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A Complete, Flexible Fuzzy-Based Approach to the Classification Problem

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ABSTRACT

We present an algebraic structure as a complete methodology of classification. Utilizing this structure, we apply an algebraic approximation to the problem of generating appropriate clusters of objects characterized by fuzzy attributes. More precisely, the values of the attributes are expressed in terms of linguistic labels, and thus are handled as fuzzy numbers. This opens new possibilities in all those fields for which the need to describe the population under analysis by means of more natural terms becomes crucial. In fact, in these cases, the application of resolution strategies based on the adoption of "standard" methods, such as a distance matrix, appears as a brutal effort to adapt quantitative methods to qualitative problems.

KEYWORDS: ***algebraic structure, fuzzy classification, linguistic approximation, relevance***

1. INTRODUCTION

The need for classification techniques has deep and extended roots. For this reason, the literature is very rich, providing us numerous approaches to this problem [1–5]. With the increasing interest of the research community in artificial intelligence, we note a major presence of nonquantitative information in systems devoted to simulating intelligent human behavior. For this reason, nowadays more attention is paid to qualitative classification methods [6–10], which are more suitable for adoption in all forms of

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natural reasoning. Our proposal belongs to this trend: a complete structure able to manipulate and classify strings which depict objects defined in our real life, i.e., fuzzy entities.

The ability to generate appropriate clusters is based on the existence of two simple but powerful operations, completely characterized in accord to the algebraic nature of the structure. In a previous definition of this structure [11], we introduced some important enhancements, extended to a new weighting technique able to treat with precision the relevance of the attributes which lead to the final classification. Thanks to these improvements, the structure gained more efficiency and flexibility, two essential characteristics for managing real problems. Nevertheless, we noted the persistence of an inconvenience in the weighting strategy: we operated in a strict binary mentality by considering the attribute as weighted or not. This restricted approach has been overcome by applying a generalization of the weighting procedure, and opening new perspectives on the classification mechanism.

The algebraic structure is introduced in Section 2. To address the need to deepen some formal aspects of the structure and to generate some critical aspects, we present in Section 3 an extension of the structure through a new definition of a basic mechanism able to operate on fuzzy objects, and we discuss the corresponding enhancements. Consequently, we face the problem of weighting the attributes. A first method is discussed to clarify the importance of the relevance of the attributes in a fuzzy-level perspective. A second proposal, aimed at ameliorating the weighting strategy, is then introduced; comparisons between the two are made in order to stress the benefits of the latter method. As a meaningful example we discuss in Section 5 an application of our classification mechanism in the field of the financial investments.

2. THE ALGEBRAIC STRUCTURE

Our classification mechanism is based on the definition of an algebraic structure useful for classifying with fuzzy attributes. The structure was initially formulated as a possible alternative to the theory of approximate reasoning [12]. Successively, several efforts have been made to enhance the properties of the structure. In [11], the basic operation of the classification process has been extended in order to handle linguistic labels; in [13], the problem of the relevance of the attributes has been faced and solved, thanks to an opportune weighting strategy [14]. In this paper, we present further important improvements which are necessary to extend and improve the classification mechanism.

2.1. Description of the Structure

Let **U** be a universe of discourse. Elements of **U** can be represented with the k-tuples $(A_1(u), \ldots, A_k(u))$, where the $A_i(u)$ are fuzzy measures of the elements of **U** whose values are ordered sets of fuzzy numbers. These numbers represent the "degree of compatibility" of the elements with respect to the fuzzy attributes. If we employ truth values, then we can use **ct** to denote completely true, **t** for true, **at** for almost true, and **f** for false, with **f** < **at** < **t** < **ct**. Each attribute A_i of the set $\{A_1, \ldots, A_k\}$ is represented by an ordered string $a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \cdots a_1^{\alpha_1}$, where the set $\{\alpha_1, \ldots, \alpha_n\}$, with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, is the set of the linguistic labels. Moreover the elements a_i are the possible subsets of **U** having as linguistic label the value α_i.

According to [15], each attribute is represented by a type-2 fuzzy set. Noting that each element of **U** has a unique evaluation with respect to a single attribute: the set $\{a_n, a_{n-1}, \ldots, a_1\}$ is an ordered partition of **U**. In this way, each attribute generates a partition, or classification, of elements of **U**.

To use the structure as a tool to classify, it is necessary to introduce an operation between the ordered strings associated with the attributes in such a way as to obtain a new string which is a finer classification of the information contained in the original strings. The idea of such an operation can be intuitively understood if we consider the twofold meaning of a digit in a number, i.e., the roles of value and position. In the case of our strings, the absolute value of the generic element a_i is the set of the elements of **U** which a_i represents, whereas the value corresponding to the position is given by the linguistic label α_i. Thus, being aware of the real nature of the elements of the strings (subsets of **U** as values, and fuzzy numbers as powers), the basic operation of our structure appears as a variant of the classical multiplication of natural numbers.

Due to the heterogeneous nature of the string, it is necessary to distinguish the operations according to the two parts of the strings, i.e., one operation to the first part composed of the subsets of **U**, and a different one applied to the second part of the string, that composed of fuzzy sets.

Now, let us formally define the operations. Let:

A and B be strings,

$\vartriangle$ be the operation between A and B,

$*$ be the operation between the first parts of the strings (ordinary sets),

$\circ$ be the operation between the second parts of the strings (fuzzy numbers).

The results $A \vartriangle B$ is an ordered string in which the first part is obtained by applying the operation $*$ to the first parts of A and B, and the second part by applying $\circ$ to the corresponding second parts of A and B in an independent and asynchronous way. More precisely,

$$\begin{gathered} A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \cdots a_1^{\alpha_1}, \qquad B = b_m^{\beta_m} b_{m-1}^{\beta_{m-1}} \cdots b_1^{\beta_1}, \\ (a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \cdots a_1^{\alpha_1}) \vartriangle (b_m^{\beta_m} b_{m-1}^{\beta_{m-1}} \cdots b_1^{\beta_1}) \\ = c_{m+n-1}^{\gamma_{m+n-1}} \cdots c_2^{\gamma_2} c_1^{\gamma_1}, \end{gathered} \tag{1}$$

where each c_i (γ_i) is generated by applying $*$ $(\circ)$ to the first (second) parts of the given strings A and B.

Before focusing our attention on the details of these operations, it is important to note that if X is a set, then $P(X)$ is the power set of X and the pair $(P(X), \text{inclusion})$ is a distributive lattice with inf $= \cap$ and sup $= \cup$ for each pair of elements of $P(X)$.

2.1.1. THE OPERATION $*$ If the lengths of the result of the operation (1) are, respectively, n and m, then the behavior of the operation $*$ is defined as follows:

$$(a_n a_{n-1} \cdots a_1) * (b_m b_{m-1} \cdots b_1) = c_{m+n-1} \cdots c_2 c_1$$

where for $n > m$ (with $i - j + 1 > 0$)

$$c_i = \begin{cases} \displaystyle\bigoplus_{j=1,\dots,i} a_j \otimes b_{i-j+1} & \text{if } 1 \le i \le m, \\ \displaystyle\bigoplus_{j=i-m+1,\dots,n} a_{i-j+1} \otimes b_j & \text{if } m+1 \le i \le m+n-1 \end{cases} \tag{2}$$

and for $n \le m$ (with $j > 0$)

$$c_i = \begin{cases} \displaystyle\bigoplus_{j=1,\dots,i} a_j \otimes b_{i-j+1} & \text{if } 1 \le i \le n, \\ \displaystyle\bigoplus_{j=i-m+1,\dots,n} a_{i-j+1} \otimes b_j & \text{if } n+1 \le i \le m+n-1; \end{cases} \tag{2'}$$

where the symbols $\oplus$ and $\otimes$ represent operations defined over $P(\mathbf{U})$ and correspond, respectively, to the well-known operations of addition and subtraction as defined for the natural numbers.

With the objective of obtaining a finer classification than those induced by A and B, the definition of the operation has been made in such a way that the following properties are satisfied: closure, commutativity, associativity, idempotence, and the existence of the zero element. Let us stress that, thanks to these properties, $(P(\mathbf{U}), *)$ becomes a commutative monoid, since there it exists only one pair of operations, $\oplus$ and $\otimes$, with $\oplus = \cup$ and $\otimes = \cap$, such that these properties are satisfied [14].

2.1.2. THE OPERATION $\circ$ The second part of the strings consists of fuzzy numbers, which are, by definition, convex fuzzy sets [16]. The reason for this choice is that (fuzzy numbers in [0, 1], $\leq$) is a distributive, semicomplemented lattice (where $\alpha_i \leq \alpha_j$ means "α_i precedes α_j" if α_i and α_j are two fuzzy numbers in [0, 1]) with inf = extended minimum and sup = extended maximum. Furthermore, these numbers can be easily handled, thanks to the well-known Dubois-Prade algorithm [17]. For the sake of simplicity, we consider a subclass of fuzzy numbers in [0, 1], those known as "flat fuzzy numbers," which include triangular and trapezoidal numbers. This decision does not affect the generality of our approach, since by exploiting the Dubois-Prade algorithm it is possible to extend the operations on triangular fuzzy numbers by handling their extremes in an appropriate way [11]:

$$(\alpha_n \alpha_{n-1} \cdots \alpha_1) \circ (\beta_m \beta_{m-1} \cdots \beta_1) = \gamma_{m+n-1} \cdots \gamma_2 \gamma_1,$$

where, for $n > m$ (with $i - j + 1 > 0$)

$$\gamma_i = \begin{cases} \bigoplus_{j=1,\ldots,i} a_j \otimes b_{i-j+1} & \text{if } 1 \leq i \leq m, \\ \bigoplus_{j=i-m+1,\ldots,n} a_{i-j+1} \otimes b_j & \text{if } m+1 \leq i \leq m+n-1, \end{cases} \tag{3}$$

and for $n \leq m$ (with $j > 0$)

$$\gamma_i = \begin{cases} \bigoplus_{j=1,\ldots,i} a_j \otimes b_{i-j+1} & \text{if } 1 \leq i \leq n \\ \bigoplus_{j=i-m+1,\ldots,n} a_{i-j+1} \otimes b_j & \text{if } n+1 \leq i \leq m+n-1, \end{cases} \tag{3'}$$

where $\oplus$ and $\otimes$ are operations defined for pairs of fuzzy numbers.

In order that the resulting string may represent a finer partition with respect to those associated to the original strings, it is necessary that the operation $\circ$ satisfy the following properties:

1. closure,
2. commutativity and associativity,
3. preservation of the ordering among the fuzzy numbers.

To guarantee these properties it is necessary to explore among the possible fuzzy operators which could be adopted to realize the operations $\oplus$ and $\otimes$. Possible candidates are:

- extended sum,
- extended product,
- extended mean.

In the first case, the operation produces the fuzzy numbers by adding the right extreme, the central point, and the left extreme of the two numbers. The remaining two operations are similar. The difference consists in applying the product and mean rather than the sum.

Among the possible combinations of these three operations, we prefer the choice $\oplus = \otimes =$ extended mean (to be denoted by the symbol ®), for which properties 1 and 3 are satisfied, even though property 2 is not assured [11]. This problem can be solved by defining an appropriate ordering of string composition in such a way to fulfil our need for classification.

2.1.3. THE LINGUISTIC APPROXIMATION To avoid the explosion of clusters (and consequently of labels) encountered during the iteration of the operation $\triangle$, it is necessary to introduce an appropriate strategy of approximation. The main goal of this strategy is to evaluate the results on the second parts, class by class, comparing them with the original labels. Let $\{\alpha_1, \ldots, \alpha_p\}$ be the set of the fuzzy numbers used to represent such labels, and β the fuzzy number to be approximated. Let us suppose that the mean value of β, denoted by m, lies in the interval $[m_i, m_{i+1}]$ whose extremes are the mean values of the fuzzy numbers α_i and α_{i+1} (for some $i = 1, 2, \ldots, p$). Letting $d = m_{i+1} - m_i$, we apply the following approximation:

1. if $m \in [m_i, m_i + d/10]$, then we approximate β with α_i;
2. if $m \in [m_i + d/10, m_i + \frac{3}{10}d]$, then we say that β is *next to* α_i, adopting the pattern $\mathrm{NT}[\alpha_i]$;
3. if $m \in [m_i + \frac{3}{10}d, m_i + \frac{7}{10}d]$, then we say that β is *included between* α_i and α_{i+1}, adopting the pattern $\mathrm{IB}[\alpha_i, \alpha_{i+1}]$;
4. if $m \in [m_i + \frac{7}{10}d, m_i + \frac{9}{10}d]$, then we say that β is *just before* α_{i+1}, adopting the pattern $\mathrm{JB}[\alpha_{i+1}]$;
5. if $m \in [m_i + \frac{9}{10}d, m_{i+1}]$, then we approximate β with α_{i+1}.

Our strategy of approximation provides an upper bound on the number of the obtainable labels. Their number can not exceed the value $4n - 3$, where n denotes the original number of linguistic labels that are taken for reference.

2.1.4. A SIMPLE EXAMPLE Let $\mathbf{U} = \{a, b, c\}$ be the universe of discourse of three individuals, and **ct**, **t**, **at**, and **f** four linguistic variables represented by these triangular fuzzy numbers:

$$\mathbf{ct} = [0.8, 1.0, 1.0], \qquad \mathbf{t} = [0.5, 0.7, 0.9], \qquad \mathbf{at} = [0.2, 0.4, 0.6],$$

$$\mathbf{f} = [0.0, 0.0, 0.2].$$

Table 1.

										$[a,c]$	$[b]$	$[-]$	$[-]$	$*$
										$[c]$	$[a]$	$[b]$	$[-]$	
										—	—	—	—	
									—	$[b]$	—	—		
								$[a]$	—	—	—			
							$[c]$	—	—	—				
$X * Y$:							$[c]$	$[a]$	$[-]$	$[b]$	$[-]$	$[-]$	$[-]$	$*$
										$[b]$	$[c]$	$[-]$	$[a]$	
							—	$[a]$	—	—	—	—	—	
						—	—	—	—	—	—	—		
					$[c]$	—	—	—	—	—	—			
				—	—	—	$[b]$	—	—	—				
$(X * Y) * Z$:				$[-]$	$[c]$	$[-]$	$[b]$	$[a]$	$[-]$	$[-]$	$[-]$	$[-]$	$[-]$	$*$
										$[-]$	$[b]$	$[c]$	$[a]$	
				—	—	—	—	$[a]$	—	—	—	—	—	
				—	$[c]$	—	—	—	—	—	—	—		
		—	—	—	$[b]$	—	—	—	—	—	—			
	—	—	—	—	—	—	—	—	—	—				
$((X * Y) * Z) * K$:	$[-]$	$[-]$	$[-]$	$[c]$	$[b]$	$[-]$	$[-]$	$[a]$	$[-]$	$[-]$	$[-]$	$[-]$	$[-]$	
	13	12	11	10	9	8	7	6	5	4	3	2	1	

Let $\{X, Y, Z, K\}$ be a set of attributes whose relationships with the universe of discourse are given in the following table:

	X	Y	Z	K
a	**ct**	**t**	**f**	**f**
b	**t**	**at**	**ct**	**a**
c	**ct**	**ct**	**t**	**at**

The strings on which we operate are

$$X = \{a, c\}^{\mathbf{ct}}\{b\}^{\mathbf{t}}\{-\}^{\mathbf{at}}\{-\}^{\mathbf{f}}, \qquad Z = \{b\}^{\mathbf{ct}}\{c\}^{\mathbf{t}}\{-\}^{\mathbf{at}}\{a\}^{\mathbf{f}},$$

$$Y = \{c\}^{\mathbf{ct}}\{a\}^{\mathbf{t}}\{b\}^{\mathbf{at}}\{-\}^{\mathbf{f}}, \qquad K = \{-\}^{\mathbf{ct}}\{b\}^{\mathbf{t}}\{c\}^{\mathbf{at}}\{a\}^{\mathbf{f}}.$$

Now, let us compute the ordered string $X \vartriangle Y \vartriangle Z \vartriangle K$.

Applying the operation on the first parts, $(((X * Y) * Z) * K)$, we obtain the calculation shown in Table 1. Applying the operation on the second parts $((X \circ Y) \circ Z) \circ K$, we obtain

			ct	**t**	**at**	**f**	$\circ$
			ct	**t**	**at**	**f**	
			ct ® f	**t ®** f	**at ® f**	**f ® f**	
		ct ® at	**t ® at**	**at ® at**	**f ® at**		
	ct ® t	**t ® t**	**at ® t**	**f ® t**			
ct ® ct	**t ® ct**	**at ® ct**	**f ® ct**				
γ_7	γ_6	γ_5	γ_4	γ_3	γ_2	γ_1	

where

$$\gamma_7 = [0.8, 1.0, 1.0], \quad \gamma_6 = [0.65, 0.85, 0.95], \quad \gamma_5 = [0.5, 0.7, 0.8333],$$
$$\gamma_4 = [0.375, 0.525, 0.675], \qquad \gamma_3 = [0.233, 0.3666, 0.566],$$
$$\gamma_2 = [0.1, 0.21, 0.4], \qquad \gamma_1 = [0.0, 0.0, 0.2].$$

Repeating the operation, we obtain

			γ_7	γ_6	γ_5
					ct
			γ_7 **® f**	γ_6 **® f**	γ_5 **® f**
		γ_7 **® at**	γ_6 **® at**	γ_5 **® at**	γ_4 **® at**
	γ_7 **® t**	γ_6 **® t**	γ_5 **® t**	γ_4 **® t**	γ_3 **® t**
γ_7 **® ct**	γ_6 **® ct**	γ_5 **® ct**	γ_4 **® ct**	γ_3 **® ct**	γ_2 **® ct**
δ_{10}	δ_9	δ_8	δ_7	δ_6	δ_5
	γ_4	γ_3	γ_2	γ_1	$\circ$
	t	**at**	**f**		
	γ_4 **® f**	γ_3 **® f**	γ_2 **® f**	γ_1 **® f**	
	γ_3 **® at**	γ_2 **® at**	γ_1 **® at**		
	γ_2 **® t**	γ_1 **® t**			
	γ_1 **® ct**				
	δ_4	δ_3	δ_2	δ_1	

where

$$\delta_{10} = [0.8000, 1.0000, 1.0000], \qquad \delta_9 = [0.6875, 0.8875, 0.9625],$$
$$\delta_8 = [0.575, 0.775, 0.8805], \qquad \delta_7 = [0.4781, 0.6468, 0.7697],$$
$$\delta_6 = [0.4072, 0.5677, 0.7156], \qquad \delta_5 = [0.3385, 0.4864, 0.6468],$$
$$\delta_4 = [0.2760, 0.3989, 0.5677], \qquad \delta_3 = [0.1722, 0.2777, 0.4777],$$
$$\delta_2 = [0.0750, 0.15, 0.35], \qquad \delta_1 = [0.0, 0.0, 0.2].$$

Finally, applying the operation $\circ$, we obtain

$$\gamma_1 = [0.0, 0.0, 0.2],$$

$$\gamma_2 = [0.0687, 0.1375, 0.3375], \qquad \gamma_3 = [0.1578, 0.2546, 0.4546],$$

$$\gamma_4 = [0.2529, 0.3658, 0.5369], \qquad \gamma_5 = [0.2952, 0.4266, 0.5927],$$

$$\gamma_6 = [0.3367, 0.4788, 0.6384], \qquad \gamma_7 = [0.375, 0.525, 0.675],$$

$$\gamma_8 = [0.4123, 0.572, 0.7141], \qquad \gamma_9 = [0.4559, 0.6221, 0.7535],$$

$$\gamma_{10} = [0.5050, 0.6761, 0.7891], \qquad \gamma_{11} = [0.5937, 0.7937, 0.8905],$$

$$\gamma_{12} = [0.6968, 0.8968, 0.9656], \qquad \gamma_{13} = [0.8, 1, 1].$$

Once we have applied the linguistic approximation for the labels belonging to nonempty sets, the result is:

$$\{a\}^{\mathbf{NT[at]}} \; \{b\}^{\mathbf{IB[at,t]}} \; \{c\}^{\mathbf{t}}.$$

2.2. Notes on the Use of the Extended Mean Operation

Since the operation $\circledR$ is commutative ($a \circledR b = b \circledR a$) but not associative [$a \circledR (b \circledR c) \neq (a \circledR b) \circledR c$], the operation $\circ$ on the second parts does not possess the properties of associativity and commutativity. This could cause difficulty in applying the operation $\circledR$ when, to combine the strings, we are obliged to fix an ordering among them, and, consequently to demonstrate the validity of such an ordering. A deeper discussion of this problem is seen in Section 3.1. Here we note only that the second parts of the strings are composed of the same elements (the set of the linguistic labels), and thus the ordering is meaningful (only the first parts of the strings are different, but the related operation $*$ is commutative and associative; furthermore, the operations $*$ and $\circ$ are independent and can be executed in parallel).

Another aspect to consider, with regard to the absence of associativity for the operation $\circledR$, relates to the ordering to follow for the expression (3). To underline this case, let us consider a naive example.

EXAMPLE 1 Let $\mathbf{U} = \{a, b, c, d, e, f\}$ be the universe of discourse and suppose that there are four attributes A, B, C, D and four linguistic labels **ct**, **t**, **at**, and **f**. The values are

$$\mathbf{ct} = \{0.8, 1.0, 1.0\}, \qquad \mathbf{t} = \{0.5, 0.7, 0.9\},$$

$$\mathbf{at} = \{0.2, 0.4, 0.6\}, \qquad \mathbf{f} = \{0.0, 0.0, 0.2\}.$$

The input data are

	A	B	C	D
a	**t**	**at**	**f**	**t**
b	**at**	**ct**	**t**	**at**
c	**ct**	**at**	**t**	**at**
d	**f**	**ct**	**ct**	**f**
e	**at**	**t**	**ct**	**f**
f	**t**	**at**	**at**	**t**

and thus the strings to treat are

$$A = \{c\}^{\mathbf{ct}}\{a, f\}^{\mathbf{t}}\{b, e\}^{\mathbf{at}}\{d\}^{\mathbf{f}},$$

$$B = \{b, d\}^{\mathbf{ct}}\{e\}^{\mathbf{t}}\{a, c, f\}^{\mathbf{at}}\{-\}^{\mathbf{f}},$$

$$C = \{d, e\}^{\mathbf{ct}}\{b, c\}^{\mathbf{t}}\{f\}^{\mathbf{at}}\{a\}^{\mathbf{f}},$$

$$D = \{-\}^{\mathbf{ct}}\{a, e, f\}^{\mathbf{t}}\{b, c\}^{\mathbf{at}}\{d\}^{\mathbf{f}}.$$

First, we operate on the first parts by generating the string $A \triangle B$, then $(A \triangle B) \triangle C$, and finally $((A \triangle B) \triangle C) \triangle D$. The result is the following ordered string of subsets of **U**:

$\{-\}$	$\cdots$	$\{e\}$	$\{b, c\}$	$\{d, f\}$	$\{a\}$	$\cdots$	$\{-\}$
13	$\cdots$	9	8	7	6	$\cdots$	1

Now we compute the linguistic labels by applying the formula (3) and showing the nonempty classes:

$$\gamma_9 = \{0.5339, 0.7082, 0.8148\} = \mathbf{t},$$

$$\gamma_8 = \{0.4977, 0.6659, 0.7801\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_7 = \{0.4657, 0.6217, 0.7408\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_6 = \{0.4162, 0.5634, 0.6987\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}].$$

Thus the resulting string is the following:

$$\{e\}^{\mathbf{t}}\{b, c, d, f\}^{\mathrm{JB}[\mathbf{t}]}\{a\}^{\mathrm{IB}[\mathbf{at}, \mathbf{t}]}.$$

Table 2.

i	$\gamma_i - \gamma_i$	i	$\gamma_i - \gamma_i$
13	.000	6	.188
12	.000	5	.165
11	.053	4	.144
10	.152	3	.061
9	.166	2	.000
8	.188	1	.000
7	.206		

Now let us consider the following new definition:

$$\underline{\gamma}_i = \begin{cases} \bigoplus\limits_{j=1,\ldots,i} \alpha_{i+j+1} \otimes \beta_j & \text{if} \quad 1 \le i \le m, \\ \bigoplus\limits_{j=i-m+1,\ldots,n} \alpha_j \otimes \beta_{i-j+1} & \text{if} \quad m+1 \le i \le m+n-1. \end{cases} \tag{4}$$

We note that the only difference from (3)is the new ordering adopted to "sum" the elements $a_i \otimes b_j$. With this modification we obtain these fuzzy numbers:

$$\underline{\gamma}_9 = \{0.3984, 0.5414, 0.6771\} = \text{IB}[\textbf{at}, \textbf{t}],$$

$$\underline{\gamma}_8 = \{0.3425, 0.4778, 0.6250\} = \text{NT}[\textbf{at}],$$

$$\underline{\gamma}_7 = \{0.2967, 0.4157, 0.5718\} = \textbf{at},$$

$$\underline{\gamma}_6 = \{0.2610, 0.3752, 0.5434\} = \textbf{at}.$$

Consequently the new string is

$$\{e\}^{\text{IB[at, t]}}\ \{b, c\}^{\text{NT[at]}}\ \{a, d, f\}^{\text{at}}.$$

We note a negative influence on the classification, focused more on the central labels and less on the external labels. Table 2 reports the corresponding data, expressed in terms of the mean value of the fuzzy numbers.

This example shows that the result on the second parts depends on the ordering established by (3). Before entering into a more detailed discussion of this problem, which will be faced in Section 4.1, we prefer to complete the discussion of the algebraic structure by introducing the concept of the weight of the attributes.

3. EXTENSION OF THE STRUCTURE

In this section some important characteristics of the structure are revisited with the objective of further improving and to extending the classification mechanism by optimizing the management of the fuzzy information handled in the structure. One extension is focused on the problem of the combinatorial explosion, which represents the most crucial issue in classifications systems. A second enhancement which will be presented in this section concerns the weighting method mentioned in the previous paragraph.

3.1. The Problem of Commutativity and Its Resolution

We have seen that the extended mean operation, used to operate on the second parts, leads to a number of nontrivial problems, especially that the result of the operation depends quantitatively on the ordering in which we apply the formula (3). Thus, it is necessary to better formalize the mechanism of such an ordering. Before discussing this aspect, we face the problem of the lack of associativity for the weighted mean (recall that the resulting operation $\vartriangle$ is neither commutative nor associative), a problem that did not occur before we introduced the operator weight in the algebraic structure. This problem is crucial; in fact, the results of the classification appear confused due to the flattening effect which arises from an approximation.

If the attributes are composed in a casual manner (for instance, following the input ordering), we will not obtain a reliable classification, especially from a qualitative standpoint. This situation becomes harder to treat if we consider the problem of what order we must fix for (3). To solve this difficulty we introduce a new operation on the second parts.

3.1.1. A NEW OPERATION ON THE SECOND PARTS Given the two strings of length n and m as before, we define the operation $\circ$ on the second parts as follows:

$$(\alpha_n \alpha_{n-1} \cdots \alpha_1) \circ (\beta_m \beta_{m-1} \cdots \beta_1) = (\gamma_{m+n-1} \cdots \gamma_2 \gamma_1),$$

where, for $n > m$ (with $i - j + 1 > 0$),

$$\gamma_i = \begin{cases} \dfrac{1}{i} \displaystyle\sum_{j=1,\ldots,i} \alpha_j \circledR \beta_{i-j+1} & \text{if } 1 \le i \le m, \\ \dfrac{1}{m+n-i} \displaystyle\sum_{j=i-m+1,\ldots,n} \alpha_{i-j+1} \circledR \beta_j & \text{if } m+1 \le i \le m+n-1, \end{cases} \tag{5}$$

and for $n \leq m$ (with $j > 0$),

$$\gamma_i = \begin{cases} \dfrac{1}{i} \displaystyle\sum_{j=1,\ldots,i} \alpha_j \circledR \beta_{i+j+1} & \text{if} \quad 1 \leq i \leq n, \\ \dfrac{1}{m+n-i} \displaystyle\sum_{j=i-m+1,\ldots,n} \alpha_{i-j+1} \circledR \beta_j & \text{if} \quad n + \leq i \leq m+n-1, \end{cases} \tag{5'}$$

where $\otimes$ is the extended mean among the triangular fuzzy numbers, and Σ represents the sum of such numbers.

The operation $\triangle$ is both commutative and associative. To better understand its behavior, let us consider again the example introduced in Section 2.2 and apply the new operation on the second parts.

For the first parts, we obtain the same result:

$\{-\}$	$\cdots$	$\{e\}$	$\{b,c\}$	$\{d,f\}$	$\{a\}$		$\{-\}$
13	$\cdots$	9	8	7	6	$\cdots$	1

Now we report the values of labels associated with nonempty classes:

$$\begin{aligned} \delta_9 &= \{0.4559, 0.6221, 0.7535\} = \text{JB}[\mathbf{t}], \\ \delta_8 &= \{0.4123, 0.5720, 0.7141\} = \text{IB}[\mathbf{at}, \mathbf{t}], \\ \delta_7 &= \{0.3750, 0.5250, 0.6750\} = \text{IB}[\mathbf{at}, \mathbf{t}], \\ \delta_6 &= \{0.3367, 0.4788, 0.6384\} = \text{NT}[\mathbf{at}]. \end{aligned}$$

Then the new string is

$$\{e\}^{\text{JB}[\mathbf{t}]}\{b,c,d,f\}^{\text{IB}[\mathbf{at},\mathbf{t}]}\{a\}^{\text{NT}[\mathbf{at}]}.$$

Let us compare this with the string when we used the "old" operation (3):

$$\{e\}^{\mathbf{t}}\{b,c,d,f\}^{\text{JB}[\mathbf{t}]}\{a\}^{\text{IB}[\mathbf{at},\mathbf{t}]}$$

and the string when we used the "old" operation (4):

$$\{e\}^{\text{IB}[\mathbf{at},\mathbf{t}]}\{b,c\}^{\text{NT}[\mathbf{at}]}\{a,d,f\}^{\mathbf{at}}.$$

Comparing these results, we note that the adoption of the new operation $\triangle$ leads to more equitable method which can be considered as intermediate between those based on (3) and (4). Moreover, we gain the properties of commutativity and associativity. One pending problem remains: the high density of the labels around a point of the interval [0, 1]. However, this point is effectively in the middle of the points corresponding to the application of the formulas (3) and (4).

4. THE PROBLEM OF RELEVANCE

In this section we discuss a mechanism suitable for improving the power and flexibility of the structure. In fact, once we have generated the strings to "multiply," we obtain a resulting string which represents a new classification, considering all the attributes in the same measure. In a practical situation, we know that the influence of an attribute on the results of the classification must be differentiated from that of the other attributes.

To better understand this situation, let us consider a typical problem, a medical diagnosis. Here, different attributes, such as "stress," "anxiety," or "cholesterol," lead to a classification of the element in the cluster "coronopathy." As is known, not all the symptoms add to the diagnosis in the same measure. It must be more appropriate to give different weights to some factors, such as the attribute cholesterol.

4.1. Weighting the Attributes

Here we show how it is possible to introduce a strategy of weighting the attributes used in our algebraic structure with the objective of improving and extending the classification mechanism.

A first step consists in modifying the values of those labels (the fuzzy numbers) in the strings which represent the most relevant attribute. The value of the modification should be chosen by experts who decide on the basis of their experience what the effect of the weight for each attribute should be. Naturally, an automatic approach is more appropriate in classification systems which do not base their functionality on the assistance of human experts.

We could increment, by a fixed amount μ, all of the linguistic labels of the string which correspond to the attribute that we want to emphasize. In this case, for the objects belonging to the strings, we will note the corresponding variation; however, this alteration is propagated over all of the classification. In order to avoid this problem, we could modulate the increment μ in such a way as to augment the highest labels and to reduce the lowest ones by normalizing the relative values.

To discuss the results of these strategies, we again consider the example in Section 2.2. The input data are shown in the following table:

	A	B	C	D
a	**f**	**at**	**f**	**t**
b	**f**	**ct**	**t**	**at**
c	**ct**	**f**	**f**	**at**
d	**f**	**ct**	**ct**	**f**
e	**f**	**t**	**ct**	**ct**
f	**f**	**at**	**at**	**t**

The strings on which we operate are

$$A = \{c\}^{\mathbf{ct}}\{-\}^{\mathbf{t}}\{-\}^{\mathbf{at}}\{a, b, d, e, f\}^{\mathbf{f}},$$

$$B = \{b, d\}^{\mathbf{ct}}\{e\}^{\mathbf{t}}\{a, f\}^{\mathbf{at}}\{c\}^{\mathbf{f}},$$

$$C = \{d, e\}^{\mathbf{ct}}\{b\}^{\mathbf{t}}\{f\}^{\mathbf{at}}\{a, c\}^{\mathbf{f}},$$

$$D = \{e\}^{\mathbf{ct}}\{a, f\}^{\mathbf{t}}\{b, c\}^{\mathbf{at}}\{d\}^{\mathbf{f}}.$$

We note that for attribute A, whose relevance is stressed, the values are significantly changed. Now let us proceed to the classification by computing the string $((A \triangle B) \triangle C) \triangle D$. The application of the operation on the first parts provides the string

$$\underset{13}{\{-\}}\{-\}\{-\}\{-\}\underset{9}{\{e\}}\{-\}\underset{7}{\{b, d\}}\{-\}\underset{5}{\{c, f\}}\underset{4}{\{a\}}\{-\}\{-\}\underset{1}{\{-\}}$$

For the second parts we compute only the labels which correspond to nonempty classes. Thus we have

$$\gamma_9 = \{0.5339, 0.7082, 0.8148\} = \mathbf{t},$$

$$\gamma_7 = \{0.4657, 0.6217, 0.7408\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_5 = \{0.3622, 0.4979, 0.6409\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}],$$

$$\gamma_4 = \{0.3167, 0.4273, 0.5739\} = \mathbf{at}.$$

Thus the resulting classification is

$$\{e\}^{\mathbf{t}}\{b, d\}^{\mathrm{JB}[\mathbf{t}]}\{c, f\}^{\mathrm{IB}[\mathbf{at}, \mathbf{t}]}\{a\}^{\mathbf{at}}.$$

Now, if we want to weight the attribute A, it is necessary to increment all the linguistic labels in our strings by a fixed amount μ. We recall that the fuzzy numbers are triangular fuzzy numbers in $[0, 1]$, and that we must obtain the same kind of results. Thus we apply the following rule: If $x = \{x_1, x_2, x_3\}$, with $x_1, x_2, x_3 \in [0, 1]$, is a fuzzy number, then the new fuzzy number is given by $\mu_x = \{y_1, y_2, y_3\}$, where $y_i = \min\{\mu + x_i, 1\}$. Applying this formula to our previous example, with $\mu = 0.2$, we obtain

$$\mu_{\mathbf{ct}} = \{1.0, 1.0, 1.0\}, \qquad \mu_{\mathbf{t}} = \{0.7, 0.9, 1.0\},$$

$$\mu_{\mathbf{at}} = \{0.4, 0.6, 0.8\}, \qquad \mu_{\mathbf{f}} = \{0.2, 0.2, 0.4\}.$$

Now we substitute the new result in the string A:

$$A' = \{c\}^{\mu_{\mathbf{ct}}}\{-\}^{\mu_{\mathbf{t}}}\{-\}^{\mu_{\mathbf{at}}}\{a, b, d, e, f\}^{\mu_{\mathbf{f}}},$$

and we apply again the operation on the second parts of the new string $((A' \vartriangle B) \vartriangle C) \vartriangle D$. The result is

$$\{e\}^{\mathrm{t}}\{b,d\}^{\mathrm{JB[t]}}\{c,f\}^{\mathrm{IB[at,t]}}\{a\}^{\mathrm{NT[at]}}.$$

Before discussing these steps, it is interesting to analyze the result obtained if we consider a variable increment. Thus, we define μ in this way: if $\alpha_n, \ldots, \alpha_1$ are linguistic labels, then we define α_k, with $k = \mathrm{int}(n/2)$, as a central label. Then we choice as the increment the quantity $\sigma = 0.2$, and we apply the increment on the values of the labels using the formula

$$\mu_i = (i - n/2)\sigma, \qquad 1 \le i \le n. \tag{6}$$

This operation yields positive increments for $i > \mathrm{int}(n/2)$, negative increments for $i < \mathrm{int}(n/2)$, and no increment for $i = \mathrm{int}(n/2)$.

If we apply this formula to the above introduced example, with $\sigma = 0.2$, the resulting new labels for the attribute A are:

$$\begin{aligned} \mu_4 &= 0.4, & \mu_{\mathrm{ct}} &= \{1.0, 1.0, 1.0\}, \\ \mu_3 &= 0.2, & \mu_{\mathrm{t}} &= \{0.7, 0.9, 1.0\}, \\ \mu_2 &= 0.0, & \mu_{\mathrm{at}} &= \{0.2, 0.4, 0.6\}, \\ \mu_1 &= -0.2, & \mu_{\mathrm{f}} &= \{0.0, 0.0, 0.0\}. \end{aligned}$$

The final string $((A' \vartriangle B) \vartriangle C) \vartriangle D$ is

$$\{e\}^{\mathrm{t}}\{b,d\}^{\mathrm{JB[t]}}\{c,f\}^{\mathrm{IB[at,t]}}\{a\}^{\mathrm{at}}.$$

Now we try to compare the two different results. Ignoring the flattening of the results due to the linguistic approximation, we note that in the first approach there is a general improvement for all the labels. However, this is less significant for the upper labels. In the second approach, we see an improvement for the upper labels, and a contrary effect for the lower labels. The intermediate labels remain essentially stable.

Neither approach generates appropriate results. The object c, which is the unique highest value for the attribute A, does not present any appreciable improvement for other objects for which the attribute is not present. The reason for this behavior must lie in the fact that we modify only the labels, without affecting the distribution of the objects. In fact, even if we were able to correct the label of the object c, we would see an incorrect modification for the object f, which belongs to the same class without sharing the same attribute A.

To solve this problem, we generate new labels and new classes for the elements of only the first part. Thanks to these new classes, we can translate to one position all the elements of the attribute that we want to weight. For such classes, we create intermediate labels with the method previously discussed. Obviously, this doubles the length of the string, with a correspondent decrease in the performance of our algorithm. The complexity of the calculus, previously equal to nm, now grows to $2nm$, and could be critical with increasing n.

Let us consider again our string

$$A = \{c\}^{\mathrm{ct}}\{-\}^{\mathrm{t}}\{-\}^{\mathrm{at}}\{a, b, d, e, f\}^{\mathrm{f}}.$$

The new form is

$$A = \{-\}\{c\}^{\mathrm{ct}}\{-\}\{-\}^{\mathrm{t}}\{-\}\{-\}^{\mathrm{at}}\{-\}\{a, b, d, e, f\}^{\mathrm{f}}.$$

Essentially, we generate the intermediate classes without computing the corresponding labels; we move all the elements of one position to the left side, thus applying the real weighting strategy, since we augment the position of the object related to the selected attribute.

Now we write out the new form of the previous string when the attribute is weighted:

$$A' = \{c\}^{\mu_{\mathrm{ct}}}\{-\}^{\mathrm{ct}}\{-\}^{\mu_{\mathrm{t}}}\{-\}^{\mu_{\mathrm{at}}}\{-\}^{\mathrm{at}}\{a, b, d, e, f\}^{\mu_{\mathrm{f}}}\{-\}^{\mathrm{f}}.$$

We must now assign the labels to the new classes. Let us consider first the method based on a fixed increment. With $\mu = 0.2$ we obtain the same values for the new labels:

$$\mu_{\mathrm{ct}} = \{1.0, 1.0, 1.0\}, \qquad \mu_{\mathrm{t}} = \{0.7, 0.9, 1.0\},$$
$$\mu_{\mathrm{at}} = \{0.4, 0.6, 0.8\}, \qquad \mu_{\mathrm{f}} = \{0.2, 0.2, 0.4\}.$$

The final string $((A' \vartriangle B) \vartriangle C) \vartriangle D$ is now

$$\begin{array}{ccccccccccc} \{-\} & \cdots & \{e\} & \{c\} & \{b, d\} & \cdots & \{f\} & \{a\} & \cdots & \{-\} \\ 17 & \cdots & 10 & 9 & 8 & \cdots & 6 & 5 & \cdots & 1 \end{array}$$

Computing the labels of the nonempty classes, we have as the final string

$$\{e\}^{\mathrm{t}}\{b, c, d\}^{\mathrm{JB[t]}}\{a, f\}^{\mathrm{IB[at,t]}}.$$

To better compare the two different approaches, we report here the string for which no attributes have been weighted (keeping in mind that we have inserted the weight in the attribute A, which has highest relevance for the object c):

$$\{e\}^{\mathrm{t}}\{b, d\}^{\mathrm{JB[t]}}\{c, f\}^{\mathrm{IB[at,t]}}\{a\}^{\mathrm{at}}.$$

We note that c has enhanced its position in the classification, since it is the unique object which obtained the highest relevance of the weighted attribute. For the other objects, we see a corresponding improvement of their position, even though it is less obvious with respect to the object c.

Now let us discuss the effect on the classification when we consider the variant increment method, as defined in Equation (6) with $\sigma = 0.2$. We generate the labels $\mu_{\mathrm{ct}}, \mu_{\mathrm{t}}, \mu_{\mathrm{at}}, \mu_{\mathrm{f}}$, which are equal to those viewed in the first part of the above introduced example:

$$\begin{aligned} \mu_4 &= 0.4, & \mu_{\mathrm{ct}} &= \{1.0, 1.0, 1.0\}, \\ \mu_3 &= 0.2, & \mu_{\mathrm{t}} &= \{0.7, 0.9, 1.0\}, \\ \mu_2 &= 0.0, & \mu_{\mathrm{at}} &= \{0.2, 0.4, 0.6\}, \\ \mu_1 &= -0.2, & \mu_{\mathrm{f}} &= \{0.0, 0.0, 0.0\}. \end{aligned}$$

Analyzing the operation on the second parts, we immediately note that this increment does not assure the ordering of the labels, and consequently the ordering must be verified and possibly "adjusted" with opportune position exchange for the labels and related classes. In our case, the label μ_{f} led to less of the label **f**, and thus we exchange the position of these two labels, with their classes, as shown in the next string:

$$A' = \{c\}^{\mu_{\mathrm{ct}}}\{-\}^{\mathrm{ct}}\{-\}^{\mu_{\mathrm{t}}}\{-\}^{\mu_{\mathrm{at}}}\{-\}^{\mathrm{f}}\{a, b, d, e, f\}^{\mu_{\mathrm{f}}}$$

Operating on the first parts, we obtain

$$\begin{array}{ccccccccccc} \{-\} & \cdots & \{c, e\} & \cdots & \{b, d\} & \cdots & \{f\} & \{a\} & \cdots & \{-\} \\ 17 & & 9 & & 7 & & 5 & 4 & & 1 \end{array}$$

Once we have computed the labels, we have as final string

$$\{c, e\}^{\mathrm{JB[t]}}\{b, d\}^{\mathrm{IB[at,t]}}\{f\}^{\mathrm{NT[at]}}\{a\}^{\mathrm{at}}.$$

The substantial difference is found in the fact that this last strategy lowered the ranking of labels associated to lower classes, i.e., the weighting mechanism gives an advantage to the elements with the most relevant attributes, a situation which validates our approach.

4.2. Rigidity of the Weighting Method

The proposed weighting strategy has some limits, in that it evaluates the relevance of the attributes as objective, not subjective, data. For example, if we are considering a population of cars, then the feature "safety" is a more subjective attribute than the feature "velocity."

We again consider the example introduced in Section 4.1 using the formula (3). Applying the "traditional" mechanism—without weighting—we obtained the string

$$\{e\}^{\mathbf{t}}\{b,d\}^{\mathrm{JB[t]}}\{c,f\}^{\mathrm{IB[at,t]}}\{a\}^{\mathbf{at}}.$$

However, if we use a weight $\mu = 0.2$ on the attribute A (for which the object c has highest value **ct**, whereas all the other ones have the lowest value **f**), then we obtain

$$\{c,e\}^{\mathrm{JB[t]}}\{b,d\}^{\mathrm{IB[at,t]}}\{f\}^{\mathrm{NT[at]}}\{a\}^{\mathbf{at}}.$$

The result has been modified: the object c has improved its position in the classification. But what happens if we decide to elevate the weight of A? Let us imagine that we are studying several models of cars, and that the attribute A represents "velocity" and C "safety." The result reflects in a logical way the opinion of an individual who does not belong to the category of drivers who prefer fast cars. In fact, the two models c and e are gathered in the same cluster. This is because the first car is the fastest but not absolutely the safest, while the second car, even though not fast, is the safest one. However, the second car, compared with the other models, satisfies reasonably well the remaining attributes. Roughly speaking, the classification reflects an individual who prefers speed, but it does not neglect the other factors. This leads to unequal classifications, since if a driver shows a maximum interest in speed, a correct classification should lead to model c alone in a cluster. In fact, even though we elevate the weight of attribute A, assigning $\mu = 0.3$, we note that models c and e are still present in the same class. This is due to the fact that the weight of attribute A influences the evaluation of the attributes present in the second parts of the strings, forgetting those existing in the first parts.

4.3. A New Mechanism for Weighting

Our goal remains to improve the classification of those objects with higher values of an attribute. However, this time we wish to do so without sacrificing the classification of the objects with lower values, i.e., the improvement of the most relevant objects must be proportioned by considering their value with respect to the weighted attribute. The effort must be focused on modifying the unary operator "weight" in such a way as to consider even those elements present in the first parts of the strings.

For simplicity, we assume that if we want to weight the attribute A, then the associated weight is $\mu > 0$ and all the remaining attributes have weights equal to zero. Let $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \cdots a_1^{\alpha_1}$ be the attribute to weight, and let μ be its weight. Then, for each class a_i with label α_i, we create on its left N_{α_i} new empty classes, with

$$N_{\alpha_i} = \mathrm{int}(m_{\alpha_i}\mu),$$

where m_{α_i} is the mean value of the fuzzy numbers represented by the label α_i.

In the second step, we move to the left the first N_{α_i} a_i together with their corresponding labels label α_i. Then, for each i, we assign to the N_{α_i} new classes those intermediate labels, between α_{i-1} and α_i, which satisfy the following formula:

$$\sigma_{ij} = \alpha_{i-1} + \frac{j(\alpha_i - \alpha_{i-1})}{N_{\alpha_i} + 1}, \quad j = 1, \dots, N\alpha_i, \qquad \text{for each } i. \tag{7}$$

Since the basic objects are fuzzy numbers, in the formula we have extended fuzzy operations. In our case,

$$A = \{c\}^{\mathbf{ct}}\{-\}^{\mathbf{t}}\{-\}^{\mathbf{at}}\{a, b, d, e, f\}^{\mathbf{f}},$$

for which, by assigning $\mu = 1$, we have

$$N_{\mathbf{ct}} = \mathrm{int}(1 \times 1) = 1, \qquad N_{\mathbf{t}} = \mathrm{int}(0.7 \times 1) = 0,$$
$$N_{\mathbf{at}} = \mathrm{int}(0.4 \times 1) = 0, \qquad N_{\mathbf{f}} = \mathrm{int}(0 \times 1) = 0.$$

We note that $N_{\mathbf{f}}$ is always equal to zero: this means that the weight given to the attribute is not correctly taken into account during the cluster generation. From the above results, we obtain

$$A' = \{c\}^{\mathbf{ct}}\{-\}^{\sigma_{41}}\{-\}^{\mathbf{t}}\{-\}^{\mathbf{at}}\{a, b, d, e, f\}^{\mathbf{f}}$$

Now we have to compute the label for the new class standing between α_3 and α_4, i.e., between **t** and **ct**. We have

$$N_{\mathbf{ct}} = 1,$$
$$\alpha_4 = \{0.8, 1.0, 1.0\}, \qquad \alpha_3 = \{0.5, 0.7, 0.9\},$$
$$\frac{\alpha_4 - \alpha_3}{N_{\mathbf{ct}} + 1} = \frac{\{0.3, 0.3, 0.1\}}{2} = \{0.15, 0.15, 0.05\},$$

for which

$$\sigma_{41} = \{0.5 + 0.15, 0.7 + 0.15, 0.9 + 0.05\} = \{0.65, 0.85, 0.95\}.$$

Now we compute the resulting string $A' \vartriangle B \vartriangle C \vartriangle D$. Operating on the first parts, we have

$$\begin{array}{l} \{-\}\{-\}\{-\}\{-\}\{-\}\{e\}\{-\}\{b, d\}\{c\}\{f\}\{a\}\{-\}\{-\}\{-\} \\ 14 \qquad\qquad\qquad\quad 9 \qquad 7 \quad 6 \;\; 5 \;\; 4 \qquad\quad\;\; 1 \end{array}$$

For the second parts,

$$\begin{aligned}
\gamma_9 &= \{0.5244, 0.6972, 0.8066\} = \mathbf{t}, \\
\gamma_7 &= \{0.4639, 0.6197, 0.7396\} = \mathrm{JB}[\mathbf{t}], \\
\gamma_6 &= \{0.4156, 0.5627, 0.6983\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}], \\
\gamma_5 &= \{0.3620, 0.4977, 0.6407\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}], \\
\gamma_4 &= \{0.3167, 0.4273, 0.5739\} = \mathbf{at}.
\end{aligned}$$

Thus, the final string is

$$\{e\}^{\mathbf{t}}\{b, d\}^{\mathrm{JB}[\mathbf{t}]}\{c, f\}^{\mathrm{IB}[\mathbf{at}, \mathbf{t}]}\{a\}^{\mathbf{at}}.$$

A clear improvement is noted for the object c in terms of its position, as wished. Less apparent is the real improvement in the label of c, and a corresponding general worsening for all the remaining labels, especially for the lowest classes, as expected.

Now, let us see what happens if we assign $\mu = 2$ to the attribute A:

$$\begin{aligned}
N_{\mathbf{ct}} &= \mathrm{int}(1 \times 2) = 2, \qquad & N_{\mathbf{t}} &= \mathrm{int}(0.7 \times 2) = 1, \\
N_{\mathbf{at}} &= \mathrm{int}(0.4 \times 2) = 0, \qquad & N_{\mathbf{f}} &= \mathrm{int}(0 \times 2) = 0.
\end{aligned}$$

Thus, we have

$$A' = \{c\}^{\mathbf{ct}}\{-\}^{\sigma_{42}}\{-\}^{\sigma_{41}}\{-\}^{\mathbf{t}}\{-\}^{\sigma_{31}}\{-\}^{\mathbf{at}}\{a, b, d, e, f\}^{\mathbf{f}}.$$

Now we have to find the labels for the new classes located between α_3 and α_4, i.e., between **t** and **ct**, and between α_2 and α_1, i.e., between **at** and **t**:

$$N_{\mathbf{ct}} = 2,$$

$$\alpha_4 = \{0.8, 1.0, 1.0\}, \qquad \alpha_3 = \{0.5, 0.7, 0.9\},$$

$$\frac{\alpha_4 - \alpha_3}{N_{\mathbf{ct}} + 1} = \frac{\{0.3, 0.3, 0.1\}}{3} = \{0.1, 0.1, 0.0333\}.$$

Thus

$$\begin{aligned}
\sigma_{42} &= \{0.5 + 0.2, 0.7 + 0.2, 0.9 + 0.0666\} = \{0.7, 0.9, 0.9666\}, \\
\sigma_{41} &= \{0.5 + 0.1, 0.7 + 0.1, 0.9 + 0.0333\} = \{0.6, 0.8, 0.9333\}.
\end{aligned}$$

Moreover,

$$\alpha_3 = \{0.5, 0.7, 0.9\}, \qquad a_2 = \{0.2, 0.4, 0.6\},$$

$$N_{\mathbf{t}} = 1,$$

$$\frac{\alpha_3 - \alpha_2}{N_{\mathbf{t}} + 1} = \frac{\{0.3, 0.3, 0.3\}}{2} = \{0.15, 0.15, 0.15\},$$

for which

$$\sigma_{31} = \{0.2 + 0.15, 0.4 + 0.15, 0.6 + 0.15\} = \{0.35, 0.55, 0.75\},$$

Calculating $A' \vartriangle B \vartriangle C \vartriangle D$, for the first parts we get

$$\underset{16}{\{-\}}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\underset{9}{\{e\}}\underset{8}{\{c\}}\underset{7}{\{b,d\}}\{-\}\underset{5}{\{f\}}\underset{4}{\{a\}}\{-\}\{-\}\underset{1}{\{-\}}$$

and for the second parts

$$\gamma_9 = \{0.5112, 0.6824, 0.7959\} = \mathbf{t},$$

$$\gamma_8 = \{0.4862, 0.6527, 0.7705\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_7 = \{0.4602, 0.6159, 0.7367\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_5 = \{0.3612, 0.4969, 0.6400\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}],$$

$$\gamma_4 = \{0.3164, 0.4270, 0.5737\} = \mathbf{at}.$$

Thus, the final string is

$$\{e\}^{\mathbf{t}}\{b, c, d\}^{\mathrm{JB}[\mathbf{t}]}\{f\}^{\mathrm{IB}[\mathbf{at}, \mathbf{t}]}\{a\}^{\mathbf{at}}.$$

We note a further improvement for the object c, as well as its label, since the object has received a final judgement equal to b and d. Moreover, an additional decrease of the lowest labels is seen, proving the soundness of the strategy. In fact, on augmenting the weight of the attribute A, for which c has highest value, the object c is more precisely placed in the highest area of the classification, a situation which did not happen with the previous approach.

Let us observe the behavior of our method when $\mu = 3$. We get

$$N_{\mathbf{ct}} = \mathrm{int}(1 \times 3) = 3, \qquad N_{\mathbf{t}} = \mathrm{int}(0.7 \times 3) = 2,$$

$$N_{\mathbf{at}} = \mathrm{int}(0.4 \times 3) = 1 \qquad N_{\mathbf{f}} = \mathrm{int}(0 \times 3) = 0.$$

Combining the above results, we have

$$A' = \{c\}^{\mathbf{ct}}\{-\}^{\sigma_{43}}\{-\}^{\sigma_{42}}\{-\}^{\sigma_{41}}\{-\}^{\mathbf{t}}\{-\}^{\sigma_{32}}\{-\}^{\sigma_{31}}\{-\}^{\mathbf{at}}\{-\}^{\sigma_{21}}\{a, b, d, e, f\}^{\mathbf{f}},$$

where, applying (7), we obtain

$$\sigma_{43} = \{0.725, 0.925, 0.975\}, \qquad \sigma_{42} = \{0.65, 0.85, 0.95\},$$

$$\sigma_{41} = \{0.575, 0.775, 0.925\}, \qquad \sigma_{32} = \{0.4, 0.6, 0.8\},$$

$$\sigma_{31} = \{0.3, 0.5, 0.7\}, \qquad \sigma_{21} = \{0.1, 0.2, 0.4\}.$$

Let us compute the string $A' \vartriangle B \vartriangle C \vartriangle D$. The first step consists in applying the operation on the first parts:

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{c\}\{-\}\{e\}\{-\}\{b,d\}\{-\}\{f\}\{a\}\{-\}\{-\}\{-\}\\ 19 \qquad\qquad\qquad\qquad\qquad\quad 11 \quad 9 \quad 7 \quad 5\ 4 \quad\quad 1\end{array}$$

We note that the object c lies alone at the top of the classification. This result satisfies our need to "model" the relevance of the attribute, as explained in the example of the cars, for which we wanted focusing on an attribute not to affect the correctness of the classification. Considering the second part, we have

$$\gamma_{11} = \{0.5257, 0.7023, 0.8132\} = \mathbf{t},$$

$$\gamma_9 = \{0.4963, 0.6647, 0.7801\} = \mathrm{JB}[\mathbf{t}],$$

$$\gamma_7 = \{0.4532, 0.6056, 0.7267\} = \mathrm{IB}[\mathbf{at}, \mathbf{t}],$$

$$\gamma_5 = \{0.3572, 0.4899, 0.6330\} = \mathrm{NT}[\mathbf{at}],$$

$$\gamma_4 = \{0.3147, 0.4239, 0.5706\} = \mathbf{at}.$$

Thus, the resulting string is

$$\{c\}^{\mathbf{t}}\{e\}^{\mathrm{JB}[\mathbf{t}]}\{b,d\}^{\mathrm{IB}[\mathbf{at},\mathbf{t}]}\{f\}^{\mathrm{NT}[\mathbf{at}]}\{a\}^{\mathbf{at}}.$$

Comparing this result with the previous one, we note that the only differences consist in a small improvement for the label of c and a clear improvement of the position of c in the final classification, as wished.

4.4. Complexity

Let us evaluate the two approaches from the standpoint of the complexity of the algorithms. We note that for the first approach, the increase in the complexity is fixed for each weighted attribute. The order of the complexity is the same as that of the multiplication of nonweighted attributes. In fact, if n is the "length" of the nonweighted attributes, i.e., the number of original classes, then the complexity of the multiplication between weighted and nonweighted attributes grows from n^2 to $2n^2$. Considering the second approach, the augmentation is also proportional to the weight μ that we want to provide to the attribute, i.e., the complexity changes from n^2 to $[n + h(\mu)]n$, according to a given growing linear function h.

For instance, if the mean values of the linguistic labels are equal to those used in the example, then $h(\mu) \cong 2\mu$. Thus, if μ is such that $h(\mu) \le n$, then the complexity is not greater than for the previous method.

However, it is important to consider if, for the same values of μ, we obtain appropriate results. To better focus on this aspect, let us analyze again the results of the preceding example. It is enough to consider only the first parts, since we do not want to be influenced by the problems of flattening due to the approximation. In this case, we have $n = 4$ and $\mu = 2$, and we obtain $h(\mu) = 3$, i.e., a quantity less than n. The result is

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{e\}\{c\}\{b,d\}\{-\}\{f\}\{a\}\{-\}\{-\}\{-\}\\ 16 \qquad\qquad\qquad\qquad\quad 9\ \ 8\quad 7 \qquad\ 5\ \ 4 \qquad\quad 1\end{array}$$

i.e., the position of c does not decrease with respect to the previous method. We recall that that string was

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{c,e\}\{-\}\{b,d\}\{-\}\{f\}\{a\}\{-\}\{-\}\{-\}\\ 17 \qquad\qquad\qquad\qquad\qquad 9 \qquad\ 7 \qquad 5\ \ 4 \qquad\quad 1\end{array}$$

If we modify the weight by assigning $\mu = 3$ to attribute A, then we obtain a "superevaluation" of the object c, which is moved to the top of the classification. This situation can be seen looking at the results provided by the first-parts operation:

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{c\}\{-\}\{e\}\{-\}\{b,d\}\{-\}\{f\}\{a\}\{-\}\{-\}\{-\}\\ 19 \qquad\qquad\qquad\qquad\quad 11 \quad\ 9 \qquad 7 \qquad 5\ \ 4 \qquad\quad 1\end{array}$$

It is important to stress that now $h(\mu) = 6$, i.e., a quantity greater than n, and thus the complexity is higher than for the first method. Suppose we assign to A the weight μ, with $2 < \mu < 3$ such that $h(\mu) = 4$. Considering our example, for $\mu = 2.5$ we obtain the results (for simplicity, we consider only the first parts):

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\{c,e\}\{-\}\{b,d\}\{-\}\{f\}\{a\}\{-\}\{-\}\{-\}\\ 17 \qquad\qquad\qquad\qquad\qquad 9 \qquad\ 7 \qquad 5\ \ 4 \qquad\quad 1\end{array}$$

and the complete string is

$$\{c,e\}^{\mathrm{JB[t]}}\{b,d\}^{\mathrm{IB[at,t]}}\{f\}^{\mathrm{NT[at]}}\{a\}^{\mathrm{at}},$$

i.e., we obtained the same strings as in Section 3. Thus, thanks to an appropriate tuning of μ, we obtain the same complexity, but with a gain of generality.

Naturally, the two methods are not to be considered as equivalent. In fact, let us suppose that the attribute A is represented by the string

$$A = \{c\}^{\mathrm{ct}}\{b\}^{\mathrm{t}}\{a\}^{\mathrm{at}}\{d\}^{\mathrm{f}}.$$

Then if we weight A with the first method, we obtain

$$A' = \{c\}\{-\}^{\mathbf{ct}}\{b\}\{-\}^{\mathbf{t}}\{a\}\{-\}^{\mathbf{at}}\{-\}^{\mathbf{f}}\{b\}.$$

However, with the second method, and with $\mu = 2.5$, we have

$$A' = \{c\}^{\mathbf{ct}}\{-\}\{-\}\{b\}^{\mathbf{t}}\{-\}\{a\}^{\mathbf{at}}\{-\}\{b\}^{\mathbf{f}}.$$

The main difference is that the classes b and a are now more distinct from the class c, i.e., the lowest classes are more penalized than with the first method.

Finally, let us discuss the behavior of this last method in our example. As seen before in Section 2.2, the "traditional" method (without weighting) returned the string

$$\{e\}^{\mathbf{t}}\{b,c,d,f\}^{\mathrm{JB[t]}}\{a\}^{\mathrm{IB[at,\,t]}}.$$

Now we apply the second strategy with increment $\sigma = 0.2$. The second parts are the same, since we obtain the same data:

$$\mu_{\mathbf{ct}} = \{1.0, 1.0, 1.0\}, \qquad \mu_{\mathbf{t}} = \{0.7, 0.9, 1.0\},$$

$$\mu_{\mathbf{at}} = \{0.2, 0{,}4, 0.6\}, \qquad \mu_{\mathbf{f}} = \{0.0, 0.0, 0.0\}.$$

We exchange the labels **f** and $\mu_{\mathbf{f}}$, together with the corresponding classes, obtaining:

$$A' = \{c\}^{\mu_{\mathrm{ct}}}\{-\}^{\mathbf{ct}}\{a,f\}^{\mu_{\mathrm{t}}}\{-\}^{\mathbf{t}}\{b,e\}^{\mu_{\mathrm{at}}}\{-\}^{\mathbf{at}}\{-\}^{\mu_{\mathrm{f}}}\{d\}^{\mathbf{f}}.$$

Computing $((A \vartriangle B) \vartriangle C) \vartriangle D$, for the first parts we have

$$\{-\}\{-\}\{-\}\{-\}\{-\}\{c\}\{e\}\{b,f\}\{a\}\{-\}\{d\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}$$

17 12 11 10 9 7 1

The complete string is

$$\{c\}^{\mathrm{NT[t]}}\{b,e,f\}^{\mathbf{t}}\{a\}^{\mathrm{JB[t]}}\{d\}^{\mathrm{IB[at,\,t]}}.$$

Now, let us apply the new approach, weighting the attribute A with a variable μ. Let us begin with $\mu = 1$; for the first parts we obtain

$$\{-\}\{-\}\{-\}\{-\}\{-\}\{c,e\}\{b\}\{d,f\}\{a\}\{-\}\{-\}\{-\}\{-\}\{-\}$$

14 9 8 7 6 1

In this case we obtain the same result, for which the complete string is

$$\{c,e\}^{\mathbf{t}}\{b,d,f\}^{\mathrm{JB[t]}}\{a\}^{\mathrm{IB[at,\,t]}}.$$

Assigning $\mu = 2$, let us execute the method again. For the first parts, the string is

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{c\}\{-\}\{e\}\{b,f\}\{a,d\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\\ 16 \qquad\qquad\quad 11 \quad 9 \quad 8 \quad 7 \qquad\qquad\qquad\quad 1\end{array}$$

whereas the complete string is

$$\{c\}^{\mathrm{NT[t]}}\{e\}^{\mathrm{t}}\{b,f,a,d\}^{\mathrm{JB[t]}}.$$

Thus, with $\mu = 2$, i.e., with a complexity less than that for the first method $[h(\mu) < n]$, we observe an attraction towards the highest position of the object c, the unique object which has the highest evaluation of the attribute A. More precisely, the label c received a greater value than with the previous method, even though this result is not evident because of the linguistic approximation.

Last, let us see what happens when $\mu = 2.5$. Considering the first parts, we have

$$\begin{array}{l}\{-\}\{-\}\{-\}\{-\}\{-\}\{c\}\{-\}\{e\}\{b,f\}\{a\}\{d\}\{-\}\{-\}\{-\}\{-\}\{-\}\{-\}\\ 17 \qquad\qquad\quad 12 \quad 10 \quad 9 \quad 8 \; 7 \qquad\qquad\qquad\quad 1\end{array}$$

whereas the complete string is

$$\{c\}^{\mathrm{NT[t]}}\{e\}^{\mathrm{t}}\{b,f,a\}^{\mathrm{JB[t]}}\{d\}^{\mathrm{IB[at,t]}}.$$

As expected, the result is essentially the same, always with the worsening for the lowest classes.

5. AN APPLICATION EXAMPLE

In this section we analyze the application of our classification methodology to financial investments. More precisely, we have interpreted some financial indicators in terms of fuzzy values and tested our mechanism by using data from a sample of firms whose securities are exchanged on the Boston Stock Exchange.

Security analysis models [18, 19] are based on the estimation of future dividends associated with securities. A traditional approach to the management of uncertainty and risk associated with investment processes is based on probability theory and on the task of identifying future revenues. The parameters of probability distribution may be computed by evaluating regression equations on historical performance data and other relevant factors. An alternative strategy consists in evaluating revenue probabilities conditioned to states of the world and weighting them with the probabilities of states themselves. The first method rests on a stability hypothesis about the trends of the variables considered. Such a hypothesis may be assumed with different degrees of confidence depending on both subjective (the analyst may have a greater or lesser propensity toward these methods) and objective (for instance, about the significance of regression results) considerations. On the other hand, the second method implies a partition into mutually exclusive scenarios which is impossible to define in a deterministic way (overlapping scenarios are always possible) and whose probabilities are in turn difficult to estimate. Due to the presence of such fuzziness, our approach can be useful in better serving the needs of financial analysts in stock selection and portfolio management.

Our case study has been developed using data available from the balance sheet information and the financial ratios about firms issuing securities. Having obtained this information, we provide a judgement on its reliability in terms of linguistic labels VH (very high), H (high), M (medium), L (low), treated operationally with fuzzy triangular numbers. For simplicity, our application example considers only four indicators for each company: return on investment, sales growth in last year, debt/equity ratio, and price per share. In Table 3 we show the values of indicators which we used to realize our classification.

Having analyzed these data, we provide a judgement expressed in terms of the linguistic labels. Table 4 reports this operation: now linguistic labels correspond to numeric values.

Now we can apply our fuzzy classification: the results appear in Table 5 and are expressed in terms of membership class (column 2), error confidence (column 3) and linguistic approximation (column 4). For this last information, we recall our notation, i.e., that the label JB means just before, the label IB means in between, and the label NT means next to (see Section 2.1.3).

This example shows how our classification methodology can be adopted for general domains. A brief comparison of our strategy with other traditional classification mechanisms can be found in [20].

6. CONCLUSIONS AND FURTHER WORK

The algebraic structure explained in this paper is a suitable tool for classification [21]. Special efforts have been made to augment the flexibility of the structure by defining a new weighting method. This new mechanism has solved a problem in a previous definition of the structure, which led to the inconveniences of nonassociativity and the noncommutativity. The enhancements introduced into the structure have dealt with the following problems:

Table 3.

Company name	Return on invested capital (%)	Sales growth last year (%)	Total debt-to-equity ratio	Recent price per share ($)
Advanced Deposition Tech Inc.	4.6	0.2	0.50	6.25
Ages Health Services Inc.	9.0	6.5	1.26	1.75
Amalgamated Automotive Ind.	0.1	2.9	0.99	5.51
American Natural Energy Co.	4.1	73.4	0.90	4.30
CAPX Corp.	6.4	291.5	0.00	2.50
Creative Technologies Corp.	80.3	136.7	1.00	6.25
CSL Lighting Manufacturing	43.8	18.1	0.92	4.75
Derma Sciences Inc.	24.1	25.6	0.02	5.00
DeWolfe Cos., Inc.	19.4	45.0	0.31	3.50
Encon Systems Inc.	12.3	380.5	0.36	5.13
Environment One Corp.	2.0	−7.6	0.43	2.25
Esquire Communications Ltd.	1.0	5.8	0.29	3.50
Exolon ESK Co.	0.1	−0.3	1.06	17.50
Interscience Computer Corp.	18.7	72.3	0.07	5.75
Manning (Greg) Auctions Inc.	9.3	31.0	0.00	2.63
Monaco Finance Inc.	5.6	34.2	0.00	8.25
MRV Communications, Inc.	14.5	67.9	0.01	9.13
Oliver Transportation, Inc.	3.8	−0.7	1.08	3.56
R2 Medical Systems, Inc.	2.9	7.9	1.00	5.00
Ride Snowboard Co.	36.0	2725.5	1.00	8.63
Skolniks Inc.	4.2	−2.6	0.06	3.56
Softpoint, Inc.	68.3	328.5	1.00	3.31
Transcor Waste Services, Inc.	6.7	41.2	1.02	2.38
Transworld Home Healthcare	8.2	51.3	0.31	8.25

6.1. Gathering of the Labels

Throughout the various examples discussed in this paper, we have noted a flattening of the new labels introduced in the algebraic structure. In fact, combining the attributes, we observed a concentration of the labels around an unspecified point in the interval [0, 1]. This phenomenon becomes more and more clear with the augmentation of the attributes involved.

The reason for this must be found in the definition of the algorithm for multiplication, and in the operations $\otimes$ and $\oplus$ (extended arithmetic mean) which compose the operation $\circ$ on the second parts. In fact, regarding the second parts, the first step consists in applying the product $(\mathbf{ct\,t\,at\,f})\circ(\mathbf{ct\,t\,at\,f})$, i.e., the set of the original labels. Analyzing the new labels, we note that those more central, such as γ_5, γ_4, and γ_3, are

Table 4.

Company Name	Return on invested capital (%)	Sales growth last year (%)	Total debt-to-equity ratio	Recent price per share ($)
Advanced Deposition Tech. Inc.	L	L	L	M
Ages Health Services Inc.	M	M	M	L
Amalgamated Automotive Ind.	L	M	L	M
American Natural Energy Co.	L	VH	L	L
CAPX Corp.	M	VH	L	L
Creative Technologies Corp.	VH	VH	M	M
CSL Lighting Manufacturing	VH	H	L	L
Derma Sciences Inc.	VH	VH	L	M
DeWolfe Cos., Inc.	H	VH	L	L
Encon Systems Inc.	H	VH	L	M
Environment One Corp.	L	L	L	L
Esquire Communications Ltd.	L	M	L	L
Exolon ESK Co.	L	L	M	H
Interscience Computer Corp.	H	VH	L	M
Manning (Greg) Auctions Inc.	M	VH	L	L
Monaco Finance Inc.	M	VH	L	M
MRV Communications, Inc.	H	VH	L	M
Oliver Transportation, Inc.	L	L	M	L
R2 Medical Systems, Inc.	L	M	M	M
Ride Snowboard Co.	VH	VH	M	M
Skolniks Inc.	L	L	L	L
Softpoint, Inc.	VH	VH	M	L
Transcor Waste Services, Inc.	M	VH	M	L
Transworld Home Healthcare	M	VH	L	M

obtained by averaging a larger number of elements (the original labels). This means that since the used operation is a mean and since the elements involved are taken from a limited set of labels, the results inevitably assume a unique mean value.

These same elements are handled successively in the applications of the operation $\circ$ and finally averaged with the original labels in such a way that the process of gathering of the labels (especially the central labels) is strongly focused on a certain "mean point." As a last step, on applying a linguistic approximation, all these labels will acquire a unique linguistic value.

As a last remark on this point, we observe that the choice of the original labels is meaningful for the above phenomena, but it is important to establish the "point of gathering." However, the gathering of the labels

Table 5.

Company name	Class	Errors	Labels
Advanced Deposition Tech. Inc.	8	0.77	JB[M]
Ages Health Services Inc.	3	0.66	[H]
Amalgamated Automotive Ind.	7	0.68	IB[VH]
American Natural Energy Co.	7	0.48	JB[H]
CAPX Corp.	3	0.46	[H]
Creative Technologies Corp.	1	−0.02	[VH]
CSL Lighting Manufacturing	2	0.38	NT[H]
Derma Sciences Inc.	1	0.15	[VH]
DeWolfe Cos., Inc.	2	0.38	NT[H]
Encon Systems Inc.	1	0.29	JB[VH]
Environment One Corp.	10	0.71	[L]
Esquire Communications Ltd.	7	0.77	[M]
Exolon ESK Co.	4	0.68	JB[H]
Interscience Computer Corp.	1	0.29	JB[VH]
Manning (Greg) Auctions Inc.	3	0.46	[H]
Monaco Finance Inc.	2	0.37	IB[H, VH]
MRV Communications, Inc.	1	0.29	JB[VH]
Oliver Transportation, Inc.	8	0.77	JB[M]
R2 Medical Systems, Inc.	3	0.66	[H]
Ride Snowboard Co.	1	−0.02	[VH]
Skolniks Inc.	10	0.71	[L]
Softpoint, Inc.	1	0.15	[VH]
Transcor Waste Services, Inc.	2	0.37	IB[H, VH]
Transworld Home Healthcare	2	0.37	IB[H, VH]

should not be sensitive to a wrong choice of the linguistic labels, so that the effect of the linguistic approximation must be very marginal. On the contrary, an important role is to be played by the definition of the mean operation within the operation $\circ$ among the labels and by the multiplication algorithm. These last two aspects need to be studied further. We are currently looking at a solution for which we use an ordinal scale for the labels in such a way as to better distribute them when the multiplication is executed. This approach does not affect the bases of our structure and seems to improve the performance.

6.2. The Weighting Strategy

The gathering of the labels affects the efficacy of the weighting mechanism presented in this work. In fact, for certain pairs of classes, we generate intermediate classes whose labels are intermediate with respect to the labels of the original classes. If the external labels are close, then the intermediate ones are still closer, with the effect of worsening the weighting mechanism.

In fact, the mechanism works well only for the first parts; the optimal behavior is not found for the second parts of the string, where we have the linguistic labels. Nevertheless, the mechanism works well globally. We argue that the reason for this anomaly is essentially the gathering of the labels. We are trying to enforce the weighting mechanism by reflecting the fact that the current mechanisms do not allow degrading of the objects which do not satisfy, or do not sufficiently satisfy, a weighted attribute.

6.3. The Concomitance of Attributes

Another important issue for a more complete and sound classification is the treatment of the concomitant presence of attributes. Rather than enter into the details of this discussion, we briefly introduce the problem with a simple example.

Let us suppose that we are treating the problem of determining the most appropriate diet for individuals who are potential candidates for coronopathy. In this case, attributes such as "diet very rich in fat," or "high consumption of alcohol" must be considered. Moreover, attention should be paid to other, less dangerous habits of such individuals. Furthermore, the simultaneous presence of "diet very rich in fat" and "high consumption of alcohol" elevates the risk of coronopathy even more. In such problems, an important role is played by the context in which we evaluate our considerations. For example, the risk of coronopathy for men 40–50 years old is judged in a different manner from that for women 20–25 years old. Thus, it is necessary to evaluate each relevant case to see the effect of concomitance on the problem. Furthermore, if we extend the problem of concomitance to more than two attributes, what are the decisions for such situations? It might be logical to apply our hypothetical operator "concomitance" to the first two, then multiply the results string by the third element, apply again the operator "concomitance" to the results, and lastly compute the possible increment or decrement due to the concomitance. This approach seems likely to have practical validity, since it takes into account the multiple presence of the attributes. In fact, our proposal essentially is based on the application of multiplication among the concomitant arguments, following the formulas provided in this paper.

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