'_0(\text{Spec}(k))$, and using Proposition 9.1. \square

Thus the class $t_p(-)$ is defined for varieties of dimension $i(p-1)$ when i is arbitrary, satisfies the degree formula when $i \leq p$, and is of p -torsion when $i \leq p-1$.

Remark 9.5. Let X be a projective variety of dimension $i(p-1)$, with $i \leq p$. Assume that there is a morphism $f: X \rightarrow Y$, for some projective variety Y , such that $f_*[X] \in \mathcal{K}(Y)$ (see (12)). Then $t_p(X) = 0 \pmod{n_Y(p)}$.

10. Correspondences modulo p

Let X and Y be varieties, with Y projective and X integral. A correspondence $\alpha: X \rightsquigarrow Y$ is an element $\alpha \in \text{CH}(X \times Y)$. The *multiplicity* of α is the image of α under the composite

$$\text{CH}(X \times Y) \rightarrow \text{CH}(\text{Spec}(k(X)) \times Y) \xrightarrow{\text{deg}} \text{CH}(\text{Spec}(k(X))) = \mathbb{Z}.$$

10.1. Strong p -incompressibility

Let p be a prime number. Following [18, p. 150], we call an integral projective variety X *strongly p -incompressible* if the existence of a correspondence $X \rightsquigarrow Y$ of multiplicity prime to p , with Y an integral projective variety of dimension $\leq \dim X$ and such that $v_p(n_Y) \geq v_p(n_X)$, implies that $\dim Y = \dim X$, and that there exists a correspondence $Y \rightsquigarrow X$ of multiplicity prime to p .

Proposition 10.1. *Let p be a prime number, and i an integer such that $0 < i \leq p$. Let X be an integral projective variety with $v_p(n_X) \geq i$. Assume that $\chi(\mathcal{O}_X)$ is prime to p . Then $\dim X \geq i(p-1)$. In case of equality, X is strongly p -incompressible.*

Proof. Assume that $\dim X < i(p-1)$. Then in particular $\dim X < p(p-1)$, and Proposition 9.1 gives

$$i \leq v_p(n_X) \leq \left\lfloor \frac{\dim X}{p-1} \right\rfloor + v_p(\chi(\mathcal{O}_X)) < i,$$

a contradiction. Hence $\dim X \geq i(p-1)$.

Now we assume that $\dim X = i(p-1)$. We proceed as in the proof of [22, Theorem 7.2] — but as we include the case $i = p$, we cannot use the relation $p \cdot t_p(X) = 0$.

Let Y be an integral projective variety of dimension $\leq \dim X$, with a correspondence $X \rightsquigarrow Y$ of multiplicity prime to p , and such that $v_p(n_Y) \geq v_p(n_X)$. We can assume that there is an integral variety Z with $\dim Z = \dim X$, and two projective morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y \hookrightarrow Y'$, such that $\text{deg } f$ is prime to p . Here Y' is the product of Y with a projective space of appropriate dimension: in particular $\dim Y' = \dim X$ and $n_{Y'} = n_Y$. Since by Theorem 9.4 we have

$$\text{deg } f \cdot t_p(X) = t_p(Z) \pmod{n_X(p)},$$

we see that $t_p(Z)$ is non-zero modulo $n_X(p)$, hence modulo $n_Y(p)$. Now by the same theorem we have

$$\deg g \cdot t_p(Y') = t_p(Z) \pmod{n_Y(p)},$$

hence $\deg g$ is non-zero and g is dominant, so that $Y = Y'$ and $\dim Y = \dim X$. Then Z gives a correspondence $Y \rightsquigarrow X$ of multiplicity prime to p .

For further reference, let us mention that we also obtain that $t_p(Y)$ is non-zero modulo $n_Y(p)$. \square

Let X be the Severi–Brauer variety of a central division algebra of degree p^n , with p a prime number. Then X is strongly p -incompressible when the characteristic of the base field is not p [22, Example 7.2, Theorem 7.2]. According to [18, Example 2.3], it is unknown whether X is strongly p -incompressible when the characteristic is p . We can treat the case $n = 1$ with Proposition 10.1.

Corollary 10.2. *The Severi–Brauer variety of a central division algebra of prime degree p is strongly p -incompressible.*

10.2. Index reduction formula

We mentioned in [13, Section 4.1] that the index reduction formula of [31, Corollary (B)] can be generalized to base fields of arbitrary characteristic. The results obtained here allow us to additionally remove all assumptions of regularity.

Proposition 10.3. *Let X and Y be integral projective varieties, of dimensions $\leq i(p - 1)$ for some $i \leq p$. Assume that $v_p(n_Y) \geq v_p(n_X) \geq i$, $p \nmid \chi(\mathcal{O}_Y)$, and $\chi(\mathcal{O}_X)$ is prime to p . Then there is no correspondence $X \rightsquigarrow Y$ of multiplicity prime to p .*

Proof. The first part of Proposition 10.1 implies that $\dim X = i(p - 1)$; hence $\dim Y \leq \dim X$. If such a correspondence exist, the last sentence of the proof of Proposition 10.1 would imply that $\chi(\mathcal{O}_Y)$ is prime to p , a contradiction. \square

10.3. Rost’s correspondences

The operations T_i can be used to extend the case $n = 1$ in [25, Theorem 9.1] to arbitrary fields.

Theorem 10.4. *Let X be a smooth projective variety of dimension $p - 1$. Assume that X has a special correspondence in the sense of [25, Definition 5.1], with $b = 1$. Assume moreover that X has no zero-cycle of degree prime to p . Then*

$$\deg(r^{(p)}(-T_X) \cdot [X]) \neq 0 \pmod{p^2}.$$

By Remark 9.3, this means that the integer $\chi(\mathcal{O}_X)$ is prime to p .

Proof. We only sketch how [25, Proof of Theorem 9.1] can be adapted. This proof only involves cycles of dimension $\leq 2p - 3$; in particular, the only Steenrod operation used is the first one. Moreover, since the degree map factors through the Chow group modulo torsion, there is no loss in using the operation with values in this group. All the properties needed are contained in Proposition 7.4. \square

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Appendix. Schemes over a regular base

Let S be a regular connected scheme of finite Krull dimension. Then, as explained in [9, Section 20.1], most results of [9] for varieties over fields extend to arbitrary schemes of finite type over S . There are only two difficulties :

- the lack of external product in general.
- one should use relative dimension \dim_S instead of the usual dimension.

In particular, the homological Chern character can be constructed in this generality. The purpose of this appendix is to explain how **Theorem 3.2** (and thus the construction of the operations $T_i, i = 0, \dots, p - 1$) can be extended to this setting.

Let \mathcal{V}_S be the category of schemes quasi-projective over S . Homological Adams operations are constructed in [28] for objects of \mathcal{V}_S . The results of Section 2 extend without difficulty to the category \mathcal{V}_S .

However, both proofs of **Proposition 3.1** used the fact that X contains a non-empty open regular subscheme. We now give a proof that works for arbitrary objects of \mathcal{V}_S , using the notations introduced in the beginning of Section 2.

Proposition A.1. *Let $l \in \mathbb{Z} - \{0\}$, $X \in \mathcal{V}_S$, and $x \in K'_0(X)_{(d)}$. Then*

$$\psi_l(x) - l^{-d} \cdot x \in K'_0(X)_{(d-1)} \otimes \mathbb{Z}[1/l].$$

Proof. As before, one can assume that X is integral and $d = \dim_S X$, and we have the exact sequence

$$0 \rightarrow K'_0(X)_{(d-1)} \otimes \mathbb{Z}[1/l] \rightarrow K'_0(X)_{(d)} \otimes \mathbb{Z}[1/l] \xrightarrow{u^*} K'_0(\text{Spec}(k(X))) \otimes \mathbb{Z}[1/l] \rightarrow 0.$$

Choose a closed embedding $X \hookrightarrow W$, with W connected and smooth over S . Let x be the generic point of X , and write W_x for $\text{Spec}(\mathcal{O}_{W,x})$. Then the map u^* in the sequence above may be identified with the map

$$f^* : K_0^X(W) \otimes \mathbb{Z}[1/l] \rightarrow K_0^{\{x\}}(W_x) \otimes \mathbb{Z}[1/l]$$

and the composite $u^* \circ \psi_l$ corresponds to

$$f^* \circ (\theta^l(-T_{W/S}) \cdot \psi_x^l) = \theta^l(-T_{W/S}) \cdot \psi_{\{x\}}^l \circ f^*.$$

The action of $\theta^l(-T_{W/S}) \in K_0(W)$ on $K_0^{\{x\}}(W_x)$ factors through $K_0(W_x)$. Since W_x is a local scheme, the rank map $K_0(W_x) \rightarrow \mathbb{Z}$ is an isomorphism. The rank of $\theta^l(-T_{W/S})$ is $l^{-\text{rank } T_{W/S}}$ [16, (2), p. 6]. Since W is flat over S , we have $\dim_S W = \text{tr. deg.}(k(W)/k(S)) = \text{rank } T_{W/S}$. Thus $\theta^l(-T_{W/S})$ acts on $K_0^{\{x\}}(W_x)$ by multiplication by $l^{-\dim_S W}$. By [12, (A4') p. 263, see also Proof of Proposition 5.3], $\psi_{\{x\}}^l$ is multiplication by $l^{\text{codim}(X,W)}$. This gives, using [9, Lemma 20.1, (2)],

$$u^* \circ \psi_l = l^{\text{codim}(X,W) - \dim_S W} \cdot u^* = l^{-\dim_S X} \cdot u^*.$$

We can now conclude using the exact sequence above. □

As a consequence, [Theorem 3.2](#) immediately extends to the category \mathcal{V}_S .

Finally, let X be a scheme of finite type over S (possibly non quasi-projective). Proceeding as in the proof of [[9](#), Lemma 18.3, (3)], we see that there is always an envelope $X' \rightarrow X$ over S , such that $X' \in \mathcal{V}_S$. Moreover we see, using [[9](#), Lemma 20.1, (3)], that $K'_0(X')_{(d)} \rightarrow K'_0(X)_{(d)}$ is surjective, for any d . Using this remark, [Theorem 3.2](#) is easily extended to the category of schemes of finite type over S .

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