Differential Geometry and its Applications 7 (1997) 1–28
North-Holland

The Darboux–Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach

Jan Cieśliński[1]

Filia Uniwersytetu Warszawskiego w Bialymstoku, Instytut Fizyki, ul. Lipowa 41, 15-424 Bialystok, Poland

Communicated by D. Krupka
Received 8 August 1994

Abstract: We show that theory of soliton surfaces, modified in an appropriate way, can be applied also to isothermic immersions in E^3. In this case the so called Sym's formula gives an explicit expression both for the isothermic immersion with prescribed fundamental forms and its Christoffel transform (dual surface). Then, applying methods of the theory of solitons, we construct the Darboux matrix for isothermic immersions reconstructing in this way the classical Darboux–Bianchi transformation for isothermic surfaces.

Keywords: Isothermic immersions, integrable geometries, soliton surfaces, Darboux transformation, Bäcklund transformation, Darboux matrix, Clifford algebras.

MS classification: 53A05, 35Q58, 15A66, 53A40, 35Q51.

1. Introduction

The connection between the classical differential geometry of surfaces and the theory of solitons is well known. The first and probably still the best example is constituted by pseudospherical surfaces in E^3 (i.e., surfaces of constant negative Gaussian curvature) which are in (almost) 1–1 correspondence with solutions of the Sine-Gordon Equation $\varphi_{,xt} = \sin\varphi$.

The transformation discovered by Bäcklund more than 100 years ago generates step by step a sequence of pseudospherical surfaces starting from any pseudospherical surface given in an explicit way. Bianchi proved the "permutability theorem" which states (roughly speaking) that the final result does not depend on the order of subsequent transformations.

On the level of the Sine-Gordon Equation this transformation "adds" some special solutions (*solitons*) onto a given background solution. Nonlinear partial differential equations studied in the theory of solitons (*integrable equations*, see [1,23]) admit, as a rule, a transformation of this kind known as the Bäcklund transformation.

The so called theory of soliton surfaces, proposed by Sym [21,22], unifies various integrable nonlinearities associating them with a specific class of surfaces. In the framework of this theory the classical Bianchi–Bäcklund transformation can be generalized in a natural way.

[1] The author was supported partially by KBN grants 2 0168 91 01 and 2 P03B 185 09. Main results of this paper were presented at the conference "Differentialgeometrie und Quantenphysik" of SFB 288 (Międzyzdroje, Poland, March 1994). *E-mail*: janek@beta.uw.bialystok.pl, janek@fuw.edu.pl.

PII S0926-2245(97)00002-8

It turns out that the soliton surfaces approach, modified in an appropriate way, can be applied also to the so called isothermic surfaces [14]. In the present paper we consider the classical Darboux–Bianchi transformation generating isothermic surfaces. The soliton surfaces approach enabled us to derive this transformation by a standard soliton technique: the construction of the so called Darboux matrix [17]. We applied a variant of the dressing method ([13, 18, 23]).

It should be pointed out that the nonlinear system (Gauss–Mainardi–Codazzi Equations) which describes isothermic immersions into E^3 has been proved to be integrable in the sense of the theory of solitons by another two independent methods: the Painlevé analysis [14] and the construction of the recursion operator [2].

2. Geometry of isothermic surfaces

Isothermic surfaces are distinguished by the fact that their curvature lines parameterized in a proper way form a conformal coordinate system. In other words, there exist local coordinates u, v in which fundamental forms read as follows:

$$I = e^{2\vartheta}(du^2 + dv^2), \qquad II = e^{2\vartheta}(k_2\, du^2 + k_1\, dv^2), \tag{1}$$

where $k_1 = k_1(u, v)$, $k_2 = k_2(u, v)$ are called principal curvatures and ϑ depends on u, v as well. Throughout this paper we consider isothermic immersions only locally: $(u, v) \in \Omega \subset \mathbb{R}^2$, where Ω is some open subset of $\mathbb{R}^2$, and we assume that the immersions have no umbilic points (i.e., $k_1 \neq k_2$).

The functions k_1, k_2 and ϑ have to satisfy the following system of nonlinear partial differential equations (Gauss–Mainardi–Codazzi Equations):

$$\vartheta_{,uu} + \vartheta_{,vv} + k_1 k_2 e^{2\vartheta} = 0, \tag{2a}$$

$$k_{1,u} + (k_1 - k_2)\vartheta_{,u} = 0, \tag{2b}$$

$$k_{2,v} + (k_2 - k_1)\vartheta_{,v} = 0. \tag{2c}$$

where comma denotes differentiation ($\vartheta_{,u} := \partial\vartheta/\partial u$ etc.).

There is also another (equivalent) characterization of isothermic surfaces. The surface is isothermic if and only if it admits an *infinitesimal* isometry preserving the mean curvature $H := \frac{1}{2}(k_1 + k_2)$. The class of isothermic surfaces contains among others: minimal surfaces and H-const surfaces. The immersions admitting *global* isometries preserving H are known as Bonnet surfaces (see [5, 8, 14]).

2.1. Christoffel transformation (dual surface)

Let $\mathbf{r} = \mathbf{r}(u, v) \in E^3$ be an isothermic surface with fundamental forms (1). It is possible to find by quadratures another isothermic surface $\bar{\mathbf{r}}$ in E^3 with fundamental forms given by

$$I = e^{-2\vartheta}(du^2 + dv^2), \qquad II = -k_2\, du^2 + k_1\, dv^2, \tag{3}$$

The surface $\tilde{\mathbf{r}}$ is known as the dual surface or Christoffel transform of $\mathbf{r}$ [3]. Its fundamental forms can be represented in the form (1), where ϑ, k_1, k_2 should be replaced by

$$\tilde{\vartheta} = -\vartheta, \qquad \tilde{k}_1 = e^{2\vartheta} k_1, \qquad \tilde{k}_2 = -e^{2\vartheta} k_2. \tag{4}$$

It can be easily seen that the Christoffel transform of $\tilde{\mathbf{r}}$ is determined by $\mathbf{r}$ modulo a rigid motion in E^3.

2.2. Classical Darboux–Bianchi transformation

The transformaton generating an infinite sequence of isothermic surfaces had been constructed in the end of XIX century. This transformation, resembling the Bäcklund transformation for pseudospherical surfaces, was studied by Bianchi and Darboux. We present its main properties following a paper of Bianchi [3].

Suppose that ϑ, k_1, k_2 define some isothermic surface (it means that they solve the system (2)). Let us introduce five auxiliary functions of u, v, namely λ, μ, ω, φ, σ, satisfying a linear system

$$\begin{aligned}
&\lambda_{,u} = -\vartheta_{,v}\mu - k_2 e^{\vartheta}\omega + m\sigma e^{\vartheta} + m e^{-\vartheta}\varphi, \\
&\mu_{,u} = \vartheta_{,v}\lambda, \qquad \varphi_{,u} = e^{\vartheta}\lambda, \qquad \omega_{,u} = k_2 e^{\vartheta}\lambda, \qquad \sigma_{,u} = e^{-\vartheta}\lambda, \\
&\mu_{,v} = -\vartheta_{,u}\lambda - k_1 e^{\vartheta}\omega + m\sigma e^{\vartheta} - m e^{-\vartheta}\varphi, \\
&\lambda_{,v} = \vartheta_{,u}\mu, \qquad \omega_{,v} = k_1 e^{\vartheta}\mu, \qquad \sigma_{,v} = -e^{-\vartheta}\mu, \qquad \varphi_{,v} = e^{\vartheta}\mu,
\end{aligned} \tag{5}$$

where m is a constant parameter.

One can easily check that the quadratic form $\lambda^2 + \mu^2 + \omega^2 - 2m\varphi\sigma$ does not depend on u, v [3]. It is convenient to choose

$$\lambda^2 + \mu^2 + \omega^2 = 2m\varphi\sigma. \tag{6}$$

The integrability conditions (assuring the existence of a non-trivial solution) for (5) are equivalent to the nonlinear system (2).

Theorem 1 (Darboux). *Let $\mathbf{r} = \mathbf{r}(u, v)$ be an isothermic surface in E^3 with fundamental forms given by* (1) *and $\mathbf{n} = \mathbf{n}(u, v)$ be the normal vector to this surface. Moreover, let λ, μ, ω, φ, σ solve the system* (5) *and satisfy the condition* (6). *Then*

$$\mathbf{r}' := \mathbf{r} - \frac{1}{m\sigma}(\lambda e^{-\vartheta}\mathbf{r}_{,u} + \mu e^{-\vartheta}\mathbf{r}_{,v} + \omega\mathbf{n}), \tag{7}$$

is an isothermic surface with fundamental forms given by

$$\begin{aligned}
I' &= \left(\frac{\varphi}{\sigma}\right)^2 e^{-2\vartheta}(du^2 + dv^2), \\
II' &= -\left(k_2\frac{\varphi}{\sigma} + \frac{\omega}{\sigma} + \frac{\omega\varphi}{\sigma^2}e^{-2\vartheta}\right)du^2 + \left(k_1\frac{\varphi}{\sigma} + \frac{\omega}{\sigma} - \frac{\omega\varphi}{\sigma^2}e^{-2\vartheta}\right)dv^2,
\end{aligned} \tag{8}$$

The above theorem gives an explicit expression for a new isothermic surface $\mathbf{r}'$ and, at the same time, a new solution ϑ', k_1', k_2' of the system (2):

$$e^{\vartheta'} = \pm\frac{\varphi}{\sigma}e^{-\vartheta}, \tag{9a}$$

$$k_1'e^{\vartheta'} = \pm\left(k_1 e^{\vartheta} + \frac{\omega}{\varphi}e^{\vartheta} - \frac{\omega}{\sigma}e^{-\vartheta}\right), \tag{9b}$$

$$k_2'e^{\vartheta'} = \mp\left(k_2 e^{\vartheta} + \frac{\omega}{\varphi}e^{\vartheta} + \frac{\omega}{\sigma}e^{-\vartheta}\right), \tag{9c}$$

where the upper (lower) sign corresponds to $m > 0$ ($m < 0$).

Bianchi proved the permutability theorem for the superposition of two such transformations [4].

3. Soliton approach to isothermic immersions

The aim of this paper is to reformulate the above classical theorem in the spirit of the theory of solitons. To be more specific, we are going to construct the so called Darboux matrix.

The starting point for this construction (and for most methods of the theory of solitons) is the so called linear problem: a system of linear partial differential equations which has to contain a parameter ("spectral parameter") [1,23]. The integrability conditions for this linear system have to be equivalent to the nonlinear system considered.

3.1. SO(4,1)-linear problem

The system (5) is a good candidate for the linear problem associated with the system (2). Assuming $m > 0$ we can change dependent variables as follows:

$$\tilde{\varphi} = \sqrt{\tfrac{1}{2}m}\,(\sigma - \varphi), \qquad \tilde{\sigma} = \sqrt{\tfrac{1}{2}m}\,(\sigma + \varphi), \tag{10}$$

and then, introducing a parameter ζ (the "spectral parameter"),

$$\zeta := \sqrt{2m}, \tag{11}$$

we obtain the following linear problem:

$$\psi_{,u} = \left(-\vartheta_{,v}\mathbf{f}_{12} - k_2 e^{\vartheta}\mathbf{f}_{13} + \zeta\sinh\vartheta\,\mathbf{f}_{14} + \zeta\cosh\vartheta\,\mathbf{f}_{15}\right)\psi, \tag{12a}$$

$$\psi_{,v} = \left(\vartheta_{,u}\mathbf{f}_{12} - k_1 e^{\vartheta}\mathbf{f}_{23} + \zeta\cosh\vartheta\,\mathbf{f}_{24} + \zeta\sinh\vartheta\,\mathbf{f}_{25}\right)\psi, \tag{12b}$$

where $\psi = (\lambda, \mu, \omega, \tilde{\varphi}, \tilde{\sigma}) \in \mathbb{R}^{4,1}$ while $\mathbf{f}_{ij}$ $(i \neq j)$ are matrices with coefficients

$$\begin{aligned}(\mathbf{f}_{ij})_{\alpha\beta} &= \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha} \qquad (\text{for } i < 5,\ j < 5),\\ (\mathbf{f}_{i5})_{\alpha\beta} &= \delta_{i\alpha}\delta_{5\beta} + \delta_{i\beta}\delta_{5\alpha},\end{aligned} \tag{13}$$

where $1 \leqslant \alpha, \beta \leqslant 4$ and δ_{jk} is Kronecker's delta. The matrices $\mathbf{f}_{ij}$ $(i < j \leqslant 5)$ form the standard basis of the Lie algebra $\mathbf{so}(4, 1)$.

Actually, ψ is a null vector in $\mathbb{R}^{4,1}$ (condition (6) assumes the form $\lambda^2+\mu^2+\omega^2+\tilde{\varphi}^2-\tilde{\sigma}^2=0$). However, from now on, we shall consider the linear problem (12) without this restriction. The integrability conditions for such linear problem are also identical with the nonlinear system (2).

The system (12) has 5 linearly independent vector solutions. The non-degenerate matrix whose columns are independent vector solutions of (12) obviously satisfies (12) as well and is known as the fundamental solution of (12).

3.2. *SP(1,1)-linear problem*

It is very convenient to take advantage of the isomorphism $\mathbf{so}(4,1)\cong\mathbf{sp}(1,1)$. We use its representation

$$\mathbf{f}_{jk} \quad\longleftrightarrow\quad \tfrac{1}{2}\mathbf{e}_{jk} := \tfrac{1}{2}\mathbf{e}_j\mathbf{e}_k. \tag{14}$$

where $\mathbf{e}_k$ are complex (in general) 4×4 matrices given by:

$$\mathbf{e}_1=\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0\end{pmatrix},\quad \mathbf{e}_2=\begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1\end{pmatrix},\quad \mathbf{e}_3=\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2\end{pmatrix},$$
$$\mathbf{e}_4=\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0\end{pmatrix},\quad \mathbf{e}_5=\begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3\end{pmatrix}. \tag{15}$$

and σ_k $(k=1,2,3)$ are standard Pauli matrices:

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{16}$$

In particular,

$$\mathbf{e}_{34}=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{pmatrix},\quad \mathbf{e}_{45}=-\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0\end{pmatrix}. \tag{17}$$

One can check by straightforward calculation that

$$\mathbf{e}_j\mathbf{e}_k+\mathbf{e}_k\mathbf{e}_j=2g_{jk}\mathbf{I}, \tag{18}$$

$$i\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5=\mathbf{I}, \tag{19}$$

$$\mathbf{e}_k^\dagger=\mathbf{e}_5\mathbf{e}_k\mathbf{e}_5, \tag{20}$$

$$\mathbf{e}_k^T=-\mathbf{e}_{34}\mathbf{e}_k\mathbf{e}_{34}, \tag{21}$$

where $j,k=1,\ldots,5$, g_{jk} are coefficients of the matrix $g=\mathrm{diag}(1,1,1,1,-1)$, $\mathbf{I}$ denotes the 4×4 identity matrix, superscript ^{T} means transposition and † means Hermitean conjugate. In particular,

$$\mathbf{e}_{jk}=-\mathbf{e}_{kj} \qquad (\text{for } k\neq j). \tag{22}$$

Moreover, to make our notation more concise, we rename the independent variables

$$u \equiv x^1, \qquad v \equiv x^2. \tag{23}$$

Using the isomorphism (14) and taking into account (18), (23) we can rewrite (12) in the form of the folowing 4×4 linear problem:

$$\Psi_{,1} = U_1 \Psi, \qquad \Psi_{,2} = U_2 \Psi, \tag{24a}$$

where Ψ is a non-degenerate complex 4×4 matrix, comma denotes differentiation ($\Psi_{,1} := \partial \Psi / \partial x^1$ etc.) and

$$U_1 = \tfrac{1}{2} \mathbf{e}_1 \big(- \vartheta_{,2} \mathbf{e}_2 - k_2 e^{\vartheta} \mathbf{e}_3 + \zeta \sinh \vartheta \; \mathbf{e}_4 + \zeta \cosh \vartheta \; \mathbf{e}_5 \big), \tag{24b}$$

$$U_2 = \tfrac{1}{2} \mathbf{e}_2 \big(- \vartheta_{,1} \mathbf{e}_1 - k_1 e^{\vartheta} \mathbf{e}_3 + \zeta \cosh \vartheta \mathbf{e}_4 + \zeta \sinh \vartheta \; \mathbf{e}_5 \big). \tag{24c}$$

Lemma 3.1. *The associative algebra of complex* 4×4 *matrices,* $M(4, \mathbb{C})$*, considered as the vector space over* $\mathbb{C}$*, is spanned by* 16 *matrices*: $\mathbf{I}$, $\mathbf{e}_k$ *and* $\mathbf{e}_{jk}$ $(j < k)$, *i.e.*,

$$M(4, \mathbb{C}) = \operatorname{span}\{\mathbf{I}, \mathbf{e}_k, \mathbf{e}_{jk}, 1 \leqslant j < k \leqslant 5\}. \tag{25}$$

Proof. $M(4, \mathbb{C})$ has dimension 16. Therefore it is enough to show that 16 matrices $\mathbf{I}$, $\mathbf{e}_k$, $\mathbf{e}_{jk}$ are linearly independent. Suppose that for some a, b^k, c^{jk}

$$a\mathbf{I} + b^k \mathbf{e}_k + c^{jk} \mathbf{e}_{jk} = 0, \tag{26}$$

where summation from 1 to 5 over repeating indices is assumed. Matrices $\mathbf{e}_j$ (15) are traceless. From (22) it follows also that $\operatorname{Tr}(\mathbf{e}_j \mathbf{e}_k) = -\operatorname{Tr}(\mathbf{e}_j \mathbf{e}_k) = 0$. Therefore, taking into account $\operatorname{Tr}(\mathbf{I}) = 4$, we have $a = 0$. By virtue of (19) and (18) the products of three or four pairwise different matrices $\mathbf{e}_k$ are traceless. Multiplying both sides of (26) by $\mathbf{e}_m$, applying (18), and computing the trace, we obtain $b^m = 0$. Similarly, multiplying both sides of (26) by $\mathbf{e}_{mn}$ and using (18) we can easily show that $c^{mn} = 0$. □

Identity (18) suggest the possibility to study our problem using Clifford algebra $\mathcal{C}(4, 1)$ generated by elements $\mathbf{e}_k$. Actually the condition (19) means that $i\mathbf{e}_k$ $(k = 1, \dots, 5)$ generate $\mathcal{C}(1, 4)^+$, the subalgebra of even elements of the Clifford algebra $\mathcal{C}(1, 4)$ (we recall that even elements are invariant under the authomorphism $\mathbf{e}_k \mapsto -\mathbf{e}_k$, $k = 1, \dots, 5$).

3.3. Group interpretation of the spectral parameter

The algebra of Lie point symmetries of the non-linear system (2) can be computed in the standard way [19]. Not entering into details we present the final result. The Lie algebra is 4-dimensional and spanned by

$$\begin{aligned} \mathbf{v}_1 &= x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - k_1 \frac{\partial}{\partial k_1} - k_2 \frac{\partial}{\partial k_2}, \qquad & \mathbf{v}_3 &= \frac{\partial}{\partial x^1}, \\ \mathbf{v}_2 &= x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - \frac{\partial}{\partial \vartheta}, & \mathbf{v}_4 &= \frac{\partial}{\partial x^2}. \end{aligned} \tag{27}$$

Let us consider the linear problem obtained from (24) by substituting $\zeta = 1$. This is a non-parametric linear problem, i.e., a linear problem without spectral parameter. The algebra of Lie point symmetries of this non-parametric linear problem (for definition see [10]) is spanned by $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. In particular, the symmetry corresponding to the vector field $\mathbf{v}_2$ is given by

$$(\Psi, \vartheta, k_1, k_2) \mapsto (e^{\frac{1}{2}t\mathbf{e}_{45}}\Psi,\ \vartheta - t,\ k_1 e^t,\ k_2 e^t), \tag{28}$$

where t is the group parameter.

The action of the one-parameter group $\exp(k\,\mathbf{v}_1)$ transforms the non-parametric linear problem into (24), where $\zeta = \exp(k)$. Thus the spectral parameter is closely related to the group parameter. We point out that it is not possible to apply an analogical procedure to the non-parametric linear problem obtained by substitution $\zeta = 0$. Every Lie point symmetry of (2) is a point symmetry of such a non-parametric linear problem.

3.4. Discrete symmetries of the linear problem

The group of all point symmetries (including discrete ones) of a given system of differential equations is much more difficult to obtain than one-parameter groups of symmetries. However, one usually can easily discover some special discrete symmetries. For instance, $u \mapsto -u$, $v \mapsto -v$, $(k_1, k_2) \mapsto (-k_1, -k_2)$ are obvious symmetries of the system (2). The Christoffel transformation is also a discrete symmetry of the system (2).

Let us consider discrete point symmetries of the linear problem (24), i.e., compositions of the constant gauge transformation

$$\Psi \mapsto G_0\Psi, \qquad U_k \mapsto G_0 U_k G_0^{-1}$$

($G_0 = \text{const}$ and $k = 1, 2$) with a discrete transformation in the space of all variables and parameters: u, v, ϑ, k_1, k_2, ζ. The following symmetries correspond to some simple choices of G_0:

$$(\Psi, u) \mapsto (\mathbf{e}_1\Psi, -u), \tag{29a}$$

$$(\Psi, v) \mapsto (\mathbf{e}_2\Psi, -v), \tag{29b}$$

$$(\Psi, k_1, k_2) \mapsto (\mathbf{e}_3\Psi, -k_1, -k_2), \tag{29c}$$

$$(\Psi, \zeta) \mapsto (\mathbf{e}_{45}\Psi, -\zeta), \tag{29d}$$

$$(\Psi, \vartheta, k_1, k_2) \mapsto (\mathbf{e}_{15}\Psi, -\vartheta,\ k_1 e^{2\vartheta},\ -k_2 e^{2\vartheta}), \tag{29e}$$

$$(\Psi, \vartheta, k_1, k_2) \mapsto (\mathbf{e}_{24}\Psi, -\vartheta,\ -k_1 e^{2\vartheta},\ k_2 e^{2\vartheta}). \tag{29f}$$

In particular, from (29ef) it follows that Christoffel transformation is a discrete symmetry of the linear problem (24).

3.5. Algebraic representation of the linear problem

In general, Ψ satisfying (24) is a $GL(4, \mathbb{C})$-valued function of x^1, x^2 and ζ. The form of the linear problem (24) implies that Ψ can be restricted to a subgroup of $GL(4, \mathbb{C})$ (the so called "reduction group," see [18]). For example:

Remark 3.2. The matrices U_1 and U_2 are traceless, $\mathrm{Tr}(U_k) = 0$. Therefore Ψ can be restricted to the subgroup $SL(4, \mathbb{C})$:

$$\det \Psi = 1, \tag{30}$$

i.e., if initial conditions $\Psi(x_0^1, x_0^2; \zeta)$ satisfy (30) then the corresponding solution $\Psi(x^1, x^2; \zeta)$ satisfies (30) as well.

Below we present more constraints restricting Ψ to some subgroups. Actually, we follow even more general approach (see [11,12]). We are going to show that the linear problem (24) is a unique consequence of a system of some algebraic (i.e., non-differential) constraints imposed on matrices U_k.

Theorem 2. *The linear problem* (24) *is* equivalent *to the following system of algebraic conditions on matrices U_1 and U_2*:

$$U_k = u_{k0} + u_{k1}\zeta, \tag{31a}$$

$$U_k(\zeta) = \mathbf{e}_{34}\, U_k^T(\zeta)\, \mathbf{e}_{34}, \tag{31b}$$

$$U_k(\zeta) = \mathbf{e}_5\, U_k^\dagger(\bar{\zeta})\, \mathbf{e}_5, \tag{31c}$$

$$U_k(\zeta) = \mathbf{e}_{45}\, U_k(-\zeta)\, \mathbf{e}_{45}\ , \tag{31d}$$

$$U_k \mathbf{e}_k + \mathbf{e}_k U_k = 0, \tag{31e}$$

$$\mathrm{Tr}(u_{k1}^2) = (-1)^{k+1}, \tag{31f}$$

$$\mathrm{Tr}(u_{11}\mathbf{e}_{15}) > 0, \tag{31g}$$

$$\mathrm{Tr}(u_{21}\mathbf{e}_{24}) < 0, \tag{31h}$$

where $k = 1, 2$ *and* $u_{10}, u_{11}, u_{20}, u_{21}$ *are* 4×4 *complex matrices which do not depend on* ζ.

Proof. One can easily check that the matrices U_1, U_2 given by (24bc) satisfy conditions (31). We are going to prove that the form (24bc) of U_1, U_2 is a unique consequence of (31). From Lemma 3.1 we have $U_m = \alpha_m \mathbf{I} + \beta_m^k \mathbf{e}_k + \gamma_m^{ij} \mathbf{e}_{ij}$, where $\alpha_m, \beta_m^k, \gamma_m^{ij}$ $(m = 1, 2)$ are some functions depending on x^1, x^2, ζ. We assume summation over k (from 1 to 5) and summation over i, j $(1 \leqslant i < j \leqslant 5)$. Taking into account (20) and (21) we rewrite (31bc) as

$$\begin{aligned} \alpha_m(\zeta)\mathbf{I} + \beta_m^k(\zeta)\mathbf{e}_k + \gamma_m^{ij}(\zeta)\mathbf{e}_{ij} &= -\alpha_m(\zeta)\mathbf{I} - \beta_m^k(\zeta)\mathbf{e}_k + \gamma_m^{ij}(\zeta)\mathbf{e}_{ij}, \\ \alpha_m(\zeta)\mathbf{I} + \beta_m^k(\zeta)\mathbf{e}_k + \gamma_m^{ij}(\zeta)\mathbf{e}_{ij} &= -\overline{\alpha_m(\bar{\zeta})}\mathbf{I} + \overline{\beta_m^k(\bar{\zeta})}\mathbf{e}_k + \overline{\gamma_m^{ij}(\bar{\zeta})}\mathbf{e}_{ij}. \end{aligned} \tag{32}$$

Therefore the constraint (31c) means that the coefficients of U_m with respect to the basis (25) are analytic functions of ζ with real coefficients while the constraint (31b) means that U_1 and U_2 are linear combinations of $\mathbf{e}_{ij}$ only. Then the conditions (31e) imply that U_k $(k = 1, 2)$ is a combination of $\mathbf{e}_{kj}$ $(j = 1, \ldots, 5)$ only.

Taking into account (31a) we can rewrite (31d) as $[u_{k0}, \mathbf{e}_{45}] = 0 = u_{k1}\mathbf{e}_{45} + \mathbf{e}_{45}u_{k1}$. Thus we obtain

$$\begin{aligned} U_1 &= a_2\mathbf{e}_{12} + a_3\mathbf{e}_{13} + (a_4\mathbf{e}_{14} + a_5\mathbf{e}_{15})\zeta, \\ U_2 &= b_1\mathbf{e}_{21} + b_3\mathbf{e}_{23} + (b_4\mathbf{e}_{24} + b_5\mathbf{e}_{25})\zeta, \end{aligned} \tag{33}$$

where a_j, b_j are some functions of x^1, x^2. Then

$$u_{11}^2 = a_4^2 \mathbf{e}_{14}^2 + a_4 a_5 (\mathbf{e}_{14}\mathbf{e}_{15} + \mathbf{e}_{15}\mathbf{e}_{14}) + a_5^2 \mathbf{e}_{15}^2 = (a_5^2 - a_4^2)\mathbf{I},$$
$$u_{21}^2 = b_4^2 \mathbf{e}_{24}^2 + b_4 b_5 (\mathbf{e}_{24}\mathbf{e}_{25} + \mathbf{e}_{25}\mathbf{e}_{24}) + b_5^2 \mathbf{e}_{25}^2 = (b_5^2 - b_4^2)\mathbf{I}.$$

where we used (18). Therefore, taking into account $\mathrm{Tr}(\mathbf{I}) = 4$, we can rewrite (31f) as $4(a_5^2 - a_4^2) = 1$ and $4(b_4^2 - b_5^2) = 1$, which is equivalent to

$$a_4 = \tfrac{1}{2}\varepsilon \sinh\vartheta, \quad a_5 = \tfrac{1}{2}\varepsilon\cosh\vartheta, \quad b_4 = \tfrac{1}{2}\varepsilon'\cosh\vartheta', \quad b_5 = \tfrac{1}{2}\varepsilon'\sinh\vartheta',$$

where $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$ and ϑ, ϑ' are some functions of x^1, x^2. The conditions (31gh) mean that $a_5 > 0$ and $b_4 > 0$, i.e., $\varepsilon = \varepsilon' = 1$.

The compatibility conditions $U_{1,2} - U_{2,1} + [U_1, U_2] = 0$ are quadratic with respect to ζ. Equating to 0 the coefficients we obtain the following system:

$$\begin{aligned}
&\tfrac{1}{2}\mathbf{e}_{12}\sinh(\vartheta - \vartheta') = 0,\\
&(\tfrac{1}{2}\vartheta_{,2}\cosh\vartheta + a_2\cosh\vartheta')\mathbf{e}_{14} + (\tfrac{1}{2}\vartheta_{,2}\sinh\vartheta + a_2\sinh\vartheta')\mathbf{e}_{15}\\
&\qquad - (\tfrac{1}{2}\vartheta'_{,1}\sinh\vartheta' + b_1\sinh\vartheta)\mathbf{e}_{24} - (\tfrac{1}{2}\vartheta'_{,1}\cosh\vartheta' + b_1\cosh\vartheta)\mathbf{e}_{25} = 0,\\
&(a_{2,2} + b_{1,1} - 2a_3 b_3)\mathbf{e}_{12} + (a_{3,2} + 2a_2 b_3)\mathbf{e}_{13} - (b_{3,1} + 2b_1 a_3)\mathbf{e}_{23} = 0.
\end{aligned}$$

Therefore $\vartheta' = \vartheta$ and the above system reduces to

$$a_2 = -\tfrac{1}{2}\vartheta_{,2}, \quad b_1 = -\tfrac{1}{2}\vartheta_{,1}, \tag{34a}$$
$$a_{3,2} - b_3\vartheta_{,2} = 0, \quad b_{3,1} - a_3\vartheta_{,1} = 0, \quad \vartheta_{,11} + \vartheta_{,22} + 4a_3 b_3 = 0. \tag{34b}$$

Moreover,

$$a_4 = b_5 = \tfrac{1}{2}\sinh\vartheta, \quad a_5 = b_4 = \tfrac{1}{2}\cosh\vartheta. \tag{34c}$$

Then it is convenient to express a_3, b_3 in terms of new variables k_1, k_2:

$$a_3 =: -\tfrac{1}{2}k_2\exp\vartheta, \quad b_3 =: -\tfrac{1}{2}k_1\exp\vartheta. \tag{34d}$$

The nonlinear system (34b) obtained this way is identical exactly with the system (2) while the remaining equations (34acd) transform (33) into the linear problem (24). □

Lemma 3.3. *If U_1, U_2 satisfy* (31bcd) *then Ψ can be restricted to the group* $\mathbf{G}$ *defined by*

$$\Psi^T(\zeta)\,\mathbf{e}_{34}\,\Psi(\zeta) = \mathbf{e}_{34}, \tag{35a}$$
$$\Psi^\dagger(\bar{\zeta})\,\mathbf{e}_5\,\Psi(\zeta) = \mathbf{e}_5, \tag{35b}$$
$$\Psi(-\zeta) = \mathbf{e}_{45}\,\Psi(\zeta)\,\mathbf{e}_{45}, \tag{35c}$$

i.e., if initial conditions $\Psi(x_0^1, x_0^2; \zeta)$ satisfy (35) *then the corresponding solution $\Psi(x^1, x^2; \zeta)$ satisfies* (35) *as well.*

Proof. The function Ψ is, by definition, a fundamental solution of (24). Therefore any other solution Ψ' of (24), being a combination of the same vector solutions, is given by $\Psi' = \Psi C$, where C is a non-singular constant matrix.

Let us substitute (31b) into (24a) and transpose both sides of the obtained equation (one should remember that $\mathbf{e}_{34}^T = -\mathbf{e}_{34}$ and $\mathbf{e}_{34}^{-1} = -\mathbf{e}_{34}$):

$$\Psi^T{}_{,k} = \Psi^T \mathbf{e}_{34} U_k \mathbf{e}_{34},$$

which is equivalent to

$$(\mathbf{e}_{34}\Psi^{-1}\mathbf{e}_{34})^T{}_{,k} = U_k(\mathbf{e}_{34}\Psi^{-1}\mathbf{e}_{34})^T.$$

Therefore $(\mathbf{e}_{34}\Psi^{-1}\mathbf{e}_{34})^T = \Psi C$. Choosing the initial conditions such that $C = -1$ we obtain (35a). Similarly, substituting (31c) into (24a) and taking Hermitean conjugate of both sides we obtain $\Psi^\dagger(\zeta)_{,k} = \Psi^\dagger(\zeta)\mathbf{e}_5 U_k(\bar\zeta)\mathbf{e}_5$ (remember that $\mathbf{e}_5^\dagger = -\mathbf{e}_5,\ \mathbf{e}_5^{-1} = -\mathbf{e}_5$). Therefore

$$(\mathbf{e}_5\Psi^{-1}(\zeta)\mathbf{e}_5)^\dagger{}_{,k} = U_k(\bar\zeta)(\mathbf{e}_5\Psi^{-1}(\zeta)\mathbf{e}_5)^\dagger,$$

and we have $(\mathbf{e}_5\Psi^{-1}(\zeta)\mathbf{e}_5)^\dagger = \Psi(\bar\zeta)C$, which (for $C = -1$) is equivalent to (35b). Finally, substituting (31d) into (24a) we obtain $\Psi(\zeta)_{,k} = \mathbf{e}_{45}U_k(-\zeta)\mathbf{e}_{45}\Psi(\zeta)$, or (because $\mathbf{e}_{45}^{-1} = \mathbf{e}_{45}$)

$$(\mathbf{e}_{45}\Psi(\zeta)\mathbf{e}_{45})_{,k} = U_k(-\zeta)(\mathbf{e}_{45}\Psi(\zeta)\mathbf{e}_{45}),$$

which implies $\mathbf{e}_{45}\Psi(\zeta)\mathbf{e}_{45} = \Psi(-\zeta)C$ and, for $C = 1$, we obtain (35c). □

3.6. Some properties of the group SP(1,1)

If $\mathbf{e}_{34}$ and $\mathbf{e}_5$ are explicitly given by (15) and (17) then the conditions (35ab) restricted to $\zeta \in \mathbb{R}$ define the standard matrix representation of the group $SP(1,1)$. Let $\mathbf{H}$ be the subgroup of $SP(1,1)$ containing the elements commuting with $\mathbf{e}_{45}$, i.e., $h \in \mathbf{H}$ iff

$$h^\dagger \mathbf{e}_5 h = \mathbf{e}_5, \qquad h^T \mathbf{e}_{34} h = \mathbf{e}_{34}, \qquad h\mathbf{e}_{45} = \mathbf{e}_{45}h. \tag{36}$$

Lemma 3.4. *The elements of the subgroup* $\mathbf{H}$ *can be parameterized as follows*:

$$h = (n_0\mathbf{I} + n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k})(\mathbf{I}\cosh\gamma + \mathbf{e}_{45}\sinh\gamma), \tag{37a}$$

where γ *and* n_k $(k = 0, 1, 2, 3)$ *are real parameters satisfying*

$$\sum_{k=0}^{3} n_k^2 = 1, \tag{37b}$$

while $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *are given by*

$$\mathbf{i} = \mathbf{e}_{32}, \qquad \mathbf{j} = \mathbf{e}_{13}, \qquad \mathbf{k} = \mathbf{e}_{21} \tag{37c}$$

(*the notation points out that* $\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ *can be interpreted as quaternions*).

Proof. By virtue of Lemma 3.1 we have $h = a\mathbf{I} + ib_k\mathbf{e}_k + c_{jk}\mathbf{e}_{jk}$, where a, b_k, c_{jk} $(j, k = 1, \ldots, 5,\ j < k)$ are complex parameters and summation over repeating indices is assumed. Using (20) and (21) we obtain easily that

$$h^\dagger = -\mathbf{e}_5(\bar a\,\mathbf{I} + i\bar b_k\mathbf{e}_k - \bar c_{jk}\mathbf{e}_{jk})\mathbf{e}_5,$$
$$h^T = -\mathbf{e}_{34}(a\mathbf{I} + ib_k\mathbf{e}_k - c_{jk}\mathbf{e}_{jk})\mathbf{e}_{34}.$$

Therefore (36) can be written as

$$(\bar{a}\,\mathbf{I} + i\bar{b}_k\mathbf{e}_k - \bar{c}_{jk}\mathbf{e}_{jk})(a\mathbf{I} + ib_k\mathbf{e}_k + c_{jk}\mathbf{e}_{jk}) = \mathbf{I}, \tag{38a}$$

$$(a\mathbf{I} + ib_k\mathbf{e}_k - c_{jk}\mathbf{e}_{jk})(a\mathbf{I} + ib_k\mathbf{e}_k + c_{jk}\mathbf{e}_{jk}) = \mathbf{I}, \tag{38b}$$

$$[ib_k\mathbf{e}_k + c_{jk}\mathbf{e}_{jk}, \mathbf{e}_{45}] = 0. \tag{38c}$$

Comparing (38a) and (38b) we obtain immediately that all coefficients a, b_k, c_{jk} are real. The condition (38c) implies $b_\mu = c_{s\mu} = 0$ for $\mu = 4, 5$ and $s = 1, 2, 3$. Thus,

$$h = a\mathbf{I} + i(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) + c_{12}\mathbf{e}_{12} + c_{13}\mathbf{e}_{13} + c_{23}\mathbf{e}_{23} + c_{45}\mathbf{e}_{45}. \tag{39}$$

Then the remaining constraint (38b) assumes the form

$$\begin{aligned}(a^2 - b_1^2 - b_2^2 - b_3^2 - c_{45}^2 + c_{12}^2 + c_{13}^2 + c_{23}^2)\mathbf{I} + 2i(c_{23}c_{45}\mathbf{e}_1 - c_{13}c_{45}\mathbf{e}_2 + c_{12}c_{45}\mathbf{e}_3)\\ + 2ib_1(c_{12}\mathbf{e}_2 + c_{13}\mathbf{e}_3) + 2ib_2(-c_{12}\mathbf{e}_1 + c_{23}\mathbf{e}_3) + 2ib_3(-c_{13}\mathbf{e}_1 - c_{23}\mathbf{e}_2) = \mathbf{I},\end{aligned}$$

where we took into account (19). Hence,

$$\begin{aligned}&ab_1 + c_{23}c_{45} - b_2c_{12} - b_3c_{13} = 0,\\ &ab_2 - c_{13}c_{45} - b_3c_{23} + b_1c_{12} = 0,\\ &ab_3 + c_{12}c_{45} + b_1c_{13} + b_2c_{23} = 0,\\ &a^2 - (b_1^2 + b_2^2 + b_3^2) - c_{45}^2 + c_{12}^2 + c_{13}^2 + c_{23}^2 = 1.\end{aligned}$$

This system can be put into the following compact form:

$$a\mathbf{b} + c_{45}\mathbf{c} = \mathbf{b} \times \mathbf{c}, \tag{40a}$$

$$1 + c_{45}^2 + \mathbf{b}\cdot\mathbf{b} = a^2 + \mathbf{c}\cdot\mathbf{c}, \tag{40b}$$

where $\mathbf{b} := (b_1, b_2, b_3)$ and $\mathbf{c} := (c_{23}, -c_{13}, c_{12})$. The cross means the vector product in E^3 while the center dot denotes the scalar product. To solve the system (40) it is enough to recall that the product $\mathbf{b}\times\mathbf{c}$ has to be orthogonal both to $\mathbf{b}$ and $\mathbf{c}$. Thus the condition (40a) means that $\mathbf{b}\times\mathbf{c} = 0$, i.e., $\mathbf{b}$ and $\mathbf{c}$ are colinear. What is more,

$$\mathbf{b} = c_{45}\mathbf{w}, \qquad \mathbf{c} = -a\mathbf{w}, \tag{41}$$

where $\mathbf{w} = (w_1, w_2, w_3) \in E^3$. Then, taking also into account (19), we may write (39) as

$$h = (a\mathbf{I} + c_{45}\mathbf{e}_{45})(\mathbf{I} + w_1\mathbf{e}_{32} + w_2\mathbf{e}_{13} + w_3\mathbf{e}_{21}), \tag{42}$$

and (40b) assumes the form $1 = (a^2 - c_{45}^2)(1 + \mathbf{w}\cdot\mathbf{w})$, or, which is equivalent,

$$a = \varepsilon(1 + \mathbf{w}\cdot\mathbf{w})^{-1/2}\cosh\gamma, \qquad c_{45} = \varepsilon(1 + \mathbf{w}\cdot\mathbf{w})^{-1/2}\sinh\gamma \tag{43}$$

where $\varepsilon = \pm 1$ and $\gamma \in \mathbb{R}$. To complete the proof we define

$$n_k := \varepsilon w_k(1 + \mathbf{w}\cdot\mathbf{w})^{-1/2}, \tag{44}$$

where $k = 0, 1, 2, 3$ and $w_0 := 1$ (note that the condition (37b) is obviously satisfied) and substitute (43) and (44) into (42). □

Let us consider the subgroup $\mathbf{H}_0 \subset \mathbf{H}$ which consists of elements commuting with both $\mathbf{e}_4$ and $\mathbf{e}_5$. The elements of $\mathbf{H}_0$ are given by

$$h = (n_0\mathbf{I} + n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}), \tag{45}$$

where n_k $(k = 0, 1, 2, 3)$ satisfy (37b). The group $\mathbf{H}_0$ is isomorphic to $SU(2)$.

Substituting $\zeta = 0$ into (35) we immediately obtain

$$\Psi(0) := \Psi(x^1, x^2; 0) \in \mathbf{H}. \tag{46}$$

Actually, for $\zeta = 0$ the linear problem (24) does not contain neither $\mathbf{e}_4$ nor $\mathbf{e}_5$. One can easily prove that there exists $\gamma_0 = \text{const}$ such that

$$\Psi(0)e^{-\gamma_0\mathbf{e}_{45}} \in \mathbf{H}_0. \tag{47}$$

In other words, $\Psi(0)$ is of the form (37) but with $\gamma = 0$. It is worth pointing out that the parameter γ_0 can be inserted into $\Psi(0) \in \mathbf{H}_0$ by the Lie point symmetry (28) with $t = 2\gamma_0$.

It will be convenient to consider also the subsets $\mathbf{H}_{\mu\nu}$ $(\mu, \nu = \pm 1)$ of $GL(4,\mathrm{C})$ defined as follows: $y \in \mathbf{H}_{\mu\nu}$ iff

$$y^\dagger \mathbf{e}_5 y = \mu\mathbf{e}_5, \qquad y^T\mathbf{e}_{34}y = \mathbf{e}_{34}, \qquad y\mathbf{e}_{45} = \nu\mathbf{e}_{45}y. \tag{48}$$

The conditions defining $\mathbf{H}_{\mu\nu}$ are very similar to (36). Actually, parameterizing the elements of $\mathbf{H}_{\mu\nu}$ by $y = y_0 h$, where y_0 is a fixed element of $\mathbf{H}_{\mu\nu}$, we immediately obtain that h has to satisfy (36). Therefore,

$$\mathbf{H}_{++} = \mathbf{H}, \quad \mathbf{H}_{+-} = i\mathbf{e}_{45}\mathbf{H}, \quad \mathbf{H}_{-+} = i\mathbf{e}_5\mathbf{H}, \quad \mathbf{H}_{--} = \mathbf{e}_4\mathbf{H}, \tag{49}$$

where $i\mathbf{e}_{45}$, $i\mathbf{e}_5$ and $\mathbf{e}_4$ are chosen elements of $\mathbf{H}_{+-}$, $\mathbf{H}_{-+}$ and $\mathbf{H}_{--}$ respectively.

Let us denote inner automorphisms of $M(4, \mathbb{C})$ by φ_h:

$$\varphi_h(g) := h^{-1}gh. \tag{50}$$

It is not difficult to obtain in the straightforward way explicit formulas for the action of the operator φ_h for $h \in \mathbf{H}$. First of all let us notice that the inverse of h (see (37)) is given by

$$h^{-1} = (n_0\mathbf{I} - n_1\mathbf{i} - n_2\mathbf{j} - n_3\mathbf{k})(\mathbf{I}\cosh\gamma - \mathbf{e}_{45}\sinh\gamma).$$

Using (18) and taking into account that $\mathbf{e}_{45}$ commutes with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ commute with $\mathbf{e}_4, \mathbf{e}_5$ we have

$$h^{-1}\mathbf{e}_1 h = (n_0^2 + n_1^2 - n_2^2 - n_3^2)\mathbf{e}_1 + 2(n_1n_2 - n_0n_3)\mathbf{e}_2 + 2(n_1n_3 + n_0n_2)\mathbf{e}_3. \tag{51a}$$

Analogical formulas for $\mathbf{e}_2$ and $\mathbf{e}_3$ are given by cyclic permutations $1 \to 2 \to 3 \to 1$. The triplet $\mathbf{i}, \mathbf{j}, \mathbf{k}$ transforms identically as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$h^{-1}\mathbf{i}h = (n_0^2 + n_1^2 - n_2^2 - n_3^2)\mathbf{i} + 2(n_1n_2 - n_0n_3)\mathbf{j} + 2(n_1n_3 + n_0n_2)\mathbf{k} \tag{51b}$$

and similar equations for $\mathbf{j}$ and $\mathbf{k}$. Moreover,

$$\begin{aligned} h^{-1}\mathbf{e}_4 h &= \mathbf{e}_4\cosh(2\gamma) + \mathbf{e}_5\sinh(2\gamma), \\ h^{-1}\mathbf{e}_5 h &= \mathbf{e}_5\cosh(2\gamma) + \mathbf{e}_4\sinh(2\gamma), \\ h^{-1}\mathbf{e}_{45} h &= \mathbf{e}_{45}. \end{aligned} \tag{51c}$$

The transformation of $\mathbf{e}_{s\mu}$ ($s = 1, 2, 3$ and $\mu = 4, 5$) is given by appropriate products of (51a) and (51c).

Therefore, we conclude that the operator φ_h ($h \in \mathbf{H}$) has the following invariant spaces: span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, span$\{\mathbf{e}_4, \mathbf{e}_5\}$, span$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, span$\{\mathbf{e}_{45}\}$ and

$$V := \operatorname{span}\{\mathbf{e}_{s\mu} \mid s = 1, 2, 3,\ \mu = 4, 5\}, \tag{52}$$

where the notation span$\{v_1, v_2, \ldots, v_n\}$ means the linear space spanned by $v_1, v_2, \ldots, v_n$. The linear space V is the sum of two null subspaces, $V = V_+ \oplus V_-$, where

$$\begin{aligned} V_+ &:= (\mathbf{I} - \mathbf{e}_{45})V = \operatorname{span}\{\mathbf{e}_s(\mathbf{e}_4 + \mathbf{e}_5) \mid s = 1, 2, 3\}, \\ V_- &:= (\mathbf{I} + \mathbf{e}_{45})V = \operatorname{span}\{\mathbf{e}_s(\mathbf{e}_4 - \mathbf{e}_5) \mid s = 1, 2, 3\}. \end{aligned} \tag{53}$$

Both V_+ and V_- can be identified with span$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \simeq \mathbf{su}(2)$ equipped with the natural structure of the Euclidean space E^3. The scalar and vector (skew) products are given respectively by

$$\begin{aligned} (a|b) &:= -4\,\mathrm{Tr}(\pi_\pm(a)\pi_\pm(b)), \\ a \times b &:= \pi_\pm^{-1}\left([\pi_\pm(a), \pi_\pm(b)]\right). \end{aligned} \tag{54}$$

where $a, b \in V_\pm$ and π_+ (π_-) projects V_+ (V_-) onto span$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in the following way:

$$\pi_\pm\Big(\tfrac{1}{4}\sum_{k=1}^{3} a_k(\mathbf{e}_{k4} \pm \mathbf{e}_{k5})\Big) := a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

In other words, we assume that $\frac{1}{4}\mathbf{e}_k(\mathbf{e}_4 \pm \mathbf{e}_5)$ ($k = 1, 2, 3$) forms an orthonormal basis in E^3.

The elements of the subgroup $\mathbf{H}_0$ commute with both $\mathbf{e}_4$ and $\mathbf{e}_5$, which implies

$$\pi_\pm(h^{-1}ah) = h^{-1}\pi_\pm(a)h, \tag{55}$$

for $h \in \mathbf{H}_0$ and $a \in V_\pm$. In particular, the bases $\mathbf{e}_s(\mathbf{e}_4 \pm \mathbf{e}_5)$ ($s = 1, 2, 3$) transform in an identical way as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (see (51a)). Thus

Corollary 3.5. *The operator φ_h is for $h \in \mathbf{H}_0$ an orthogonal transformation (rotation) both in $V_+ \simeq E^3$ and in $V_- \simeq E^3$.*

3.7. Spectral description of isothermic surfaces

The fundamental forms define a surface uniquely (modulo a rigid motion). However, it is usually very difficult to recover the explicit expression for the position vector to the surface from the knowledge of its fundamental forms.

The existence of an associated linear problem with the spectral parameter can help greatly to solve that problem. The formula

$$R = \Psi^{-1}\Psi_{,\zeta}, \tag{56}$$

proposed by Sym [20] defines (for a fixed ζ) a surface $R = R(x^1, x^2; \zeta)$ immersed in an appropriate Lie algebra. In many interesting cases (see for example [5, 16, 21, 22]) this formula allows to reconstruct a surface which is given implicitly by its fundamental forms.

In the case of isothermic surfaces the Sym's formula needs some modification. Indeed, R given by (56) is immersed in 10-dimensional Lie algebra $\mathbf{sp}(1,1)$. Actually one can prove that taking

$$R(0) := R(x^1, x^2; 0) \equiv \Psi^{-1}\Psi_{,\zeta}|_{\zeta=0}$$

we may confine ourselves to some subspace of dimension 6.

Lemma 3.6. *$R(0)$ defines a surface immersed in the 6-dimensional linear space V given by (52).*

Proof. Differentiating (35) with respect to ζ we obtain, for $\zeta \in \mathbb{R}$,

$$\Psi^T_{,\zeta}\mathbf{e}_{34}\Psi + \Psi^T\mathbf{e}_{34}\Psi_{,\zeta} = 0,$$
$$\Psi^\dagger_{,\zeta}\mathbf{e}_5\Psi + \Psi^\dagger\mathbf{e}_5\Psi_{,\zeta} = 0,$$
$$-\Psi_{,\zeta}(-\zeta) = \mathbf{e}_{45}\Psi_{,\zeta}(\zeta)\mathbf{e}_{45}.$$

Using once more (35) we transform immediately these formulas into

$$R^T\mathbf{e}_{34} + \mathbf{e}_{34}R = 0, \tag{57a}$$
$$R^\dagger\mathbf{e}_5 + \mathbf{e}_5R = 0, \tag{57b}$$
$$R(-\zeta) = -\mathbf{e}_{45}R(\zeta)\mathbf{e}_{45}. \tag{57c}$$

The conditions (57ab) define Lie algebra $\mathbf{sp}(1,1)$ (compare (31bc)) and mean that R is a linear combination of $\mathbf{e}_{jk}$ with real coefficients (compare the proof of Theorem 2, especially equations (32)). From (57c) we conclude that $R(0)$ anti-commutes with $\mathbf{e}_{45}$. Thus $R(0)$ is a combination of $\mathbf{e}_{s\mu}$ ($s = 1, 2, 3,\ \mu = 4, 5$), which completes the proof. □

By the way, the Lie algebra of the linear problem (12) has also dimension 6 (for more details see [14]).

It turns out that to obtain an isothermic immersion we have to project $R(0)$ onto $V_+ \simeq E^3$ or $V_- \simeq E^3$.

Theorem 3. *Suppose that $\Psi = \Psi(x^1, x^2; \zeta) \in \mathbf{G}$ (see (35)) solves (24), where (ϑ, k_1, k_2) is a fixed solution of (2), while $\Psi(0) := \Psi(x^1, x^2; 0) \in \mathbf{H}_0$. Then*

$$\mathbf{r} := \tfrac{1}{2}(\mathbf{I} - \mathbf{e}_{45})\,\Psi^{-1}\Psi_{,\zeta}|_{\zeta=0} \tag{58}$$

is V_+-valued function describing in an explicit way the isothermic surface implicitly defined by the fundamental forms (1). Moreover

$$\bar{\mathbf{r}} := \tfrac{1}{2}(\mathbf{I} + \mathbf{e}_{45})\,\Psi^{-1}\Psi_{,\zeta}|_{\zeta=0} \tag{59}$$

is V_--valued function which gives explicitly the Christoffel transform of $\mathbf{r}$, i.e., the fundamental forms of $\bar{\mathbf{r}}$ are given by (3).

Proof. Lemma 3.6 and definition (53) immediately imply $\mathbf{r} \in V_+$ and $\bar{\mathbf{r}} \in V_-$. Taking into account (24a) we have

$$\begin{aligned} R_{,j}(0) &= \Psi^{-1}(0)U_{j,\zeta}|_{\zeta=0}\Psi(0), \\ R_{,jk}(0) &= \Psi^{-1}(0)(U_{j,\zeta k} + [U_{j,\zeta}, U_k])|_{\zeta=0}\Psi(0), \end{aligned} \tag{60}$$

(for $j, k = 1, 2$) and the following derivatives of $\mathbf{r}$ can be easily computed:

$$\begin{aligned}
\mathbf{r}_{,1} &= \tfrac{1}{4} e^{\vartheta} \Psi^{-1}(0)(\mathbf{e}_{14} + \mathbf{e}_{15})\Psi(0),\\
\mathbf{r}_{,2} &= \tfrac{1}{4} e^{\vartheta} \Psi^{-1}(0)(\mathbf{e}_{24} + \mathbf{e}_{25})\Psi(0),\\
\mathbf{r}_{,11} &= \vartheta_{,1}\mathbf{r}_{,1} - \vartheta_{,2}\mathbf{r}_{,2} - \tfrac{1}{4} k_2 e^{2\vartheta} \Psi^{-1}(0)(\mathbf{e}_{34} + \mathbf{e}_{35})\Psi(0),\\
\mathbf{r}_{,22} &= \vartheta_{,2}\mathbf{r}_{,2} - \vartheta_{,1}\mathbf{r}_{,1} - \tfrac{1}{4} k_1 e^{2\vartheta} \Psi^{-1}(0)(\mathbf{e}_{34} + \mathbf{e}_{35})\Psi(0),\\
\mathbf{r}_{,12} &= \vartheta_{,2}\mathbf{r}_{,1} + \vartheta_{,1}\mathbf{r}_{,2}.
\end{aligned}$$

Then we define

$$\mathbf{n} := -\tfrac{1}{4}\Psi^{-1}(0)(\mathbf{e}_{34} + \mathbf{e}_{35})\Psi(0),$$

a unit vector orthogonal both to $\mathbf{r}_{,1}$ and $\mathbf{r}_{,2}$. Similarly,

$$\begin{aligned}
\bar{\mathbf{r}}_{,1} &= -\tfrac{1}{4} e^{-\vartheta} \Psi^{-1}(0)(\mathbf{e}_{14} - \mathbf{e}_{15})\Psi(0),\\
\bar{\mathbf{r}}_{,2} &= \tfrac{1}{4} e^{-\vartheta} \Psi^{-1}(0)(\mathbf{e}_{24} - \mathbf{e}_{25})\Psi(0),\\
\bar{\mathbf{r}}_{,11} &= -\vartheta_{,1}\bar{\mathbf{r}}_{,1} + \vartheta_{,2}\bar{\mathbf{r}}_{,2} + \tfrac{1}{4} k_2 \Psi^{-1}(0)(\mathbf{e}_{34} - \mathbf{e}_{35})\Psi(0),\\
\bar{\mathbf{r}}_{,22} &= -\vartheta_{,2}\bar{\mathbf{r}}_{,2} + \vartheta_{,1}\bar{\mathbf{r}}_{,1} - \tfrac{1}{4} k_1 \Psi^{-1}(0)(\mathbf{e}_{34} - \mathbf{e}_{35})\Psi(0),\\
\bar{\mathbf{r}}_{,12} &= -\vartheta_{,2}\bar{\mathbf{r}}_{,1} - \vartheta_{,1}\bar{\mathbf{r}}_{,2},\\
\bar{\mathbf{n}} &:= -\tfrac{1}{4}\Psi^{-1}(0)(\mathbf{e}_{34} - \mathbf{e}_{35})\Psi(0).
\end{aligned}$$

To prove the theorem it is enough to compute the fundamental forms

$$I := (\mathbf{r}_{,i}|\mathbf{r}_{,j})\, dx^i\, dx^j, \qquad II := (\mathbf{r}_{,ij}|\mathbf{n})\, dx^i\, dx^j,$$

(summation over repeating indices is assumed) for surfaces $\mathbf{r}$ and $\bar{\mathbf{r}}$ taking into account the assumption $\Psi(0) \in \mathbf{H}_0$ and also (54), (55) and Corollary 3.5. $\square$

4. The Darboux–Bäcklund transformation

Soliton surfaces and associated integrable systems admit the so called Darboux–Bäcklund transformation which generalizes classical results of Bianchi, Bäcklund and Darboux and generates a multi-parameter family of solutions (for example the so called "N-soliton surfaces" [9, 22]).

The Darboux–Bäcklund transformation applied to the linear problem (24a) is a gauge transformation

$$\tilde{\Psi} = D\Psi, \tag{61a}$$

where $D = D(x^1, x^2; \zeta)$ is a matrix such that the structure of corresponding matrices $\tilde{U}_j$ ($j = 1, 2$)

$$\tilde{U}_j = D_{,j}D^{-1} + DU_jD^{-1}, \tag{61b}$$

is identical to the structure of matrices U_j. For instance, if U_j are given by (24bc) then $\tilde{U}_1, \tilde{U}_2$ should have the same form as U_1, U_2 but the functions ϑ, k_1, k_2 entering U_j have to be replaced by some new functions $\tilde{\vartheta}$, $\tilde{k}_1$, $\tilde{k}_2$. It is obvious that these functions satisfy the nonlinear system (2). The matrix D with this property is called Darboux matrix.

The transformation for soliton surfaces (56), induced by the transformation (61a), namely

$$\tilde{R} = R + \Psi^{-1} D^{-1} D_{,\zeta} \Psi, \tag{61c}$$

in many cases turns out to generalize the classical Bäcklund transformation for pseudospherical surfaces (see [16,21,22]).

4.1. The form of the Darboux matrix

The well known construction of Zakharov and Shabat (see, for example [23]), valid for a large class of integrable systems, yields the following form of the simplest (i.e., that with a single simple pole) Darboux matrix

$$D = f N \left(I + \frac{\lambda_1 - \mu_1}{\zeta - \lambda_1} P \right), \tag{62a}$$

where f is (scalar) constant with respect to x^1, x^2, $N = N(x^1, x^2)$ is a matrix known as the "unimodular normalization matrix" ($\det N = 1$), I is the identity matrix, λ_1 and μ_1 are complex parameters and P is the projector ($P^2 = P$) given by

$$\begin{aligned} \ker P &= \Psi(x^1, x^2; \lambda_1)\, [V_{\mathrm{ker}}], \\ \operatorname{im} P &= \Psi(x^1, x^2; \mu_1)\, [V_{\mathrm{im}}], \end{aligned} \tag{62b}$$

where $[V_{\mathrm{ker}}]$, $[V_{\mathrm{im}}]$ are constant vector spaces (elements of appropriate Grassmanians) represented by matrices V_{ker} and V_{im} respectively.

The scalar factor f is introduced to separate the determinant of the normalization matrix. It is worth mentioning that equations (61b) are invariant with respect to f even if we admit $f = f(\zeta)$.

Usually N, λ_1, μ_1, $[V_{\mathrm{ker}}]$ and $[V_{\mathrm{im}}]$ cannot be arbitrary. Indeed, the problem of constructing D resolves itself to finding appropriate restrictions.

4.2. Algebraic representation of the linear problem as a method to construct the Darboux matrix

Theorem 2 represents the linear problem (24) in the form of a system of algebraic constraints on matrices U_1 and U_2. It allows us to *define* the Darboux matrix as follows (see [11,12,13]).

Definition 4.1. D is the Darboux matrix for the system (24) iff the corresponding gauge transformation (61b) preserves the system of algebraic constraints (31).

This definition can also be used to reconstruct the above results of Zakharov and Shabat [13]. We confine ourselves to the simplest (or "one-soliton") Darboux matrix by making an assumption that both D and D^{-1} have a single simple pole with respect to ζ (denoted by λ_1 and μ_1, respectively). Then, assuming $\lambda_1 \neq \mu_1$, we obtain the form (62a) of D. To preserve the analytic dependence of U_k ($k = 1, 2$) on ζ (in our case: (31a)) we have to choose P in the form (62b).

The constraints like (31bcd) impose some restrictions on Ψ known as "reduction group" [18]. To find the corresponding restrictions on D we use the following propositions ([12, 13]), which hold for any $n \times n$ matrix linear problem of the form (24a).

Proposition 4.2. *The reduction* $\mathrm{Tr}(U_k) = 0$ $(k = 1, 2)$ *and* $\det \Psi = 1$ *implies*

$$f = \left(\frac{\zeta - \mu_1}{\zeta - \lambda_1}\right)^{d/n} \tag{63}$$

where $d = \dim(\mathrm{im}\, P)$.

Proposition 4.3. *Let H be a constant matrix such that $H^\dagger$ is proportional to H. The reduction* $U_k^\dagger(\bar{\zeta}) = -HU_k(\zeta)H^{-1}$ $(k = 1, 2)$ *and* $\Psi^\dagger(\bar{\zeta})\, H\, \Psi(\zeta) = H$ *implies that either*

$$\mu_1 = \bar{\lambda}_1, \quad N^\dagger = \nu H\, N^{-1} H^{-1}, \quad P^\dagger = HPH^{-1}, \quad [V_{\mathrm{im}}]^\perp = [H V_{\mathrm{ker}}], \tag{64a}$$

where $^\perp$ denotes the orthogonal complement and $\nu = \exp(2\pi i m d/n)$ *(m is an integer), or*

$$\begin{aligned} &d = \tfrac{1}{2}n, \qquad \mu_1 \in \mathbb{R}, \qquad \lambda_1 \in \mathbb{R}, \\ &N^\dagger = \pm H\, N^{-1} H^{-1}, \qquad P^\dagger = H(I - P)H^{-1}, \\ &[V_{\mathrm{im}}]^\perp = [H V_{\mathrm{im}}], \qquad [V_{\mathrm{ker}}]^\perp = [H V_{\mathrm{ker}}]. \end{aligned} \tag{64b}$$

Proposition 4.4. *Let B is a constant matrix such that B^T is proportional to B. The reduction* $U_k^T(\zeta) = -BU_k(\zeta)B^{-1}$ $(k = 1, 2)$ *and* $\Psi^T(\zeta)B\Psi(\zeta) = B$ *implies*

$$\begin{aligned} &d = \tfrac{1}{2}n, \quad N^T = BN^{-1}B^{-1}, \quad P^T = B(I - P)B^{-1}, \\ &[\overline{V}_{\mathrm{im}}]^\perp = [B V_{\mathrm{im}}], \quad [\overline{V}_{\mathrm{ker}}]^\perp = [B V_{\mathrm{ker}}]. \end{aligned}$$

Proposition 4.5. *The reduction* $U_k(-\zeta) = JU_k(\zeta)J^{-1}$ $(k = 1, 2)$ *(J is a constant matrix such that $J^{-1} = J$) and* $\Psi(-\zeta) = J\Psi(\zeta)J^{-1}$ *implies*

$$d = \tfrac{1}{2}n, \quad \mu_1 = -\lambda_1, \quad N = \pm JNJ^{-1}, \quad P = J(I - P)J^{-1}, \quad [V_{\mathrm{ker}}] = [J V_{\mathrm{im}}].$$

Remark 4.6. The constraints on P, rewritten in terms of $\ker P$ and $\mathrm{im}\, P$, have exactly the same form as the corresponding constraints on $[V_{\mathrm{ker}}]$ and $[V_{\mathrm{im}}]$.

These propositions can be obtained on use of the Zakharov–Shabat–Mikhailov approach [18, 23]. The proofs are rather straightforward [13]: the above constraints restrict both Ψ and $\tilde{\Psi}$ to the same subgroup and, as a consequence, $D = \tilde{\Psi}\, \Psi^{-1}$ must belong to this subgroup as well.

Before checking the remaining constraints (31efgh), let us give convenient explicit expressions for the transformation of the coefficients u_{kj}. We consider here only a particular case sufficient for the purposes of this paper (for more general formulas see [12, 13]).

Proposition 4.7. *If D is given by* (62), $\lambda_1 = -\mu_1 = i\kappa_1$, *and U_1, U_2 are linear in ζ (i.e.,* $U_k = u_{k0} + u_{k1}\zeta$*), then the transformation* (61b) *assumes the form* $(k = 1, 2)$

$$\tilde{u}_{k1} = N\, u_{k1} N^{-1}, \tag{65a}$$

$$\tilde{u}_{k0} = N\, (u_{k0} + \kappa_1 [u_{k1}, A])\, N^{-1} + N_{,k} N^{-1}, \tag{65b}$$

where

$$A := i(I - 2P). \tag{66}$$

Moreover, A satisfies differential equations

$$A_{,k} = [u_{k0}, A] + \tfrac{1}{2}\kappa_1 \, [[u_{k1}, A], A]\,. \tag{67}$$

Proof. Substituting (62a) into (61b) we obtain expressions with apparent poles in λ_1 and μ_1. Conditions for vanishing of these poles are given by (67). Then equations (61b) become linear in ζ and by equating to zero the corresponding coefficients we obtain (65). □

Actually, while proving Proposition 4.7 we have shown that the form (31a) of the matrices U_k is preserved by the transformation (61b) iff A satisfies (67). It is worth pointing out that P of the form (62b) satisfies (67).

4.3. *The construction of the Darboux matrix for isothermic surfaces*

We proceed to the detailed construction of the Darboux matrix for the system (24) using the general results given in the previous section. To use Propositions 4.2–4.5 we identify, by virtue of Theorem 2, $n = 4$, $H = \mathbf{e}_5$, $B = \mathbf{e}_{34}$, $J = \mathbf{e}_{45}$. Moreover, we consider $\Psi(\zeta) \in SP(1,1)$ for $\zeta \in \mathbb{R}$, which means that $\det \Psi = 1$ (see also Remark 3.2). Then Proposition 4.2 implies

$$f \, \det N = \left(\frac{\zeta - \mu_1}{\zeta - \lambda_1}\right)^{d/4},$$

where $d = \dim(\mathrm{im} P)$. In the sequel we shall assume that $\det N = 1$.

From Proposition 4.5 it follows that $d = 2$, $\mu_1 = -\lambda_1$. Proposition 4.3 implies either (64a) or (64b). In this paper we confine ourselves exclusively to the first case: $\bar{\mu}_1 = \lambda_1$. Therefore,

$$\lambda_1 = i\kappa_1, \qquad \mu_1 = -i\kappa_1, \tag{68}$$

where κ_1 is a real parameter. Thus, the coefficient f is given by

$$f = \sqrt{\frac{\zeta + i\kappa_1}{\zeta - i\kappa_1}} \tag{69}$$

4.3.1. *The projector P.*

Propositions 4.3 (subcase (64a)), 4.4 and 4.5 impose the following constraints on the projector P:

$$P^{\dagger} = -\mathbf{e}_5 \, P \, \mathbf{e}_5, \qquad P^T = -\mathbf{e}_{34} \, (\mathbf{I} - P) \, \mathbf{e}_{34}, \qquad \mathbf{I} - P = \mathbf{e}_{45} \, P \, \mathbf{e}_{45}. \tag{70}$$

It is convenient to use the non-degenerate matrix $A := i(\mathbf{I} - 2P)$ (see (66)) satisfying

$$A^2 = -\mathbf{I}. \tag{71}$$

Then the system (70) can be rewritten as

$$A^{\dagger} = \mathbf{e}_5 A \mathbf{e}_5, \qquad A^T = \mathbf{e}_{34} A \mathbf{e}_{34}, \qquad A = -\mathbf{e}_{45} A \mathbf{e}_{45}. \tag{72}$$

It turns out to be possible to solve the above algebraic equations and to derive an explicit expression for P.

Proposition 4.8. *The projector P satisfying the constraints* (70) *can be represented as follows*:

$$
\begin{aligned}
P &= \tfrac{1}{2}(\mathbf{I} + iA), \\
A &= (p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3)(\mathbf{e}_4 \cosh\chi + \mathbf{e}_5 \sinh\chi) ,
\end{aligned}
\tag{73a}
$$

where $p_k = p_k(x^1, x^2) \in \mathbb{R}$ $(k = 1, 2, 3)$ *are subject to the constraint*

$$
p_1^2 + p_2^2 + p_3^2 = 1 \tag{73b}
$$

and $\chi = \chi(x^1, x^2) \in \mathbb{R}$.

Proof. Taking into account (71) and comparing (72) with (48) we easily see that (72) means $A \in \mathbf{H}_{+-}$, i.e., (compare (49) and (37a))

$$
A = i\mathbf{e}_5(\cosh\chi + \mathbf{e}_{45}\sinh\chi)(p_0 + p_1\mathbf{e}_{32} + p_2\mathbf{e}_{13} + p_3\mathbf{e}_{21}),
$$

where $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$. Moreover

$$
A^{-1} = i\mathbf{e}_5(\cosh\chi + \mathbf{e}_{45}\sinh\chi)(p_0 - p_1\mathbf{e}_{32} - p_2\mathbf{e}_{13} - p_3\mathbf{e}_{21}).
$$

The condition (71), i.e., $A^{-1} = -A$, implies $p_0 = 0$. Finally, from (19) we have: $i\mathbf{e}_5\mathbf{e}_{32} = \mathbf{e}_{14}$ etc., which completes the proof. □

If U_1 and U_2 are given by (24bc) then the matrix equation (67), written in terms of χ and p_k, assumes the form of the following system of equations:

$$
\begin{aligned}
p_{1,1} &= -\vartheta_{,2}p_2 - k_2 e^{\vartheta} p_3 + \kappa_1(p_2^2 + p_3^2)\sinh(\chi - \vartheta), \\
p_{2,1} &= \vartheta_{,2}p_1 - \kappa_1 p_1 p_2 \sinh(\chi - \vartheta), \\
p_{3,1} &= k_2 e^{\vartheta} p_1 - \kappa_1 p_1 p_3 \sinh(\chi - \vartheta), \\
p_{1,2} &= \vartheta_{,1}p_2 + \kappa_1 p_1 p_2 \cosh(\chi - \vartheta), \\
p_{2,2} &= -\vartheta_{,1}p_1 - k_1 e^{\vartheta} p_3 - \kappa_1(p_1^2 + p_3^2)\cosh(\chi - \vartheta), \\
p_{3,2} &= k_1 e^{\vartheta} p_2 + \kappa_1 p_2 p_3 \cosh(\chi - \vartheta), \\
\chi_{,1} &= -\kappa_1 p_1 \cosh(\vartheta - \chi), \\
\chi_{,2} &= -\kappa_1 p_2 \sinh(\vartheta - \chi).
\end{aligned}
\tag{74}
$$

The constraints on $\ker P$ and $\operatorname{im} P$, imposed by Propositions 4.3–4.5 (we constantly assume the subcase (64a)), read as follows (see also Remark 4.6)

$$
(\operatorname{im} P)^{\perp} = \mathbf{e}_5 \ker P, \tag{75a}
$$

$$
(\operatorname{im} P)^{\perp} = \mathbf{e}_{34}\overline{\operatorname{im} P}, \tag{75b}
$$

$$
(\ker P)^{\perp} = \mathbf{e}_{34}\overline{\ker P}, \tag{75c}
$$

$$
\ker P = \mathbf{e}_{45} \operatorname{im} P. \tag{75d}
$$

Proposition 4.9. *The kernel and image of the projector P can be represented (in the generic case) by the following* 4×2 *matrices*:

$$\operatorname{im} P = \left[\begin{pmatrix} \bar{a} & b \\ b & -a \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right], \qquad \ker P = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -a & b \\ b & \bar{a} \end{pmatrix}\right], \tag{76}$$

where a and b are, respectively, complex and real functions of x^1, x^2. The constant linear subspaces $[V_{\text{ker}}]$ and $[V_{\text{im}}]$ (elements of $G_{2,4}(\mathbb{C})$) are represented by 4×2 *matrices of the same form, with functions a, b are replaced by constants a_0 and b_0.*

Proof. Let assume that the last two rows of the 4×2 matrix representing $\operatorname{im} P$ are linearly independent. Then

$$\operatorname{im} P = \left[\begin{pmatrix} X \\ I \end{pmatrix}\right],$$

where $X \in M(2, \mathbb{C})$. From (75d) we compute $\ker P$:

$$\ker P = \left[\begin{pmatrix} -\sigma_1 \\ -\sigma_1 X \end{pmatrix}\right].$$

Then the remaining equations (75) can be rewritten as follows:

$$\left[\begin{pmatrix} I \\ -X^\dagger \end{pmatrix}\right] = \left[\begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}\begin{pmatrix} -\sigma_1 \\ -\sigma_1 X \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} I \\ -X^\dagger \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \overline{X} \\ I \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} -X^\dagger\sigma_1 \\ \sigma_1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} -\sigma_1 \\ -\sigma_1\overline{X} \end{pmatrix}\right].$$

If $[V_1], [V_2] \in G_{2,4}(\mathbb{C})$, then $[V_1] = [V_2]$ implies $[V_1] = [V_2 C]$ for any $C \in GL(2, \mathbb{C})$. Therefore, taking into account also $\sigma_3\sigma_1 = i\sigma_2$, we can easily reduce the above equations to $X^\dagger = -\sigma_2 X \sigma_2$, $X^\dagger = \overline{X}$. Substituting $\overline{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we finally obtain $d = -\bar{a}$, $c = b = \bar{b}$. To complete the proof we use Remark 4.6. □

Constructing the Darboux matrix it is convenient to express the projector P explicitly in terms of its kernel and image (compare (62b)).

Proposition 4.10. *The coefficients p_k and χ of the projector P can be expressed in terms of the coordinates a, b parameterizing* $\ker P$ *and* $\operatorname{im} P$ *as follows*:

$$p_1 = \frac{1 - |a|^2 - b^2}{\sqrt{\Delta}}, \quad p_2 = \frac{2 \operatorname{Re} a}{\sqrt{\Delta}}, \quad p_3 = \frac{2 \operatorname{Im} a}{\sqrt{\Delta}},$$
$$\exp\chi = \left(\frac{|a|^2 + (b+1)^2}{|a|^2 + (b-1)^2}\right)^{1/2}, \tag{77a}$$

where

$$\Delta := (|a|^2 + (b+1)^2)(|a|^2 + (b-1)^2). \tag{77b}$$

Proof. The matrix P is defined by

$$P(\widehat{\ker P}, \widehat{\operatorname{im} P}) = (0, \widehat{\operatorname{im} P}), \tag{78}$$

where $\widehat{\ker P}$, $\widehat{\operatorname{im} P}$ denote any matrices representing $\ker P$ and $\operatorname{im} P$. Assuming that $\widehat{\ker P}$, $\widehat{\operatorname{im} P}$ are given by (76) and using the basis (25) we rewrite (78) as $PK = \frac{1}{2}K(\mathbf{I} + i\mathbf{e}_{14})$, where $K := (\mathbf{I} - \mathbf{e}_{12}\operatorname{Re} a - \mathbf{e}_{13}\operatorname{Im} a - \mathbf{e}_{45}b)$. Therefore,

$$P = \tfrac{1}{2}(\mathbf{I} + iK\mathbf{e}_{14}K^{-1}). \tag{79}$$

Computing $K(\mathbf{I} + \mathbf{e}_{12}\operatorname{Re} a + \mathbf{e}_{13}\operatorname{Im} a - \mathbf{e}_{45}b) = 1 + b^2 + |a|^2 - 2\mathbf{e}_{45}b$, and taking into account that $(1 + b^2 + |a|^2 - 2\mathbf{e}_{45}b)(1 + b^2 + |a|^2 + 2\mathbf{e}_{45}b) = \Delta$, where Δ is given by

$$\Delta = (1 + b^2 + |a|^2)^2 - 4b^2 = (|a|^2 + (b+1)^2)(|a|^2 + (b-1)^2), \tag{80}$$

we conclude that $K^{-1} = \Delta^{-1}(\mathbf{I} + \mathbf{e}_{12}\operatorname{Re} a + \mathbf{e}_{13}\operatorname{Im} a - \mathbf{e}_{45}b)(1 + b^2 + |a|^2 + 2\mathbf{e}_{45}b)$. Finally, from (79) we obtain that $A \equiv i(\mathbf{I} - 2P)$ is given by:

$$A = \Delta^{-1}\big((1 - |a|^2 - b^2)\mathbf{e}_1 + 2\operatorname{Re} a\mathbf{e}_2 + 2\operatorname{Im} a\mathbf{e}_3\big)\big((1 + |a|^2 + b^2)\mathbf{e}_4 + 2b\mathbf{e}_5\big).$$

Comparing this result with (73) we obtain (77). □

4.3.2. The normalization matrix N. Propositions 4.3, 4.4 and 4.5 imply the following constraints on N:

$$N^\dagger \mathbf{e}_5 N = \mu\mathbf{e}_5, \qquad N^T\mathbf{e}_{34}N = \mathbf{e}_{34}, \qquad N = \nu\,\mathbf{e}_{45}\,N\,\mathbf{e}_{45}, \tag{81}$$

where $\mu = \pm 1$ and $\nu = \pm 1$. It means (compare (48)) that

$$N \in \mathbf{H}_{\mu\nu} \tag{82}$$

i.e., the form of N is given by (49) and (37).

Proposition 4.11. *Suppose that the Darboux–Bäcklund transformation preserves the constraints* (31a–d), *in particular, P is given by* (73) *and N is of the form* (82). *Then the requirement to preserve the constraints* (31egh) *implies*

$$N = \pm N_0\,(\cosh(\chi + \gamma_0) + \mathbf{e}_{45}\sinh(\chi + \gamma_0)), \tag{83}$$

where $N_0 \in \{\mathbf{I}, \mathbf{e}_3, i\mathbf{e}_{15}, \mathbf{e}_{24}\}$ and $\gamma_0 = \mathrm{const} \in \mathbb{R}$.

Proof. Considering (31e) it is convenient to treat separately equations for u_{k1} and u_{k0}. Therefore, $\tilde{u}_{k1}\mathbf{e}_k + \mathbf{e}_k\tilde{u}_{k1} = 0$, for $k = 1, 2$. Using (65a) we can rewrite it as

$$u_{k1}N^{-1}\mathbf{e}_kN + N^{-1}\mathbf{e}_kNu_{k1} = 0. \tag{84}$$

By virtue of (49) N is of the form $N = \mathbf{y}_0 h$, where $\mathbf{y}_0 \in \{\mathbf{I}, \mathbf{e}_4, i\mathbf{e}_5, i\mathbf{e}_{45}\}$ and h is given by (37). Obviously we have $\mathbf{y}_0^{-1}\mathbf{e}_j\mathbf{y}_0 = \mathbf{e}_j$ ($j = 1, 2, 3$), Therefore, $N^{-1}\mathbf{e}_kN \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (compare (51a)). From (31be) it follows that u_{k1} commutes with $\mathbf{e}_j$ for $j \neq k$ (compare the proof of Theorem 2) and anti-commutes with $\mathbf{e}_k$. Therefore (84) implies that $N^{-1}\mathbf{e}_k N$ is proportional to $\mathbf{e}_k$. Actually,

$$N^{-1}\mathbf{e}_k N = \pm \mathbf{e}_k \tag{85}$$

because φ_N is an orthogonal transformation in span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (see (51a)). Comparing once more (51a) (and its cyclic permutations) we conclude that exactly one coefficient n_k $(k = 0, 1, 2, 3)$ is different from zero (it has to be equal ± 1). Thus N is of the form

$$N = \pm \mathbf{y}_0 \mathbf{q}_0 e^{\gamma \mathbf{e}_{45}}, \tag{86a}$$

where

$$\mathbf{y}_0 \in \{\mathbf{I}, \mathbf{e}_4, i\mathbf{e}_5, i\mathbf{e}_{45}\}, \qquad \mathbf{q}_0 \in \{\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}. \tag{86b}$$

Let us proceed to the next condition, $\tilde{u}_{k0}\mathbf{e}_k + \mathbf{e}_k \tilde{u}_{k0} = 0$. Using (65b) and taking into account (85) we obtain

$$(u_{k0} + \kappa_1 [u_{k1}, A])\mathbf{e}_k + N^{-1} N_{,k} \mathbf{e}_k + \mathbf{e}_k (u_{k0} + \kappa_1 [u_{k1}, A]) + \mathbf{e}_k N^{-1} N_{,k} = 0. \tag{87}$$

The formula (85) can be rewritten as $N\mathbf{e}_k = \pm \mathbf{e}_k N$. Therefore

$$N_{,j}\mathbf{e}_k = \pm \mathbf{e}_k N_{,j}, \qquad (j, k = 1, 2), \tag{88}$$

and, using also (31e) and (85), we transform (87) into

$$N_{,k} = -\tfrac{1}{2}\kappa_1 N [u_{k1}, A - \mathbf{e}_k A \mathbf{e}_k]. \tag{89}$$

Let us evaluate the right-hand side of the formula (89) for $k = 1, 2$. First of all, we recall that

$$u_{11} = \tfrac{1}{2}\mathbf{e}_{15} e^{\vartheta \mathbf{e}_{45}}, \qquad u_{21} = \tfrac{1}{2}\mathbf{e}_{24} e^{\vartheta \mathbf{e}_{45}}. \tag{90}$$

Moreover, using (73a) we compute in the straightforward way $A - \mathbf{e}_1 A \mathbf{e}_1 = 2p_1 \mathbf{e}_{14} e^{\chi \mathbf{e}_{45}}$, $A - \mathbf{e}_2 A \mathbf{e}_2 = 2p_2 \mathbf{e}_{25} e^{\chi \mathbf{e}_{45}}$. Therefore, (89) assumes the form

$$N_{,1} = -\kappa_1 p_1 \cosh(\vartheta - \chi) N \mathbf{e}_{45}, \qquad N_{,2} = -\kappa_1 p_2 \sinh(\vartheta - \chi) N \mathbf{e}_{45},$$

and substituting from (86) we finally obtain

$$\gamma_{,1} = -\kappa_1 p_1 \cosh(\vartheta - \chi), \qquad \gamma_{,2} = -\kappa_1 p_2 \sinh(\vartheta - \chi).$$

Comparing this result with equations for χ (see (74)) we obtain $\gamma = \chi + \gamma_0$, where γ_0 is a constant.

Let us consider constraints (31gh). Using (65a) we have

$$\mathrm{Tr}(u_{11} N^{-1} \mathbf{e}_{15} N) > 0, \quad \mathrm{Tr}(u_{21} N^{-1} \mathbf{e}_{24} N) < 0.$$

Then, taking into account (86) and (90) we obtain

$$\begin{aligned} &\mathrm{Tr}\left(e^{(2\gamma - \vartheta)\mathbf{e}_{45}} \mathbf{e}_{15} (\mathbf{y}_0 \mathbf{q}_0)^{-1} \mathbf{e}_{15} \mathbf{y}_0 \mathbf{q}_0\right) > 0, \\ &\mathrm{Tr}\left(e^{(2\gamma - \vartheta)\mathbf{e}_{45}} \mathbf{e}_{24} (\mathbf{y}_0 \mathbf{q}_0)^{-1} \mathbf{e}_{24} \mathbf{y}_0 \mathbf{q}_0\right) < 0. \end{aligned} \tag{91}$$

From $\mathrm{Tr}(\mathbf{e}_{45}) = 0$ and $\mathbf{e}_{45}^2 = 1$ we have $\mathrm{Tr}\left(e^{(2\gamma-\vartheta)\mathbf{e}_{45}}\right) > 0$. Let us recall also that $\mathbf{e}_{15}^2 = 1$ and $\mathbf{e}_{24}^2 = -1$. Moreover, $\mathbf{y}_0\mathbf{q}_0$ either commutes or anticommutes with any element of the form $\mathbf{e}_{jk}$. Therefore (91) is equivalent to

$$[\mathbf{y}_0\mathbf{q}_0, \mathbf{e}_{15}] = 0, \qquad [\mathbf{y}_0\mathbf{q}_0, \mathbf{e}_{24}] = 0, \tag{92}$$

i.e., $\mathbf{y}_0\mathbf{q}_0 \in \mathrm{span}\{\mathbf{I}, \mathbf{e}_3, \mathbf{e}_{15}, \mathbf{e}_{24}\}$. Taking into account (86b) (compare also (19) and (37c)) we complete the proof. □

4.3.3. The Darboux matrix. Now we proceed to the last constraint, (31f). From (65a) we obtain $\mathrm{Tr}(\tilde{u}_{k1}^2) = \mathrm{Tr}(u_{k1}^2)$, i.e., $\mathrm{Tr}(u_{k1}^2)$ is invariant under the Darboux–Bäcklund transformation and any constraints of the form $\mathrm{Tr}(u_{k1}^2) = f_k$, where $f_k = f_k(x^1, x^2)$ are given functions, are preserved. In particular, we can prescribe $f_k := (-1)^{k+1}$.

Therefore, taking into account the results of previous paragraphs, we can formulate the following theorem.

Theorem 4. *A matrix of the form* (62), *where* $\lambda_1 = -\mu_1 = i\kappa_1$, *preserves the system of algebraic conditions* (31) *if and only if* P *and* N *are given by* (73) *and* (83) *respectively.*

In other words, such matrix is the Darboux matrix for the linear system (24) and the corresponding transformation (61b) generates solutions of the nonlinear system (2). Taking into account also (69) we obtain the following form of D:

$$D = \frac{N(\zeta\mathbf{I} - \kappa_1 A)}{\sqrt{\zeta^2 + \kappa_1^2}}, \tag{93}$$

where N, A are given by (73),(83) and the square root should be treated as two-valued function of the parameter ζ. Moreover

$$D^{-1} = \frac{(\zeta\mathbf{I} + \kappa_1 A)N^{-1}}{\sqrt{\zeta^2 + \kappa_1^2}}, \tag{94}$$

which can be immediately checked using $A^2 = -\mathbf{I}$.

However, not every Darboux matrix generates a new isothermic immersion: according to Theorem 3 the function Ψ defines an isothermic immersion if $\Psi(0) \in \mathbf{H}_0$. Therefore to obtain the Darboux–Bäcklund transformation for isothermic surfaces we have to impose one more restriction on the Darboux matrix:

$$D(x^1, x^2; 0) \in \mathbf{H}_0, \tag{95}$$

which can be rewritten, substituting $\zeta = 0$ into (93), as $\mp NA \in \mathbf{H}_0$, or, taking into account (83) and (73), $\mp N_0\left(\sum_{k=1}^3 p_j\mathbf{e}_j\right)\left(\mathbf{e}_4\cosh\gamma_0 + \mathbf{e}_5\sinh\gamma_0\right) \in \mathbf{H}_0$. The elements of $\mathbf{H}_0$ have to commute with both $\mathbf{e}_4$ and $\mathbf{e}_5$, which implies $\gamma_0 = 0$, $N_0\mathbf{e}_4 + \mathbf{e}_4 N_0 = 0$, $N_0\mathbf{e}_5 - \mathbf{e}_5 N_0 = 0$. Thus, taking into account Proposition 4.11, N_0 has to be equal to $\mathbf{e}_{24}$. Therefore the constraint (95) restricts the matrix N to the following form:

$$N = \pm\mathbf{e}_2(\mathbf{e}_4\cosh\chi + \mathbf{e}_5\sinh\chi). \tag{96}$$

Corollary 4.12. *The simplest (i.e., "one-soliton") Darboux matrix for the system* (24), *generating isothermic surfaces, is given by*

$$D = \mathbf{e}_2\left(\frac{\kappa_1}{\sqrt{\zeta^2+\kappa_1^2}}\left(\sum_{k=1}^{3} p_k\mathbf{e}_k\right) + \frac{\zeta}{\sqrt{\zeta^2+\kappa_1^2}}\,(\cosh\chi\,\mathbf{e}_4 + \sinh\chi\,\mathbf{e}_5)\right). \tag{97}$$

The factor ± 1 (see (96)) is omitted because the square roots in the formula (97) imply the ambiguity of sign anyway. To be even more explicit we can use (77). Finally we obtain

$$D = \mathbf{e}_2\frac{\kappa_1\big((1-|a|^2-b^2)\mathbf{e}_1 + 2\,\mathrm{Re}\,a\mathbf{e}_2 + 2\,\mathrm{Im}\,a\mathbf{e}_3\big) + \zeta\big((1+|a|^2+b^2)\mathbf{e}_4 + 2b\mathbf{e}_5\big)}{\sqrt{(\zeta^2+\kappa_1{}^2)\big((1+|a|^2+b^2)^2-(2b)^2\big)}}, \tag{98a}$$

where κ_1 is a real parameter and a, b are functions defined by

$$\begin{pmatrix} \bar{a} & b \\ b & -a \end{pmatrix} := AB^{-1}, \tag{98b}$$

where A and B are 2×2 matrix functions given by

$$\begin{pmatrix} A \\ B \end{pmatrix} := \Psi(x^1, x^2; -i\kappa_1)\begin{pmatrix} X_0 \\ I \end{pmatrix}, \tag{98c}$$

and, finally,

$$X_0 = \begin{pmatrix} \bar{a}_0 & b_0 \\ b_0 & -a_0 \end{pmatrix}, \tag{98d}$$

where $a_0 \in \mathbb{C}$ and $b_0 \in \mathbb{R}$ are constant. Formula (98) expresses the Darboux matrix solely in terms of the function Ψ corresponding to the given isothermic surface $\mathbf{r}$.

4.4. Darboux–Bäcklund transformation for isothermic surfaces

The transformation (65) with A, N given by (73a), (96) and U_k given by (24bc) can be rewritten as the following system of equations:

$$\tilde{\vartheta} = 2\chi - \vartheta, \tag{99a}$$

$$\tilde{k}_1 e^{\tilde{\vartheta}} = -k_1 e^{\vartheta} - 2\kappa_1 p_3 \cosh(\chi - \vartheta), \tag{99b}$$

$$\tilde{k}_2 e^{\tilde{\vartheta}} = k_2 e^{\vartheta} - 2\kappa_1 p_3 \sinh(\chi - \vartheta), \tag{99c}$$

$$\chi_{,1} + \kappa_1 p_1 \cosh(\chi - \vartheta) = 0, \tag{99d}$$

$$\chi_{,2} - \kappa_1 p_2 \sinh(\chi - \vartheta) = 0, \tag{99e}$$

$$\tilde{\vartheta}_{,1} = -\vartheta_{,1} - 2\kappa_1 p_1 \cosh(\chi - \vartheta), \tag{99f}$$

$$\tilde{\vartheta}_{,2} = -\vartheta_{,2} + 2\kappa_1 p_2 \sinh(\chi - \vartheta). \tag{99g}$$

The equations (99abc) give the explicit formulas for $\tilde{\vartheta}$, $\tilde{k}_1$, $\tilde{k}_2$ (we point out that p_1, p_2, p_3 and χ can be expressed in terms of the "old" wave function Ψ, see (77) and (98bc)).

The equations (99de) are satisfied by virtue of (74), while (99fg) can be obtained by differentiating (99a) and using (99de). Therefore the equations (99defg) follow from (99abc) and provide no new information.

The Darboux–Bäcklund transformation for corresponding soliton surfaces (56), i.e., the transformation (61c) with D given by (93) (see also (94)) assumes the form

$$\tilde{R} = R + \frac{\kappa_1^2}{(\zeta^2+\kappa_1^2)^2}\Psi^{-1}(\zeta)\,(\mathbf{I}\zeta+\kappa_1 A)\,\Psi(\zeta).$$

We are mostly interested (see Theorem 3) in the subcase $\zeta = 0$:

$$\tilde{R}(0) = R(0) + \frac{1}{\kappa_1}\Psi^{-1}(0)A\Psi(0). \tag{100}$$

In particular, the transformations for isothermic surfaces (58) and their Christoffel transforms (59) read as

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{2}{\kappa_1}e^{\chi}\,\Psi^{-1}(0)\Big(\frac{1}{4}\sum_{k=1}^{3}p_k(\mathbf{e}_{k4}+\mathbf{e}_{k5})\Big)\Psi(0), \tag{101}$$

$$\tilde{\bar{\mathbf{r}}} = \bar{\mathbf{r}} + \frac{2}{\kappa_1}e^{-\chi}\,\Psi^{-1}(0)\Big(\frac{1}{4}\sum_{k=1}^{3}p_k(\mathbf{e}_{k4}-\mathbf{e}_{k5})\Big)\Psi(0), \tag{102}$$

One should remember that $\Psi(0)\in\mathbf{H}_0$ and, because of that, $\varphi_{\Psi(0)}$ is a rotation in both V_+ and V_- (see Corollary 3.5).

When constructing the Darboux matrix we required $D(\zeta)\in\mathbf{G}$ and $D(0)\in\mathbf{H}_0$. Therefore $\tilde{\Psi}(\zeta)\in\mathbf{G}$, $\tilde{\Psi}(0)\in\mathbf{H}_0$ and, by virtue of Theorem 3 we obtain the following result.

Corollary 4.13. *The surface described by* $\tilde{\mathbf{r}}$ *is isothermic and* $\tilde{\bar{\mathbf{r}}}$ *is the Christoffel transform of* $\tilde{\mathbf{r}}$.

Moreover, using Proposition 4.9, we can express $\tilde{\mathbf{r}}$ and $\tilde{\bar{\mathbf{r}}}$ as follows:

$$\tilde{\mathbf{r}}_\pm = \mathbf{r}_\pm + \frac{2}{\kappa_1}\Psi^{-1}(0)\frac{\big((1-|a|^2-b^2)\mathbf{e}_1+2\,\mathrm{Re}\,a\mathbf{e}_2+2\,\mathrm{Im}\,a\mathbf{e}_3\big)}{|a|^2+(b\mp 1)^2}\frac{(\mathbf{e}_4\pm\mathbf{e}_5)}{4}\Psi(0), \tag{103}$$

where we introduced the notation $\mathbf{r}_+ := \mathbf{r}$, $\mathbf{r}_- := \bar{\mathbf{r}}$.

4.5. "One-soliton" isothermic surfaces

Applying the Darboux–Bäcklund transformation to the trivial solution of a nonlinear evolution equation one obtains solution known as a single soliton.

The function Ψ_0 corresponding to the trivial solution $\vartheta = k_1 = k_2 = 0$ is given by

$$\Psi_0(\zeta) = \big(\cosh(\tfrac{1}{2}\zeta u)+\mathbf{e}_{15}\sinh(\tfrac{1}{2}\zeta u)\big)\big(\cos(\tfrac{1}{2}\zeta v)+\mathbf{e}_{24}\sin(\tfrac{1}{2}\zeta v)\big), \tag{104}$$

where we came back to the original parameters, $u\equiv x^1$ and $v\equiv x^2$.

The corresponding surface (58) is a plane parameterized by cartesian coordinates u, v,

$$\mathbf{r}_0 = \tfrac{1}{4}(u\mathbf{e}_1+v\mathbf{e}_2)(\mathbf{e}_4+\mathbf{e}_5) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}, \tag{105}$$

where we identified V_+ with E^3. The action of the Darboux–Bäcklund transformation results in

$$\mathbf{r}_1 = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \frac{1}{\kappa_1(\cosh\tau\cosh\gamma_1 - \cos\xi)} \begin{pmatrix} 2\sin\xi \\ -2\sinh\tau\cosh\gamma_1 \\ -2\sinh\gamma_1 \end{pmatrix}, \tag{106}$$

where $\xi = \kappa_1 u + \xi_1$, $\tau = \kappa_1 v + \tau_1$; κ_1 is a constant parameter ("spectral parameter") and, finally, γ_1, τ_1, ξ_1 are constant parameters related to a_0, b_0. The fundamental forms of the surface $\mathbf{r}_1$ are defined by

$$e^{\vartheta} = \frac{\cosh\gamma_1\cosh\tau + \cos\xi}{\cosh\gamma_1\cosh\tau - \cos\xi}, \tag{107a}$$

$$k_1 = \frac{2\kappa_1\sinh\gamma_1\cosh\gamma_1\cosh\tau}{(\cosh\gamma_1\cosh\tau + \cos\xi)^2}, \tag{107b}$$

$$k_2 = \frac{2\kappa_1\sinh\gamma_1\cos\xi}{(\cosh\gamma_1\cosh\tau + \cos\xi)^2}. \tag{107c}$$

The mean curvature of the surface $\mathbf{r}_1$

$$H = \frac{\kappa_1\sinh\gamma_1}{\cosh\gamma_1\cosh\tau + \cos\xi} \tag{108}$$

has a constant sign.

5. Classical results versus the dressing method

It is interesting to compare the transformation obtained by the dressing method (see (99abc) and (101)) with the classical Darboux–Bianchi transformation given by (9) and (7).

First of all, using the formulas expressing $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{n}$ by $\Psi(0)$ and ϑ (see the proof of Theorem 3), we can rewrite (101) as

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{2}{\kappa_1} e^{\chi}\left(p_1 e^{-\vartheta}\mathbf{r}_{,1} + p_2 e^{-\vartheta}\mathbf{r}_{,2} - p_3\mathbf{n}\right). \tag{109}$$

Taking into account (23) we see that the formula (109) is identical to (7), i.e., $\tilde{\mathbf{r}} \equiv \mathbf{r}'$, iff one identifies

$$\lambda/m\sigma = -2e^{\chi}p_1/\kappa_1, \tag{110a}$$

$$\mu/m\sigma = -2e^{\chi}p_2/\kappa_1, \tag{110b}$$

$$\omega/m\sigma = 2e^{\chi}p_3/\kappa_1. \tag{110c}$$

Moreover, substituting (110abc) into (6), we have

$$\varphi/m\sigma = 2e^{2\chi}/\kappa_1^2. \tag{110d}$$

Then, substituting (110cd) into (99bc) we obtain (9bc) where the lower sign has to be chosen, which means that $m < 0$. Finally, comparing (99a) with (9a) and (110d) we can express m in terms of κ_1:

$$m = -\tfrac{1}{2}\kappa_1^2. \tag{110e}$$

In other words, in formula (11) we should take $\zeta = i\kappa_1$.

A comment is in order. According to a result of Zakharov and Shabat (see (62ab)) the Darboux matrix is built of the wave function Ψ evaluated at $\zeta = \lambda_1$ and $\zeta = \mu_1$. In our case, because of (68), (98c) and (35c), the Darboux matrix can be expressed solely in terms of Ψ evaluated at $i\kappa_1$.

Concluding remarks. The results of this paper strongly suggest that the Clifford algebra $\mathcal{C}(1, 4)$ is a natural structure to describe isothermic immersions in E^3. We used a specific matrix representation because the existing methods to construct the Darboux–Bäcklund transformation are given in terms of matrices. It is obvious that the construction of the Darboux matrix can be extended to Clifford algebras, which will be done in subsequent papers (compare [15]).

Another interesting problem is to find an interpretation of the Sym's formula for $\zeta \neq 0$ and to apply Sym's formula to isothermic immersions in 3-dimensional spaces of constant curvature (S^3 and H^3).

Acknowledgements

I would like to thank Prof. Antoni Sym for suggesting me to apply the methods of the theory of solitons to isothermic surfaces and for many stimulating discussions. I am grateful also to Adam Doliwa for useful remarks.

Added in proof

After submission of this paper several new results in the field appeared: isothermic surfaces have been interpreted as the so called curved flats in the symmetric space $O(4, 1)/O(3) \times O(1, 1)$ [7], discrete isothermic surfaces were introduced and discussed [6], the Bäcklund transformation in the discrete case has been constructed [15] etc.

References

[1] M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, (SIAM, Philadelphia, 1981).

[2] T. van Bemmelen, Symmetry computations with total derivative operators, Ph.D. Thesis, Dep. Appl. Math., Twente Univ., Enschede 1993.

[3] L. Bianchi, Ricerche sulle superficie isoterme and sulla deformazione delle quadriche, *Annali di Matematica, Ser. III* **11** (1905) 93–157.

[4] L. Bianchi, Il teorema di permutabilità per le transformazioni di Darboux delle superficie isoterme, *Rendiconti della Accademia dei Lincei* **13** (1904) (1) 359–367.

[5] A.I. Bobenko, Surfaces in terms of 2 by 2 matrices. Old and new integrable cases, in: A.P. Fordy and J.C. Wood, eds., *Harmonic maps and integrable systems*, Aspects of Mathematics 23 (Vieweg, Braunschweig, 1994).

[6] A. Bobenko and U. Pinkall, Discrete isothermic surfaces, *J. Reine Angew. Math.* **475** (1996) 187–208.

[7] F. Burstall, U. Hertrich-Jeromin, F. Pedit and U. Pinkall, Curved flats and isothermic surfaces, *Math. Zeit.*, to appear.

[8] S.S. Chern, Deformation of surfaces preserving principal curvatures, in: I. Chavel and H.M. Farkas, eds., *Differential Geometry and Complex Analysis* (Springer, 1985) 155–163.

[9] J. Cieśliński, An effective method to compute N-fold Darboux matrix and N-soliton surfaces, *J. Math. Phys.* **32** (1991) 2395–2399.

[10] J. Cieśliński, Lie point symmetries as a tool to isolate integrable geometries, in: M. Boiti et al., eds., *Nonlinear Evolution Equations and Dynamical Systems* (World Scientific, Singapore, 1992) 260–268.

[11] J. Cieśliński, A new approach to the Darboux–Bäcklund transformation and its application to the Bianchi system, Preprint IP-WUD N. 13, Bialystok, 1992.
[12] J. Cieśliński, Algebraic representation of the linear problem as a method to construct the Darboux–Bäcklund transformation, *Chaos, Solitons & Fractals* **5** (1995) 2303–2313.
[13] J. Cieśliński, An algebraic method to construct the Darboux matrix, *J. Math. Phys.* **36** (1995) 5670–5706.
[14] J. Cieśliński, P. Goldstein and A. Sym, Isothermic surfaces in E^3 as soliton surfaces, *Phys. Lett. A* **205** (1995) 37–43.
[15] J. Cieśliński, The Bäcklund transformation for discrete isothermic surfaces, Proc. Conf. *Symmetries and Integrability of Discrete Equations*, Canterbury, England, July 1996 (Cambridge Univ. Press) to appear.
[16] D. Levi and A. Sym, Integrable systems describing surfaces of non-constant curvature, *Phys. Lett. A* **149** (1990) 381–387.
[17] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin–Heidelberg, 1991).
[18] A.V. Mikhailov, The reduction problem and the inverse scattering method, *Physica D* **3** (1981) 73–117.
[19] P. Olver, *Applications of Lie groups to Differentiall Equations* (Springer, 1986).
[20] A. Sym, Soliton surfaces, *Lett. Nuovo Cim.* **33** (1982) 394–400.
[21] A. Sym, Soliton Surfaces II. Geometric unification of solvable nonlinearities, *Lett. Nuovo Cim.* **36** (1983) 307–312.
[22] A. Sym, Soliton surfaces and their application. Soliton geometry from spectral problems, in: R. Martini, ed., *Geometric Aspects of the Einstein Equations and Integrable Systems*, Lecture Notes in Physics 239 (Springer, Berlin, 1985) 154–231.
[23] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaievsky, *Theory of Solitons* (Nauka, Moscow, 1980).