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Coxeter complexes and graph-associahedra [☆]

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Abstract

Given a graph Γ, we construct a simple, convex polytope, dubbed *graph-associahedra*, whose face poset is based on the connected subgraphs of Γ. This provides a natural generalization of the Stasheff associahedron and the Bott–Taubes cyclohedron. Moreover, we show that for any simplicial Coxeter system, the minimal blow-ups of its associated Coxeter complex has a tiling by graph-associahedra. The geometric and combinatorial properties of the complex as well as of the polyhedra are given. These spaces are natural generalizations of the Deligne–Knudsen–Mumford compactification of the real moduli space of curves.

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1. Introduction

The Deligne–Knudsen–Mumford compactification of the real moduli space of curves $\overline{\mathcal{M}}_0^n(\mathbb{R})$ appears in many areas, from operads [8,16], to combinatorics [10,13], to group theory [5,6]. One reason for this is an intrinsic tiling of $\overline{\mathcal{M}}_0^n(\mathbb{R})$ by the *associahedron*, the Stasheff polytope [15]. The motivation for this work comes from a remarkable fact, first noticed by Kapranov, involving Coxeter complexes: Blowing up certain faces of the Coxeter complex of type A yields a double cover of $\overline{\mathcal{M}}_0^n(\mathbb{R})$. Extending this to the Coxeter complex of affine type $\widetilde{A}$ results in a moduli space tessellated by the *cyclohedron* [9], the Bott–Taubes polytope associated to knot invariants [2]. Davis et al. have shown these spaces to be aspherical, where all the homotopy properties are completely encapsulated in their fundamental groups [5]. This paper looks at analogues of $\overline{\mathcal{M}}_0^n(\mathbb{R})$ for all simplicial Coxeter groups W, which we denote as $\mathcal{C}(W)_\#$.

Section 2 begins with the study of graph-associahedra. For any graph Γ, we construct a simple, convex polytope whose face poset is based on the connected subgraphs of Γ (Theorem 2.6). This provides a natural generalization of the associahedron and the cyclohedron. Some combinatorial properties of this polytope are also explored (Theorem 2.9).

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Section 3 provides the background of Coxeter complexes and proves that graph-associahedra tile $\mathcal{C}(W)_\#$ (Theorem 3.7). A gluing map of these polytopes is also provided (Theorem 3.8). Section 4 finishes by looking at the geometry of $\mathcal{C}(W)_\#$. In particular, we show that each blown-up cell of $\mathcal{C}(W)_\#$ resolves into a product of lower-dimensional blown-up Coxeter complexes (Theorem 4.2).

2. Constructing graph-associahedra

2.1. The motivating example will be the associahedron.

Definition 2.1. Let $\mathfrak{A}(n)$ be the poset of bracketings of a path with n nodes, ordered such that $a \prec a'$ if a is obtained from a' by adding new brackets. The *associahedron* K_n is a convex polytope of dimension $n-2$ whose face poset is isomorphic to $\mathfrak{A}(n)$.

The associahedron K_n was originally defined by Stasheff for use in homotopy theory in connection with associativity properties of H-spaces [15, Section 2]. The construction of the polytope K_n is given by Lee [14] and Haiman (unpublished). The vertices of K_n are enumerated by the Catalan numbers. Fig. 1(a) shows the 2-dimensional K_4 as the pentagon. Each edge of K_4 has one set of brackets, whereas each vertex has two. Note that Fig. 7(a) shows $\mathcal{C}(A_3)_\#$ tiled by 24 K_4 pentagons. We give an alternate definition of K_n with respect to *tubings*.

Definition 2.2. Let Γ be a graph. A *tube* is a proper nonempty set of nodes of Γ whose induced graph is a proper, connected subgraph of Γ. There are three ways that two tubes t_1 and t_2 may interact on the graph.

(1) Tubes are *nested* if $t_1 \subset t_2$.
(2) Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.
(3) Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in Γ.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of Γ is a set of tubes of Γ such that every pair of tubes in T is compatible. A k*-tubing* is a tubing with k tubes.

Lemma 2.3. *Let Γ be a path with $n-1$ nodes. The face poset of K_n is isomorphic to the poset of all valid tubings of Γ, ordered such that tubings $T \prec T'$ if T is obtained from T' by adding tubes.*

Fig. 1(b) shows the faces of associahedron K_4 labeled with tubings. The proof of the lemma is based on a trivial bijection between bracketings and tubings on paths.

2.2. For a graph Γ with n nodes, let $\triangle_\Gamma$ be the $n-1$ simplex in which each facet (codimension 1 face) corresponds to a particular node. Each proper subset of nodes of Γ corresponds to a unique face of $\triangle_\Gamma$, defined by the intersection of the faces associated to those nodes. The empty set corresponds to the face which is the entire polytope $\triangle_\Gamma$.

Definition 2.4. For a given graph Γ, truncate faces of $\triangle_\Gamma$ which correspond to 1-tubings in increasing order of dimension. The resulting polytope $\mathcal{P}\Gamma$ is the *graph-associahedron*.

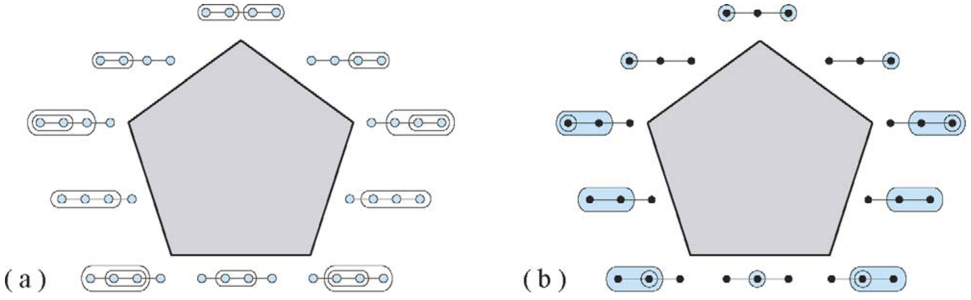

Fig. 1. Associahedron K_4 labeled with (a) bracketings and (b) tubings.

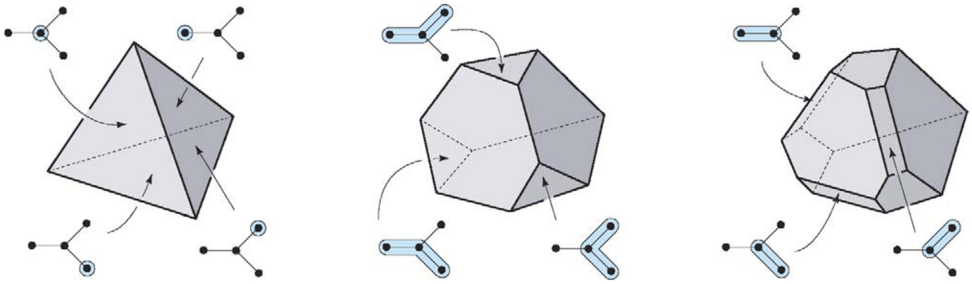

Fig. 2. Iterated truncations of the 3-simplex based on an underlying graph.

This definition is well-defined: Theorem 2.6 below guarantees that truncating any ordering of faces of the same dimension produces the same poset/polytope. Note also that $\mathcal{P}\Gamma$ is a simple, convex polytope.

Example 2.5. Fig. 2 shows a 3-simplex tetrahedron truncated according to a graph. The facets of $\mathcal{P}$() are labeled with 1-tubings. One can verify that the edges correspond to all possible 2-tubings and the vertices to 3-tubings.

Theorem 2.6. *$\mathcal{P}\Gamma$ is a simple, convex polytope whose face poset is isomorphic to set of valid tubings of Γ, ordered such that $T \prec T'$ if T is obtained from T' by adding tubes.*

The proof of this theorem is given at the end of the section. Note that simplicity and convexity of $\mathcal{P}\Gamma$ follows from its construction. Stasheff and Schnider [16, Appendix B] proved the following motivating examples. They follow immediately from Theorem 2.6.

Corollary 2.7. *When Γ is a path with $n-1$ nodes, $\mathcal{P}\Gamma$ is the associahedron K_n. When Γ is a cycle with $n-1$ nodes, $\mathcal{P}\Gamma$ is the cyclohedron W_n.*

2.3. For a given tube t and a graph Γ, let Γ_t denote the induced subgraph on the graph Γ. By abuse of notation, we sometimes refer to Γ_t as a tube.

Definition 2.8. Given a graph Γ and a tube t, construct a new graph Γ_t^* called the *reconnected complement*: If V is the set of nodes of Γ, then $V - t$ is the set of nodes of Γ_t^*. There is an edge between nodes a and b in Γ_t^* if either $\{a, b\}$ or $\{a, b\} \cup t$ is connected in Γ.

Fig. 3 illustrates some examples of 1-tubings on graphs along with their reconnected complements.

Theorem 2.9. *The facets of $\mathcal{P}\Gamma$ correspond to the set of 1-tubings on Γ. In particular, the facet associated to a 1-tubing $\{t\}$ is combinatorially equivalent to $\mathcal{P}\Gamma_t \times \mathcal{P}\Gamma_t^*$.*

Proof. We know from Theorem 2.6 that a facet of $\mathcal{P}\Gamma$ is given by a 1-tubing $\{t\}$. The faces contained in this facet are the tubings T of Γ that contain t. Now if $t_i \subset t$ is a tube of Γ_t then it is also a tube of Γ. Consider the map

$$\rho : \{\text{tubes of } \Gamma_t^*\} \to \{\text{tubes of } \Gamma \text{ containing } t\}$$

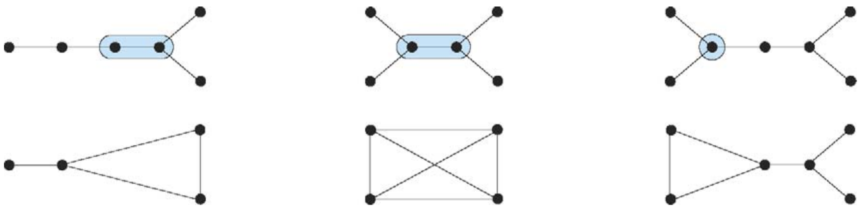

Fig. 3. Examples of 1-tubings and their reconnected complements.

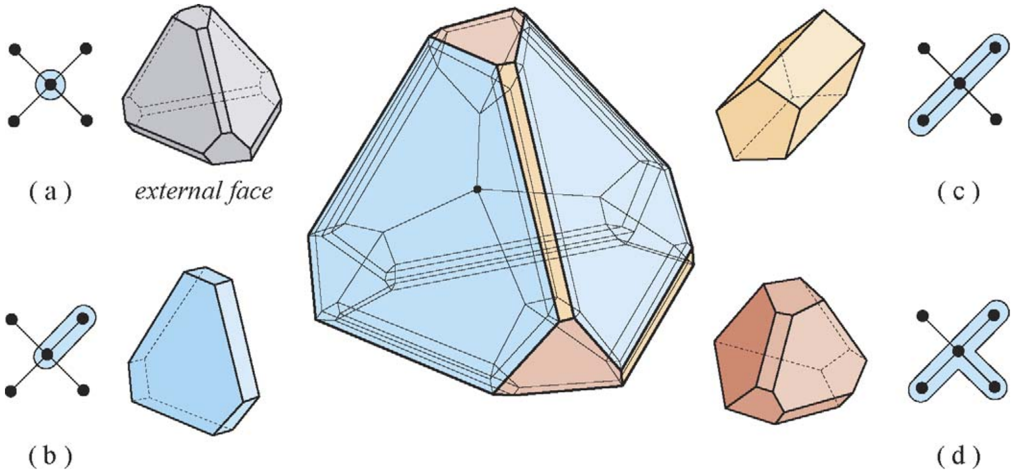

Fig. 4. The Schlegel diagrams of a 4-polytope along with its four types of facets.

where

$$\rho(t') = \begin{cases} t' \cup t & \text{if } t' \cup t \text{ is connected in } \Gamma, \\ t' & \text{otherwise.} \end{cases}$$

Note that ρ is a bijection and it preserves the validity of tubings. That is, two tubes t_1 and t_2 are compatible in Γ_t^* if and only if $\rho(t_1)$ and $\rho(t_2)$ are compatible. Define the natural map

$$\widehat{\rho}: \{\text{tubings on } \Gamma_t^*\} \times \{\text{tubings on } \Gamma_t\} \to \{\text{tubings on } \Gamma\}$$

where

$$\widehat{\rho}(T_i \times T_j) = \{t\} \cup \bigcup_{t_i \in T_i} \{\rho(t_i)\} \cup \bigcup_{t_j \in T_j} \{t_j\}.$$

It is straightforward to show that this is an isomorphism of posets. □

Example 2.10. Fig. 4 shows the Schlegel diagram of the 4-dimensional polytope $\mathcal{P}($$)$. It is obtained from the 4-simplex by first truncating four vertices, each of which become a 3-dimensional facet, as depicted in Fig. 4(d) along with its 1-tubing. Then six edges are truncated, becoming facets of type Fig. 4(c); note that Theorem 2.9 shows the structure of the facet to be the product of the associahedron K_4 of Fig. 1(b) and an interval. Finally four 2-faces of the original 4-simplex are truncated to result in the polytope of Fig. 4(b); this is the product of the cyclohedron W_3 (hexagon) and an interval. Four of the original five facets of the 4-simplex have become the polyhedron of Fig. 4(d), whereas the fifth (external) facet is the 3-dimensional *permutohedron*, as shown in Fig. 4(a).

2.4. The remaining section is devoted to the proof of Theorem 2.6, which follows directly from Lemmas 2.14 and 2.15 below. First we must define a poset operation analogous to truncation.

We define an initial partial ordering $\prec_0$ on tubes by saying that $t_i \prec_0 t_j$ if and only if $t_i \subset t_j$. We also define a partial ordering on a set of tubings $\mathfrak{T}$ induced by any partial ordering of tubes of Γ: Given tubings $T_I, T_J \in \mathfrak{T}$, then $T_I \prec T_J$ if and only if for all $t_j \in T_J$, there exists t_i such that $t_j \prec t_i \in T_I$. We write this partially ordered set of tubings as $(\mathfrak{T}, \prec)$. Note that $\triangle_\Gamma$ is isomorphic to $(\mathfrak{T}_0, \prec_0)$: The set of nonnested tubings of Γ with order induced by $\prec_0$.

Definition 2.11. Given a poset of tubings $(\mathfrak{T}, \prec)$, we can produce a set $(\mathfrak{T}', \prec')$ by *promoting* the tube t_*. Let

$$\mathfrak{T}_* = \bigl\{T \cup \{t_*\} \mid T \in \mathfrak{T} \text{ and } T \cup \{t_*\} \text{ is a valid tubing of } \Gamma\bigr\}$$

and let $\mathfrak{T}' = \mathfrak{T} \cup \mathfrak{T}_*$. Let $\prec'$ be defined so that t_* is incomparable to any other tube, and for any tubes t_a, t_b not equal to t_*, let $t_a \prec' t_b$ if and only if $t_a \prec t_b$ (see Fig. 5).

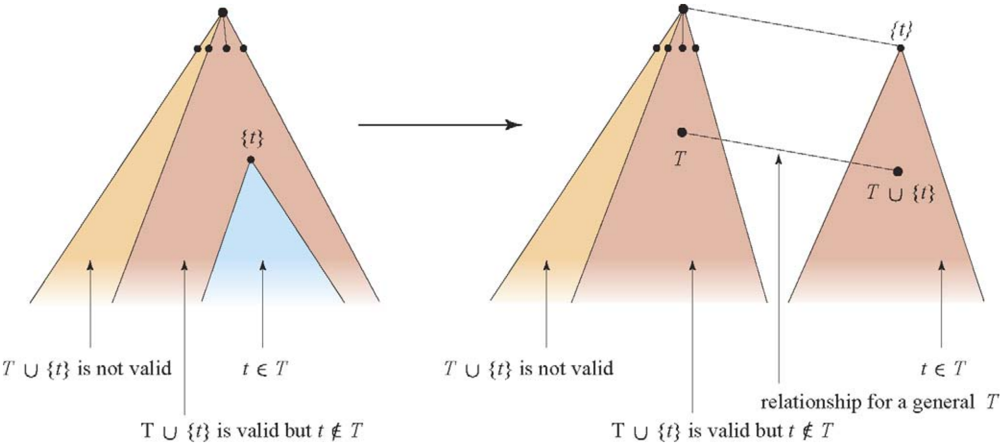

Fig. 5. A sketch of the poset lattice before and after promotion of tube $\{t\}$. Regions shaded with like colors are isomorphic as posets.

Let $\{t_i\}$ be the set of tubes in Γ for $i \in \{1, \dots, k\}$ ordered in decreasing size. Notice these correspond to the faces of $\triangle_\Gamma$ in increasing order of dimension. Let $(\mathfrak{T}_i, \prec_i)$ be the resulting set after consecutively promoting the tubes $t_1, \dots, t_i$ in $(\mathfrak{T}_0, \prec_0)$. The following two lemmas explicitly define the tubings and the ordering of $(\mathfrak{T}_i, \prec_i)$. Both are trivial inductions from the definition of promotion.

Lemma 2.12. *$\mathfrak{T}_i$ is the set of all valid tubings of the form $T_0 \bigcup_{j=1}^m \{t_{q_j}\}$ where $T_0 \in \mathfrak{T}_0$ and $\{q_j\} \subseteq \{1, \dots, i\}$.*

Lemma 2.13. *If a or b is less than or equal to i, then $t_a \prec_i t_b$ if and only if $a = b$. If both a and b are greater than i, then $t_a \prec_i t_b$ if and only if $t_a \prec_0 t_b$.*

As a special case we can state the following:

Lemma 2.14. *$(\mathfrak{T}_k, \prec_k)$ is isomorphic as a poset to the set of tubings of Γ, ordered such that $T \prec T'$ if and only if T can be obtained by adding tubes to T'.*

Proof. Applying Lemma 2.12 to the case $i = k$ shows $\mathfrak{T}_k$ is the set of all tubings of Γ. Lemma 2.13 shows that $T \prec_k T'$ if and only if $T \supset T'$. $\square$

The only step that remains is to show the equivalence of promotion to truncation when performed in this order. The following lemma accomplishes this.

Lemma 2.15. *Let f_i be a face of $\triangle_\Gamma$ corresponding to the tube t_i. Let $\mathcal{P}_i$ be the polytope created by consecutively truncating faces $f_1, \dots, f_i$ of $\triangle_\Gamma$. Then $(\mathfrak{T}_k, \prec_k) \cong P_k$.*

Proof. For consistency, we refer to $\triangle_\Gamma$ by $\mathcal{P}_0$. Since $\mathcal{P}_0$ is convex, so is $\mathcal{P}_i$. Thus we may define these polytopes as intersections of halfspaces. Denote the hyperplane that defines the halfspace H_a^+ by H_a. If X is the halfspace set for a polytope $\mathcal{P}$ then there is a natural poset map

$$\Psi : \mathcal{P} \to \Omega(X)^{op} : f \mapsto X_f$$

where $\Omega(X)^{op}$ is the set of subsets of X ordered under reverse inclusion and X_f is the subset such that $f = \mathcal{P} \cap \bigcap_{a \in X_f} H_a$. Note that Ψ is an injection with its image as all the sets X' such that $\mathcal{P} \cap \bigcap_{a \in X'} H_a$ in nonempty. By truncating $\mathcal{P}$ at f_*, a new halfspace H_*^+ is added with the following properties:

(1) A vertex of $\mathcal{P}$ is in H_*^+ if and only if it is not in f_*.
(2) No vertices of $\mathcal{P}$ are in H_*.

This produces the truncated polytope $\mathcal{P}_* = H_*^+ \cap \bigcap_{a\in X} H_a^+$. Let $\mathcal{P}_0$ be defined by $\bigcap_{a\in X_0} H_a^+$ where X_0 is the set of indices for the defining halfspaces. Let H_i^+ be the halfspace with which we intersect $\mathcal{P}_{i-1}$ to truncate f_i. The halfspace set for $\mathcal{P}_i$ is

$$X_i = X_{i-1} \cup \{i\} = X_0 \cup \{1, \ldots, i\}.$$

We define the map $\Psi_i : \mathcal{P}_i \to \Omega(X_i)^{op}$ which takes a face of $\mathcal{P}_i$ to the set of hyperplanes that contain it.

We now produce an order preserving injection Φ_i from $\mathfrak{T}_i$ to $\Omega(X_i)^{op}$. Let ϕ_0 be the map from tubes of Γ to $\Omega(X_0)$ that takes a tube t_i to $\Psi_0(f_i)$. Define

$$\phi_i(t_j) = \begin{cases} \{j\} & \text{if } j \leqslant i, \\ \phi_0(t_j) & \text{if } j > i. \end{cases}$$

This allows us to define a new map

$$\Phi_i : \mathfrak{T}_i \to \Omega(X_i)^{op} : T_J \mapsto \bigcup_{t_j \in T_J} \phi_i(t_j).$$

It follows from the definition that this is an order preserving injection. An induction argument shows that $\Phi_i(\mathfrak{T}_i) = \Psi_i(\mathcal{P}_i)$. Since Ψ_i and Φ_i are order preserving and injective, we have that $\Psi_i^{-1} \circ \Phi_i : \mathfrak{T}_i \to \mathcal{P}_i$ is an isomorphism of posets. □

3. Tiling Coxeter complexes

3.1. We begin with some standard facts and definitions about Coxeter systems. Most of the background used here can be found in Bourbaki [3] and Brown [4].

Definition 3.1. Given a finite set S, a *Coxeter group* W is given by the presentation

$$W = \langle s_i \in S \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = m_{ji}$ and $2 \leqslant m_{ij} \leqslant \infty$.

Associated to any Coxeter system (W, S) is its *Coxeter graph* Γ_W: Each node represents an element of S, where two nodes s_i, s_j determine an edge if and only if $m_{ij} \geqslant 3$. A Coxeter group is *irreducible* if its Coxeter graph is connected and it is *locally finite* if either W is finite or each proper subset of S generates a finite group. A Coxeter group is *simplicial* if it is irreducible and locally finite. The classification of simplicial Coxeter groups and their Coxeter graphs are well-known [3, Chapter 6]. Unless stated otherwise, the Coxeter groups discussed below are assumed to be simplicial.

Every simplicial Coxeter group has a realization as a group generated by reflections acting faithfully on a variety [4, Chapter 3]. The geometry of the variety is either spherical, Euclidean, or hyperbolic, depending on the group. Every conjugate of a generator s_i acts on the variety as a reflection in some hyperplane, dividing the variety into simplicial chambers. This variety, along with its cellulation is the *Coxeter complex* corresponding to W, denoted $\mathcal{C}W$. The hyperplanes associated to the generators s_i of W all border a single chamber, called the *fundamental chamber* of $\mathcal{C}W$. The W-action on the chambers of $\mathcal{C}W$ is transitive, and thus we may associate an element of W to each chamber; generally, the identity is associated to the fundamental chamber.

Notation. For a spherical Coxeter complex $\mathcal{C}W$, we define the *projective Coxeter complex* $\mathbb{PC}(W)$ to be $\mathcal{C}W$ with antipodal points on the sphere identified. These complexes arise naturally in blow-ups, as shown in Theorem 4.2.

Example 3.2. The Coxeter group of type A_n has n generators, and $m_{ij} = 3$ if $i = j \pm 1$ and 2 otherwise. Thus A_n is isomorphic to the symmetric group $\mathbb{S}_{n+1}$ and acts on the intersection of the unit sphere in $\mathbb{R}^{n+1}$ with the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$. Each s_i is the reflection in the plane $x_i = x_{i+1}$. Fig. 6(a) shows the Coxeter complex $\mathcal{C}A_3$, a 2-sphere cut into 24 triangles.

The B_n Coxeter group has n generators with the same m_{ij} as A_n except that $m_{12} = 4$. The group B_n is the symmetry group of the n-cube, and acts on the unit sphere in $\mathbb{R}^n$. Each generator s_i is a reflection in the hyperplane $x_{i-1} = x_i$, except s_1 which is the reflection in $x_1 = 0$. Fig. 6(b) shows the Coxeter complex $\mathcal{C}B_3$, the 2-sphere tiled by simplices.

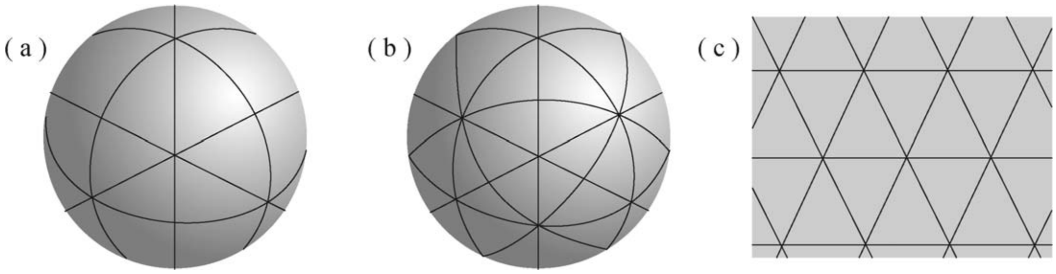

Fig. 6. Coxeter complexes $\mathcal{C}A_3$, $\mathcal{C}B_3$, and $\mathcal{C}\widetilde{A}_2$.

The $\widetilde{A}_n$ Coxeter group has $n+1$ generators, with $m_{ij}=3$ if $i=j\pm 1$, and $m_{(1)(n+1)}=3$. Every other m_{ij} equals two. The group $\widetilde{A}_n$ acts on the hyperplane defined by $x_1+x_2+\cdots+x_{n+1}=0$ in $\mathbb{R}^{n+1}$. Each s_i is the reflection in $x_i=x_{i+1}$, except s_{n+1} which is the reflection in $x_{n+1}=x_1+1$. Fig. 6(c) shows the Coxeter complex $\mathcal{C}\widetilde{A}_2$, the plane with the corresponding hyperplanes.

3.2. The collection of hyperplanes $\{x_i=0 \mid i=1,\dots,n\}$ of $\mathbb{R}^n$ generates the *coordinate* arrangement. A crossing of hyperplanes is *normal* if it is locally isomorphic to a coordinate arrangement. A construction which transforms any crossing into a normal crossing involves the algebro-geometric concept of a blow-up; see Section 4.1 for a definition.

A general collection of blow-ups is usually noncommutative in nature; in other words, the order in which spaces are blown-up is important. For a given arrangement, De Concini and Procesi [7, Section 3] establish the existence (and uniqueness) of a *minimal building set*, a collection of subspaces for which blow-ups commute for a given dimension, and for which every crossing in the resulting space is normal. We denote the minimal building set of an arrangement $\mathcal{A}$ by $\mathrm{Min}(\mathcal{A})$. Let α be an intersection of hyperplanes in an arrangement $\mathcal{A}$. Denote $\mathcal{H}\alpha$ to be the set of all hyperplanes that contain α. We say $\mathcal{H}\alpha$ is *reducible* if it is a disjoint union $\mathcal{H}\beta \sqcup \mathcal{H}\gamma$, where $\alpha=\beta\cap\gamma$ for intersections of hyperplanes β and γ.

Lemma 3.3. [7, Section 2] $\alpha\in\mathrm{Min}(\mathcal{A})$ *if and only if* $\mathcal{H}\alpha$ *is irreducible.*

If reflections in $\mathcal{H}\alpha$ generate a Coxeter group (finite reflection group), it is called the *stabilizer* of α and denoted W_α. For a Coxeter complex $\mathcal{C}W$, we denote its minimal building set by $\mathrm{Min}(\mathcal{C}W)$. The relationship between the set $\mathrm{Min}(\mathcal{C}W)$ and the group W is given by the following.

Lemma 3.4. [5, Section 3] $\alpha\in\mathrm{Min}(\mathcal{C}W)$ *if and only if* W_α *is irreducible.*

Definition 3.5. The *minimal blow-up* of $\mathcal{C}W$, denoted as $\mathcal{C}(W)_\#$, is obtained by blowing-up along elements of $\mathrm{Min}(\mathcal{C}W)$ in *increasing* order of dimension.

The construct $\mathcal{C}(W)_\#$ is well-defined: Lemma 4.9 below guarantees that blowing-up any ordering of subspaces in $\mathrm{Min}(\mathcal{C}W)$ of the same dimension produces the same cellulation.

Example 3.6. Fig. 7(a) shows the blow-ups of the sphere $\mathcal{C}A_3$ of Fig. 6(a) at nonnormal crossings. Each blown up point has become a hexagon with antipodal identification and the resulting manifold is $\mathcal{C}(A_3)_\#$. Fig. 8 shows the local structure at a blow-up, where each crossing is now normal. The minimal blow-up of the projective Coxeter complex of type A_3 is shown in Fig. 7(b), with the four points blown up in $\mathbb{RP}^2$. Fig. 7(c) shows the minimal blow-up of $\mathcal{C}\widetilde{A}_2$ of Fig. 6(c).

3.3. Given the construction of graph-associahedra above, we turn to applying them to the chambers tiling $\mathcal{C}(W)_\#$.

Theorem 3.7. *Let W be a simplicial Coxeter group and Γ_W be its associated Coxeter graph. Then $\mathcal{P}\Gamma_W$ is the fundamental domain for $\mathcal{C}(W)_\#$.*

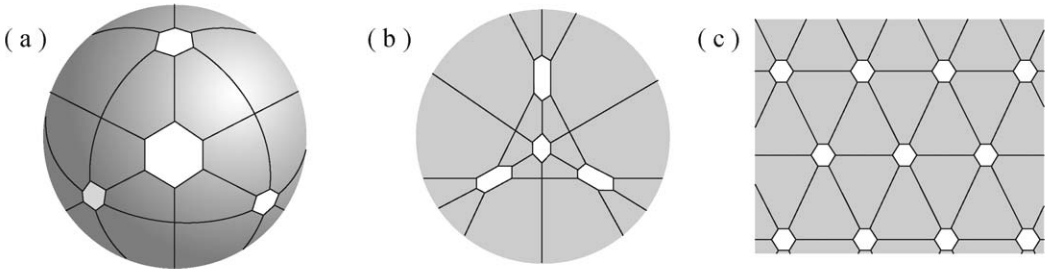

Fig. 7. Minimal blow-ups of (a) $\mathcal{C}A_3$, (b) $\mathbb{P}\mathcal{C}(A_3)$ and (c) $\mathcal{C}\widetilde{A}_2$.

Proof. It is a classic result of geometric group theory that each chamber of a simplicial Coxeter complex $\mathcal{C}W$ is a simplex. The representation of W can be chosen such that the generators correspond to the reflections through the supporting hyperplanes of a fixed chamber. In other words, a fundamental chamber of $\mathcal{C}W$ is the simplex $\triangle_{\Gamma_W}$ such that each facet of $\triangle_{\Gamma_W}$ is associated to a node of Γ_W.

Let f be a face of $\triangle_{\Gamma_W}$ and let α be the *support* of f, the smallest intersection of hyperplanes of $\mathcal{C}W$ containing f. As in the previous section, the face f corresponds to a subset S of the nodes of Γ_W. The nodes in S represent the generators of W that stabilize α. These elements generate W_α, and the subgraph induced by S is the Coxeter graph of W_α.

By Lemma 3.4, α is an element of $\mathrm{Min}(\mathcal{C}W)$ if and only if W_α is irreducible. But W_α is irreducible if and only if Γ_{W_α} is connected, that is, when the set of nodes of Γ_{W_α} is a tube of Γ_W. Note that blowing up α in $\mathcal{C}W$ truncates the face f of $\triangle_{\Gamma_W}$. Thus performing minimal blow ups of $\mathcal{C}W$ is equivalent to truncating the faces of $\triangle_{\Gamma_W}$ that correspond to tubes of Γ_W. By definition, the resulting polytope is $\mathcal{P}\Gamma_W$.

Remark. The *maximal* building set is the collection of *all* crossings, not just the nonnormal ones. The fundamental chambers of the maximal blow-up of $\mathcal{C}W$ will be tiled by permutohedra, obtained by iterated truncations of all faces of the simplex.

Remark. The generalized associahedra of Fomin and Zelevinsky [11] are fundamentally different than graph-associahedra. Although both are motivated from type A_n (the classical associahedra of Stasheff), they are distinct in all other cases. For example, the cyclohedron is the generalized associahedron of type B_n, whereas it is the type $\widetilde{A}_n$ graph-associahedron.

3.4. The construction of the Coxeter complex $\mathcal{C}W$ implies a natural W-action. This action, restricted to the chambers is faithful and transitive, so we can identify each chamber with the group element that takes the fundamental chamber to it. The faces of the chambers of $\mathcal{C}W$ have different types (according to their associated tubings in Γ_W). A transformation is *type preserving* if it takes each face to a face of the same type. We call the W-action type preserving because each w induces a type preserving transformation of $\mathcal{C}W$.

We may use this action to define a W-action on $\mathcal{C}(W)_\#$. There is a hyperplane-preserving isomorphism between $\mathcal{C}W - \bigcup \mathrm{Min}(\mathcal{C}W)$ and $\mathcal{C}(W)_\# - \bigcup \mathrm{Min}(\mathcal{C}W)$. We define the W-action on $\mathcal{C}(W)_\#$ to agree with the W-action on $\mathcal{C}W$ in $\mathcal{C}(W)_\# - \bigcup \mathrm{Min}(\mathcal{C}W)$. We define the action on the remainder of $\mathcal{C}(W)_\#$ by requiring that for all subvarieties V of $\mathcal{C}(W)_\# - \bigcup \mathrm{Min}(\mathcal{C}W)$, the action of w takes the closure of V to the closure of wV. The W-action defined this way is type preserving, and the stabilizer of each hyperplane α is the group W_α.

Given $\mathcal{C}(W)_\#$, we may associate an element $s_f \in W$ to each facet f of the fundamental chamber. We call this element the *reflection* in that facet. If α is the hyperplane of $\mathcal{C}(W)_\#$ that contains f, then s_f is a reflection in α. This corresponds to the reflection across α in $\mathcal{C}W$, which is the longest word in W_α [4, Section 3]. For a face f of the fundamental chamber, define $W_f = \langle s_{f_i} \rangle$ and $s_f = \prod s_{f_i}$, where the f_i's are the facets of the fundamental chamber that contain f. We denote the face corresponding to f of the chamber labeled w by $w(f)$.

Theorem 3.8. *$\mathcal{C}(W)_\#$ can be constructed from $|W|$ copies of $\mathcal{P}\Gamma_W$, labeled by the elements of W and with the face $w(f)$ identified to $w'(f)$ whenever $w^{-1}w' \in W_f$. The facet directly opposite w through $w(f)$ is ws_f.*

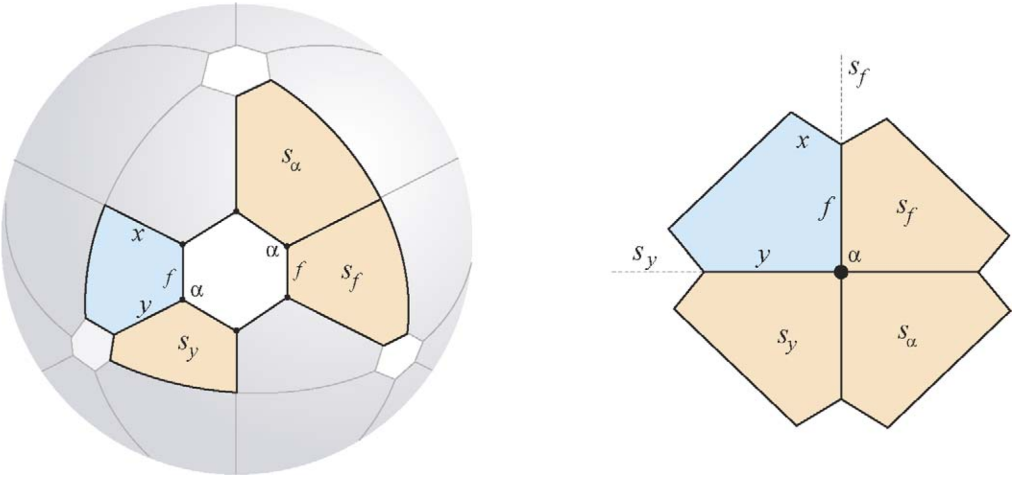

Fig. 8. Reflections locally around $\mathcal{C}(A_3)_\#$.

Remark. One may be tempted to think that whenever $w(f)$ is identified with $w'(f)$, the map between them is the restriction of the identity map between the chambers w and w'. However, Davis et al. [6, Section 8] show that this is not the case, and compute the actual gluing maps between faces. For this reason they call the elements s_f "mock reflections". The gluing map may also be computed by applying the theorem above to subfaces of f.

Proof of Theorem 3.8. Since the W-action is type preserving, a chamber w contains a face f if and only if w preserves f. Recall that W_f is generated by reflections in facets that contain f. Thus f is contained only in chambers whose elements correspond to W_f. The chamber that lies directly across f from the fundamental chamber corresponds to the longest word in W_f. Minimal blow-ups of $\mathcal{C}W$ resolve nonnormal crossings, so W_f is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$, where F has codimension d. Thus the longest word in W_f is the product of generators s_{f_i}.

For every subspace $\alpha \in \operatorname{Min}(\mathcal{C}W)$ and every $w \in W$, the subspace $w(\alpha)$ is also in $\operatorname{Min}(\mathcal{C}W)$. Thus we may extend the adjacency relation to chambers other than the fundamental chamber analogously. Since the W-action preserves containment, a face $w'(f)$ is identified with $w(f)$ if and only if $w^{-1}w' \in W_f$. Similarly, w respects reflection across F so the chamber directly across $w(f)$ from w is ws_f. □

Example 3.9. Consider the Coxeter group A_3. Denote two facets of the fundamental chamber of $\mathcal{C}A_3$ by x and y, whose reflections have the property that $(s_x s_y)^3 = 1$. Note that $\mathcal{C}(A_3)_\#$ is tiled by 24 copies of the associahedron $\mathcal{P}(A_3)$. Let f be the facet adjacent to x and y in the fundamental chamber of $\mathcal{C}(A_3)_\#$ and let α be the intersection of y and f, as in Fig. 8. Then $s_f = s_x s_y s_x$, and the fundamental chamber meets $s_y s_x$, $s_x s_y s_x$, and s_y at α. If we travel directly across α from the fundamental chamber, we arrive in $s_\alpha = s_x s_y s_x \cdot s_y = s_y s_x$.

4. Geometry of minimal blow-ups

4.1. One of our objectives is to describe the geometric structures of $\operatorname{Min}(\mathcal{C}W)$ before and after blow-ups. This final section proves Theorem 4.2 which describes $\mathcal{C}(W)_\#$ seen from the viewpoint of $\mathcal{C}W$.

We recall elementary notions of local structures, along with fixing notation: The tangent space of a variety V at p is denoted $T_p(V)$. For a Coxeter complex $\mathcal{C}W$, the tangent space has a natural Euclidean geometry which it inherits from the embedding of $\mathcal{C}W$ in $\mathbb{R}^n$ (with the hyperbolic simplicial Coxeter groups being viewed as acting on the hyperboloid model inside $\mathbb{R}^n$). Two nonzero subspaces of $T_p(V)$ are *perpendicular* if each vector in the first is perpendicular to each vector in the second, under the Euclidean geometry of $T_p(V)$. The *tangent bundle* of a variety V on a subvariety U is

$$T_U(V) = \{(p, v) \mid p \in U,\ v \in T_p(V)\}.$$

If $U = V$, we write $T(V)$. The *normal space* of U at p is

$$N_p(U) = \{v \mid v \in T_p(V),\ v \perp T_p(U)\}$$

and the *normal bundle* of U at a subvariety $W \subset U$ is

$$N_W(U) = \{(p, v) \mid p \in U,\ v \in N_p(V)\}.$$

If $W = U$, we write $N(U)$.

Definition 4.1. [12] The *blow-up* of a variety V along a codimension k intersection α of hyperplanes is the closure of $\{(x, f(x)) \mid x \in V\}$ in $V \times \mathbb{P}^{k-1}$. The function $f : V \to \mathbb{P}^{k-1}$ is defined by $f : p \mapsto [f_1(p) : f_2(p) : \cdots : f_k(p)]$, where the f_i define hyperplanes of $\mathcal{H}\alpha$ whose intersection is α.

We denote the blow-up of V along α by $V_{\#\alpha}$. There is a natural projection map

$$\pi : V_{\#\alpha} \to V : (x, y) \mapsto x$$

which is an isomorphism on $V - \alpha$. The hyperplanes of $V_{\#\alpha}$ are the closures $\pi^{-1}(h - \alpha)$ for each hyperplane h of V and one additional hyperplane $\pi^{-1}(\alpha)$. Thus $V - \alpha$ and $V_{\#\alpha} - \pi^{-1}(\alpha)$ are isomorphic not only as varieties but as cellulations.[1] The hyperplane α of $V_{\#\alpha}$ has a natural identification with the projectified normal bundle of α in V. The intersection of a hyperplane h with α is the part of α that corresponds to $T_\alpha(h) \subset N(\alpha)$.

4.2. An arrangement of hyperplanes of a variety V cut V into regions. We say that the hyperplanes give a *cellulation* of V. Two cellulations are equivalent if there is a hyperplane-preserving isomorphism between the two varieties. Let α be an intersection of hyperplanes. We say that hyperplanes h_i *cellulate* α to mean the intersections $h_i \cap \alpha$ give a cellulation of α, denoted by $\mathcal{C}\alpha$. The notation $\mathcal{C}\alpha$ will always refer to the cellulation of α in the original complex, rather than its image in subsequent blow-ups. Let $\mathrm{Min}(\mathcal{C}\alpha)$ denote the minimal building set of $\mathcal{C}\alpha$, and let $\mathcal{C}(\alpha)_\#$ denote the blow-up of the minimal building set of α.

Theorem 4.2. *Let $\mathcal{C}W$ be the Coxeter complex of a simplicial Coxeter group W and let $\alpha \in \mathrm{Min}(\mathcal{C}W)$. The blow-up of α in $\mathcal{C}(W)_\#$ is equivalent to the product $\mathcal{C}(\alpha)_\# \times \mathbb{P}\mathcal{C}(W_\alpha)_\#$.*

Example 4.3. There are $2\binom{n+1}{n-k}$ dimension k elements of $\mathrm{Min}(\mathcal{C}A_n)$. Each of these elements become $\mathcal{C}(A_{k+1})_\# \times \mathbb{P}\mathcal{C}(A_{n-k-1})_\#$ in $\mathcal{C}(A_n)_\#$. Fig. 9(d) shows the projective Coxeter complex $\mathbb{P}\mathcal{C}(A_4)_\#$ after minimal blow-ups. This is the Deligne–Knudsen–Mumford compactification $\overline{\mathcal{M}}{}_0^6(\mathbb{R})$ of the real moduli space of curves with six marked points. It is the real projective sphere $\mathbb{RP}^3$ with five points and ten lines blown-up. Each of the five blown-up points are $\mathbb{P}\mathcal{C}(A_3)_\#$, shown in Fig. 9(b) as $\mathcal{C}(A_4)_\#$ before projecting through the antipodal map. Each of the ten lines, each line defined by two distinct points in $\mathrm{Min}(\mathcal{C}A_4)$, becomes $\mathbb{P}\mathcal{C}(A_2)_\# \times \mathbb{P}\mathcal{C}(A_2)_\#$, a 2-torus depicted in Fig. 9(c). Note that there are also ten codimension 1 subspaces $\mathbb{P}\mathcal{C}(A_3)_\#$ pictured in Fig. 9(a), defined by three distinct points in $\mathrm{Min}(\mathcal{C}A_4)$.

Remark. Lemmas 4.5 and 4.6 are enough to provide the results of Theorem 4.2 for the maximal blow-up of $\mathcal{C}W$.

Remark. Extensions of these results to configuration spaces are given in [1, Section 3].

4.3. The proof of Theorem 4.2 requires two definitions and four preliminary lemmas.

Definition 4.4. Let β and γ be intersections of hyperplanes in a cellulation of V. We say that β is *strongly perpendicular* to γ and write $\beta \perp \gamma$ if for all p in $\beta \cap \gamma$, all three of the following subspaces span $T_p(V)$ *and* any two of them are perpendicular:

(1) $T_p(\beta \cap \gamma)$,
(2) $N_p(\beta)$, and
(3) $N_p(\gamma)$.

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[1] We give each hyperplane h of $V_{\#\alpha}$ the same name as its projection $\pi(h)$.

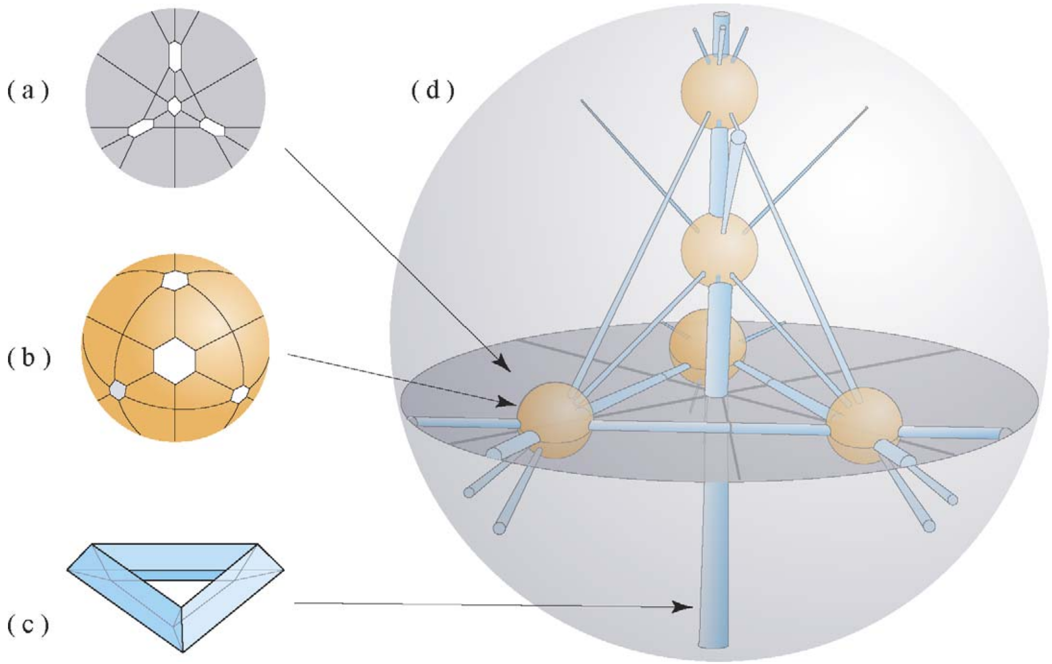

Fig. 9. The projective Coxeter complex (d) $\mathbb{P}\mathcal{C}(A_4)_\#$ along with components (a) $\mathbb{P}\mathcal{C}(A_3)_\#$, (b) $\mathcal{C}(A_3)_\#$ and (c) $\mathbb{P}\mathcal{C}(A_2)_\# \times \mathbb{P}\mathcal{C}(A_2)_\#$.

Note that this directly implies that $T_p(\beta)$ is the span of $T_p(\beta \cap \gamma) \cup N_p(\gamma)$. For an intersection of hyperplanes β, the normal space $N_p(\beta)$ is the span of the normal spaces of the elements of $\mathcal{H}\beta$ at p; if β contains γ, then $N_p(\gamma)$ contains $N_p(\beta)$. This shows immediately that if $\mathcal{H}\beta$ reduces to $\mathcal{H}\beta_1 \sqcup \mathcal{H}\beta_2$, then $\beta_1 \perp \beta_2$.

Lemma 4.5. *For every intersection of hyperplanes β in a Coxeter complex $\mathcal{C}W$, the set $\mathcal{H}\beta$ has a unique maximal decomposition $\mathcal{H}\beta = \mathcal{H}\beta_1 \sqcup \cdots \sqcup \mathcal{H}\beta_k$ where*

(1) *each $\mathcal{H}\beta_i$ is irreducible,*
(2) *$\beta_i \perp \beta_j$ for all $i \neq j$, and*
(3) *$\bigcap_{i \in S} \beta_i$ properly contains β for any proper subset $S \subset \{1, 2, \ldots, k\}$.*

Proof. If the normal spaces of two hyperplanes h_1, h_2 of $\mathcal{H}\beta$ are not perpendicular, write $h_1 \sim h_2$. Then $\sim$ is a symmetric, reflexive relation on $\mathcal{H}\beta$. Let $\approx$ be the unique smallest equivalence relation containing $\sim$ as a subset of $\mathcal{H}\beta \times \mathcal{H}\beta$.

No two hyperplanes $h_1 \sim h_2$ can be separated by any reduction of $\mathcal{H}\beta$. To prove this, suppose they could, and let $\mathcal{H}\beta$ reduce to $\mathcal{H}\beta_1 \sqcup \mathcal{H}\beta_2$ with h_1 in β_1 and h_2 in β_2. Then since W_β is a Coxeter group, the reflection of h_1 across h_2 must be in $\mathcal{H}\beta$. By hypothesis, the resulting hyperplane must contain β_1 or β_2. The former implies that $\beta_1 \subset h_2$ and the latter implies $\beta_2 \subset h_1$, yielding a contradiction. Since $\approx$ is the smallest transitive relation containing $\sim$, the hyperplanes h_1, h_2 cannot be separated whenever $h_1 \approx h_2$.

However, if $\approx$ partitions $\mathcal{H}\beta$ into at least two classes, then we may separate $\mathcal{H}\beta$ into $H_1 \sqcup H_2$ with each partition contained in either H_1 or H_2. Clearly $\bigcap H_1 \cap \bigcap H_2 = \beta$. To verify that $\mathcal{H}(\bigcap H_i) = H_i$, note that no element h_1 of H_1 may contain $\bigcap H_2$. If it does, then for all p in β, we have $N_p(h_1)$ contained in $N_p(\bigcap H_2)$, and thus in the span of $\{N_p(h_2)\}$ for h_2 in H_2. This violates the pairwise perpendicularity in our choice of H_1, H_2. Thus the equivalence relation $\approx$ partitions $\mathcal{H}\beta$ into a unique maximal decomposition and therefore the $\mathcal{H}\beta_i$'s are irreducible. Also, no proper subset S of the β_i can intersect in exactly β, since then $\bigcup \mathcal{H}\beta_i$ for β_i in S would be $\mathcal{H}\beta$. But $\mathcal{H}\beta$ must reduce to $\bigcup \mathcal{H}\beta_i$ for $\beta_i \in S$ and $\bigcup \mathcal{H}\beta_j$ for $\beta_j \notin S$ by the argument above.

When $\mathcal{H}\gamma$ reduces to $\mathcal{H}\gamma_1 \sqcup \mathcal{H}\gamma_2$, we have $\gamma_1 \perp \gamma_2$. Furthermore, since $N_p(\gamma_1)$ is the span of the normal spaces of $\mathcal{H}\gamma_1$, and $\mathcal{H}\gamma_1 \subset \mathcal{H}\gamma$, then for any $\gamma_3 \perp \gamma$ (with nonempty intersection), it follows that $\gamma_3 \perp \gamma_1$. Thus by induction, $\beta_i \perp \beta_j$ for $i \neq j$. □

4.4. The following two lemmas describe the effect of a blow-up on a cellulation. The first lemma combines several facts that follow directly from the definitions of hyperplanes and blow-ups. Note that as we perform blow-ups of $\mathcal{C}W$, the set of hyperplanes that contain a given β may change. However, $\mathcal{H}\beta$ is always assumed to refer to the set of hyperplanes that contain β in $\mathcal{C}W$.

Lemma 4.6. *Let β be a subvariety of V with cellulation C_1. Suppose the tangent spaces of the hyperplanes $\mathcal{H}\beta$ cellulate the normal bundle at each point p with cellulation C_2.*

1. *The subvariety β of $V_{\#\beta}$ is a product $C_1 \times \mathbb{P}C_2$.*
2. *The tangent space $T_p(V_{\#\beta})$ for $p \in \beta$ retains a local Euclidean structure. Roughly speaking, $n-1$ of the coordinate vectors are in $T_p(\beta)$, and the other is parallel to the 1-dimensional subspace of $N_{\pi(p)}(\beta)$ that corresponds to p.*
3. *For each hyperplane h of V that meets β at a subvariety $\gamma \neq \beta$, the hyperplane h of $V_{\#\beta}$ meets β at $\gamma \times \mathbb{P}C_2$.*
4. *For each hyperplane h of V that properly contains β, the hyperplane h of $V_{\#\beta}$ meets β at $C_1 \times h'$, where h' is the image of $T_p(h)$ in $\mathbb{P}C_2$. Also $\beta \perp h$.*

Lemma 4.7. *Let β be a subvariety of V with cellulation C_1. If $\beta \perp \gamma$ in V, then $\mathcal{C}\beta$ in $V_{\#\gamma}$ is equivalent to $(C_1)_{\#(\beta\cap\gamma)}$.*

Proof. The normal bundle $N_{\beta\cap\gamma}(\gamma)$ is contained in $T_{\beta\cap\gamma}(\beta)$ since $\beta \perp \gamma$. Thus $N(\beta \cap \gamma)$ and $N(\gamma)$ have the same intersection with $T(\beta)$. Since blow-ups replace a variety with its projectified normal bundle, the blow-ups along γ and $\beta \cap \gamma$ produce equivalent cellulations of β. □

Finally we establish the tools that will allow us to change the order in which we blow up elements of $\mathrm{Min}(\mathcal{C}W)$. The following definition and lemma give a class of orderings that produce the same cellulation as minimal blow-ups.

Definition 4.8. Given a variety V with intersections of hyperplanes β, γ, the blow-ups along β and γ *commute* if the cellulations $(V_{\#\beta})_{\#\gamma}$ and $(V_{\#\gamma})_{\#\beta}$ are equivalent, and the induced map on the hyperplanes preserves their labels.

Lemma 4.9. *Let $x_1, x_2, \ldots, x_k$ be an ordering of the elements of* $\mathrm{Min}(\mathcal{C}W)$ *such that $i \leqslant j$ whenever x_i is contained in x_j. Then blowing up $\mathcal{C}W$ along the x_i in order gives a cellulation equivalent to $\mathcal{C}(W)_{\#}$. The induced map on the hyperplanes also preserves labels.*

Proof. First we verify that if $\beta \perp \gamma$, then the blow-ups along β and γ commute. Since $\beta \perp \gamma$, the bundle $N_{\beta\cap\gamma}(\beta)$ is contained in $T_{\beta\cap\gamma}(\gamma)$ and $N_{\beta\cap\gamma}(\gamma)$ is contained in $T_{\beta\cap\gamma}(\beta)$. Define the maps $\pi_\beta : V_{\#\beta} \to V$ and $\pi_{\beta\gamma} : (V_{\#\beta})_{\#\gamma} \to V_{\#\beta}$. Then $\pi_\beta^{-1}(V-\gamma) = \pi_\beta^{-1}(V) - \pi_\beta^{-1}(\gamma)$ since $N_{\beta\cap\gamma}(\beta) \subset T_{\beta\cap\gamma}(\gamma)$. Thus $(V_{\#\beta})_{\#\gamma}$ is the closure of $\pi_{\beta\gamma}(\pi_\beta^{-1}(V-\gamma))$, which is the closure of $\pi_{\beta\gamma}^{-1}(\pi_\beta^{-1}(V-\gamma-\beta))$. Similar reasoning shows that $(V_{\#\gamma})_{\#\beta}$ is the closure of $\pi_{\gamma\beta}^{-1}(\pi_\gamma^{-1}(V-\beta-\gamma))$. Since the π's are isomorphisms on $V-\beta-\gamma$, we have a natural isomorphism between $(V_{\#\beta})_{\#\gamma}$ and $(V_{\#\gamma})_{\#\beta}$.

Now take β, γ to be elements of $\mathrm{Min}(\mathcal{C}W)$ such that neither contains the other. By Lemma 3.3, the arrangements $\mathcal{H}\beta$ and $\mathcal{H}\gamma$ are irreducible. Applying Lemma 4.5 shows that if $\mathcal{H}(\beta \cap \gamma)$ is reducible, then $\beta \perp \gamma$ and $\mathcal{H}\beta \sqcup \mathcal{H}\gamma$ is the unique reduction.

If $\mathcal{H}(\beta \cap \gamma)$ is irreducible, then $\beta \cap \gamma$ is in $\mathrm{Min}(\mathcal{C}W)$. After the blow-up along $\beta \cap \gamma$, the resulting spaces β and γ do not intersect by Lemma 4.6, and thus (vacuously) $\beta \perp \gamma$. If $\beta \cap \gamma$ is not in $\mathrm{Min}(\mathcal{C}W)$, then $\beta \perp \gamma$. In either case, the blow-ups along β and γ commute. Thus we may transpose any two elements that do not contain each other in the ordering of $\mathrm{Min}(\mathcal{C}W)$ and get an equivalent cellulation (with matching hyperplane labels) after blowing up all of $\mathrm{Min}(\mathcal{C}W)$. Repeating this procedure proves the statement of the lemma. □

4.5. We have now assembled all the lemmas needed for the proof of the theorem.

Proof of Theorem 4.2.

We begin by applying Lemma 4.9. Divide the elements of $\text{Min}(\mathcal{C}W)$ into three sets:

(1) $\{\alpha\}$,
(2) $X = \{\beta\colon\ \beta \not\supset \alpha\}$, and
(3) $Y = \{\beta\colon\ \beta \subset \alpha\}$.

We reorder the elements of $\text{Min}(\mathcal{C}W)$ as follows: First we blow up the elements of X, ordered by the dimension of $\beta \cap \alpha$, followed by blowing up along α. Finally blow up the elements of Y in order of dimension, as usual. Note that this is a valid application of Lemma 4.9, since if β contains γ, then $\beta \cap \alpha$ contains $\gamma \cap \alpha$.

We next produce a bijection ϕ between the set X' of elements x_i in X that intersect α in $(\cdots((\mathcal{C}W)_{\#x_1})_{\#x_2}\cdots)_{\#x_{i-1}}$ and the elements of $\text{Min}(\mathcal{C}\alpha)$ in $\mathcal{C}W$. We show that the map $\phi : X' \to \text{Min}(\mathcal{C}\alpha) : \beta \mapsto \beta \cap \alpha$ is a bijection, and that blowing up the elements of X has the same effect on the cellulation of α as blowing up the elements of $\text{Min}(\mathcal{C}\alpha)$.

(1) Suppose $\beta \in X'$ and $\beta \subset \alpha$, and thus $\mathcal{H}\alpha \subset \mathcal{H}\beta$. Since β is in $\text{Min}(\mathcal{C}W)$, the group W_β is an irreducible spherical Coxeter group by Lemma 3.4, and the arrangement $\mathcal{H}\alpha$ is irreducible in $\mathcal{H}\beta$ by Lemma 3.3. For all spherical Coxeter groups W_β, the elements of $\mathcal{H}\beta$ intersect α in an irreducible arrangement.[2] Therefore $\beta = \beta \cap \alpha \in \text{Min}(\mathcal{C}\alpha)$.
(2) Suppose $\beta \in X'$ and $\beta \not\subset \alpha$, then $\beta \cap \alpha$ is not in $\text{Min}(\mathcal{C}W)$, thus $\mathcal{H}(\beta \cap \alpha)$ reduces. Lemma 4.5 guarantees that $\beta \perp \alpha$. Thus by Lemma 4.7, blowing up β is equivalent to blowing up $\beta \cap \alpha$ in the cellulation of α.
(3) We now produce an function $\psi : \text{Min}(\mathcal{C}\alpha) \to X'$ that will be the inverse to ϕ. For $\beta \in \text{Min}(\mathcal{C}\alpha)$, either $\beta \in \text{Min}(\mathcal{C}W)$ or $\mathcal{H}\beta$ is reducible. If $\beta \in \text{Min}(\mathcal{C}W)$, then let $\psi(\beta) = \beta$. If not, then $\mathcal{H}\beta$ must reduce to $\mathcal{H}\alpha_0 \sqcup \mathcal{H}\alpha_1 \sqcup \cdots \sqcup \mathcal{H}\alpha_m$. Without loss of generality, assume α contains α_0.
Since $N_\beta(\alpha_i)$ is contained in $T_\beta(\alpha_0)$ for $i \neq 0$, the normal spaces of the elements of $\mathcal{H}\beta - \mathcal{H}\alpha_0$ are the same in α as they are in $\mathcal{C}W$. Thus in α, we know that $\alpha_i \perp \alpha_j$ for $i, j \neq 0, i \neq j$. Furthermore, the normal space $N_p(\alpha_0)$ in α is a subset of $N_p(\alpha_0)$ in V. Thus if $N_p(\alpha_0)$ in α is nonzero, it is perpendicular to each $N_p(\alpha_i)$ in α. Thus the set of hyperplanes of α induced by $\mathcal{H}\beta - \mathcal{H}\alpha$ reduces to the disjoint union induced by $(\mathcal{H}\alpha_0 - \mathcal{H}\alpha) \sqcup \mathcal{H}\alpha_1 \sqcup \cdots \sqcup \mathcal{H}\alpha_m$. To satisfy the hypothesis that $\beta \in \text{Min}(\mathcal{C}\alpha)$, it is necessary that $\alpha_0 = \alpha$ and $m = 1$. Thus we define $\psi(\beta) = \alpha_1$. It is straightforward to check that ψ is the inverse of ϕ, so ϕ is a bijection.
(4) By our choice of ordering, the elements of X' are blown up in the same order as elements of $\text{Min}(\mathcal{C}\alpha)$ under minimal blow-ups. Furthermore, the subvariety α has equivalent cellulations in the blow-up along $\phi(\beta)$ and in the blow-up along β. This follows trivially if $\beta \in \alpha$, and from Lemma 4.7 if not.

Thus, after blowing up all the elements of X in $\mathcal{C}W$, the cellulation of α is equivalent to $\mathcal{C}(\alpha)_\#$. By Lemma 4.6, the result after blowing up α is equivalent to $\mathcal{C}(\alpha)_\# \times \mathbb{P}\mathcal{C}(W_\alpha)$. Furthermore, for each element $y \in Y$, we have

$$y \perp \alpha \quad \text{and} \quad y \cap \alpha = \mathcal{C}(\alpha)_\# \times y',$$

where y' is the image of y in $\mathbb{P}\mathcal{C}(W_\alpha)$. Since the elements of Y are ordered by dimension, they are also ordered by their dimension in $\mathcal{C}W_\alpha$. Lemma 4.7 guarantees that blowing up the elements of Y produces a cellulation of α equivalent to $\mathcal{C}(\alpha)_\# \times \mathbb{P}\mathcal{C}(W_\alpha)_\#$. □

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[2] This can be checked by hand for the simpler cases, and a detailed decomposition of types A_n, B_n and D_n is given in [1, Section 5]. For the larger complexes (E_6, E_7, E_8), we can exploit the appearance of A_n as a subgroup.

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