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Exact boundary observability for nonautonomous quasilinear wave equations

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ABSTRACT

By means of a direct and constructive method based on the theory of semiglobal C^2 solution, the local exact boundary observability is shown for nonautonomous 1-D quasilinear wave equations. The essential difference between nonautonomous wave equations and autonomous ones is also revealed.

1. Introduction

This paper deals with the following 1-D quasilinear nonautonomous wave equation

$$u_{tt} - c^2(t, x, u, u_x, u_t)u_{xx} = f(t, x, u, u_x, u_t), \tag{1.1}$$

where c, f are suitably smooth functions with respect to their arguments, $c = c(t, x, u, u_x, u_t) > 0$ is the propagation speed of the nonlinear wave, and f satisfies

$$f(t, x, 0, 0, 0) \equiv 0, \tag{1.2}$$

$u = 0$ is an equilibrium of system (1.1). Here we emphasize that the wave speed c explicitly depends on time which will bring some new phenomena in the features of observability, as we will see later.

The boundary conditions can be one of the following physically meaningful inhomogeneous boundary conditions

$$x = 0: \quad u = h(t), \tag{1.3a}$$

$$x = 0: \quad u_x = h(t), \tag{1.3b}$$

$$x = 0: \quad u_x - \alpha u = h(t), \tag{1.3c}$$

$$x = 0: \quad u_x - \beta u_t = h(t) \tag{1.3d}$$

and a similar one of the following boundary conditions

$$x = L: \quad u = \bar{h}(t), \tag{1.4a}$$

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$$x = L: \quad u_x = \bar{h}(t), \tag{1.4b}$$

$$x = L: \quad u_x + \bar{\alpha} u = \bar{h}(t), \tag{1.4c}$$

$$x = L: \quad u_x + \bar{\beta} u_t = \bar{h}(t), \tag{1.4d}$$

where $\alpha, \beta, \bar{\alpha}$, and $\bar{\beta}$ are positive constants.

The initial condition is given by

$$t = t_0: \quad (u, u_t) = \big(\varphi(x), \psi(x)\big), \quad 0 \leqslant x \leqslant L, \tag{1.5}$$

where $(\varphi, \psi) \in C^2[0, L] \times C^1[0, L]$, and the conditions of C^2 compatibility are supposed to be satisfied at the points $(t_0, 0)$ and (t_0, L), respectively.

The *exact observability* problem which we are interested in can be described as follows: can we find $T > 0$ and some suitable observation $k(t)$ (the value of u or u_x of the solution $u = u(t, x)$ to the mixed problem (1.1) and (1.3)–(1.5)), such that the initial data φ can be uniquely determined by the observation $k(t)$ together with the known given boundary functions $(h(t), \bar{h}(t))$ on the time interval $[t_0, t_0 + T]$? Moreover, can we have an estimate (*observability inequality*) on φ in terms of $k(t)$ and $(h(t), \bar{h}(t))$? More precisely, noting that $u = 0$ is an equilibrium of system (1.1), we will focus on the local exact boundary observability for the nonautonomous mixed problem (1.1) and (1.3)–(1.5) in a neighborhood (in C^2-sense) of $u = 0$.

Exact controllability and observability for wave equations (and other partial differential equations) have been intensively studied since Russell [18] and Lions [14]. Classical techniques to derive observability estimates for linear wave equations are mainly the following: *Multiplier Methods* (see [7,14,16]), *Carleman Estimates* (see [5,6,23,24]), *Microlocal Analysis* (see [1,2]), *Spectral Method* (see [15,19,20]), etc. Due to the duality arguments (see [7,14,18,26]), we know that exact controllability of a linear system can be reduced to the observability estimate of its dual system. However, in general, the duality principle dose not hold for nonlinear dynamical systems (see [4,9]). Consequently, one has to study controllability and observability for the nonlinear systems separately. With usual energy estimates and perturbation method, Pan, Teo and Zhang [17] studied observability (in that paper, it is called state observation problem) for a semilinear wave equation, and they also gave a conceptual algorithm of resolution. Concerning the controllability for nonlinear wave equations, there are also some results (see [8,27,28] for semilinear case, and [25] for quasilinear case). For autonomous 1-D quasilinear wave equations, Li and his collaborators established a complete theory on exact boundary controllability and observability, by means of a direct and constructive method which is based on the theory of semiglobal C^2 solution (see [9,10,13]).

To our knowledge, there are few results on controllability and observability for nonautonomous wave equations, in which the wave operator (the principle part of the wave equation) depends explicitly on time. Cavalcanti [3] established exact boundary controllability for n-D linear nonautonomous wave equation by utilizing *Hilbert Uniqueness Method* of Lions [14]. In [3], the assumption that the wave speed is larger than a positive constant is vital to obtain the main results. However, as is pointed out in Section 2, the degenerate case that the propagation speed approximates zero may produce some delicate new phenomena in observability (and also in controllability, see [22]).

In this paper, we establish local exact boundary observability, by Li's method (with some modification), for some nonautonomous 1-D quasilinear wave equations. The exact boundary controllability for these equations has already been established by Wang [22]. Li's method is said to be a direct and constructive one, because it treats the quasilinear system directly without any linearization and fixed point (or compactness) arguments and the observability inequality can be obtained by solving some well-posed mixed problems. This method is based on the theory of so-called *semiglobal classical solution* (see [9,12,21]) which guarantees the well-posedness of classical solution on a preassigned (possibly quite large) time interval $[t_0, t_0 + T]$. Li's method is very useful in 1-D case and it can be used for various kinds of (linear or nonlinear) boundary conditions.

Compared with the results in [10], the main difficulties that we encounter here lie in two parts: to get the existence and uniqueness of semiglobal C^2 solution to the nonautonomous mixed problem (1.1) and (1.3)–(1.5); and to have a better estimate on observation time T which is no more as easy as the autonomous case [10]. Moreover, we have to pay attention to the influence of the boundary functions $(h(t), \bar{h}(t))$, while [10] considers only the situation that $h \equiv \bar{h} \equiv 0$. We point out also that the results obtained in this paper cover all the results in [10].

The organization of this paper is as follows: by a simple example, we show the possible features of the exact observability for nonautonomous quasilinear wave equations in Section 2. The fundamental theory of semiglobal C^2 solution to the nonautonomous mixed problem (1.1) and (1.3)–(1.5) is introduced in Section 3. Adopting Li's method, the main results, Theorems 4.1–4.2, are proved in Section 4. Finally, some remarks are given in Section 5.

For the convenience of statement, we denote in the whole paper that

$$l = \begin{cases} 2 & \text{for (1.3a)}, \\ 1 & \text{for (1.3b)–(1.3d)} \end{cases} \tag{1.6}$$

and

$$\bar{l} = \begin{cases} 2 & \text{for (1.4a)}, \\ 1 & \text{for (1.4b)–(1.4d)}. \end{cases} \tag{1.7}$$

We also denote C as a positive constant which is independent of the solution and C can be different constants in different situations.

2. Features of exact observability for nonautonomous quasilinear wave equations

The results in [10] show that, for autonomous quasilinear wave equations, one can choose proper boundary observed values to uniquely determine any given small initial value (φ, ψ) at $t = 0$, provided that the observability time $T > 0$ is large enough. By translation, this conclusion still holds if the observation starts at any initial time $t = t_0$ instead of $t = 0$. Hence, the observability time T can be chosen to be independent of t_0 in the autonomous case.

In nonautonomous cases, however, the exact boundary observability usually depends on the selection of the initial time. Consider the linear nonautonomous wave equation

$$u_{tt} - \big(c(t)\big)^2 u_{xx} = 0, \tag{2.1}$$

which is a special case of (1.1) as c depends only on time t. One can see that:

1) the two-sided exact boundary observability holds for Eq. (2.1) on the time interval $[t_0, t_0 + T]$ if and only if $\int_{t_0}^{t_0+T} c(t)\,dt \geqslant L$;
2) the one-sided exact boundary observability holds for Eq. (2.1) on the time interval $[t_0, t_0 + T]$ if and only if $\int_{t_0}^{t_0+T} c(t)\,dt \geqslant 2L$.

By the different choices of $c(t)$, it is easy to see that there are three possibilities: the exact boundary observability for Eq. (2.1) holds

1) only for some initial time $t_0 \in \mathbb{R}$, but not for the others;
2) for none of the initial time $t_0 \in \mathbb{R}$;
   or
3) for all the initial time $t_0 \in \mathbb{R}$.

However, there is only the possibility 3) in autonomous case as shown by Remark 5.4.

Moreover, the observability time T for Eq. (2.1) usually depends on the initial time t_0, that is to say, the exact boundary observability holds only when $T > T(t_0)$. On the other hand, the observability time T might be independent of t_0 in some special cases, for instance, if $c(t)$ is a suitable periodic function.

In summary, the exact boundary observability for nonautonomous hyperbolic systems is much more complicated than that in autonomous cases, and we should pay more attention on it.

3. Semiglobal C^2 solution to 1-D nonautonomous quasilinear wave equations

In this section, we establish the theory on the semiglobal C^2 solution to the mixed initial-boundary value problem (1.1) and (1.3)–(1.5) on the domain

$$R(t_0, T_0) = \big\{(t, x) \,\big|\, t_0 \leqslant t \leqslant t_0 + T_0,\ 0 \leqslant x \leqslant L\big\}, \tag{3.1}$$

where $T_0 > 0$ is a preassigned and possibly quite large number.

Suppose that the conditions of C^2 compatibility are satisfied at the points $(t, x) = (t_0, 0)$ and (t_0, L), respectively. In order to get the semiglobal C^2 solution to the mixed problem for (1.1) with various kinds of boundary conditions in a unified manner, we reduce the problem to a corresponding mixed problem for a first order quasilinear hyperbolic system (cf. [13,22]).

Let

$$v = u_x, \qquad w = u_t \tag{3.2}$$

and

$$U = (u, v, w)^T. \tag{3.3}$$

Eq. (1.1) can be reduced to the following first order quasilinear system

$$\begin{cases} u_t = w, \\ v_t - w_x = 0, \\ w_t - c^2(t, x, u, v, w)v_x = f(t, x, u, v, w). \end{cases} \tag{3.4}$$

Accordingly, the initial condition (1.5) becomes

$$t = t_0\colon\quad U = \big(\varphi(x), \varphi'(x), \psi(x)\big)^T, \quad 0 \leqslant x \leqslant L. \tag{3.5}$$

Since $c(t,x,u,u_x,u_t)>0$, (3.4) is a strictly hyperbolic system with three distinct real eigenvalues

$$\lambda_1=-c(t,x,u,v,w)<\lambda_2\equiv 0<\lambda_3=c(t,x,u,v,w), \tag{3.6}$$

and the corresponding left eigenvectors can be taken as

$$\begin{cases} l_1(t,x,U)=\big(0,c(t,x,u,v,w),1\big),\\ l_2(t,x,U)=(1,0,0),\\ l_3(t,x,U)=\big(0,-c(t,x,u,v,w),1\big). \end{cases} \tag{3.7}$$

Setting

$$v_i=l_i(t,x,U)U \quad (i=1,2,3), \tag{3.8}$$

namely,

$$\begin{cases} v_1=c(t,x,u,v,w)v+w,\\ v_2=u,\\ v_3=-c(t,x,u,v,w)v+w, \end{cases} \tag{3.9}$$

we have

$$\begin{cases} v_1+v_3=2w,\\ v_1-v_3=2c(t,x,u,v,w)v. \end{cases} \tag{3.10}$$

The boundary condition (1.3a) can be rewritten as

$$x=0:\quad v_1+v_3=2h'(t) \tag{3.11}$$

together with the following condition of C^0 compatibility

$$h(t_0)=\varphi(0). \tag{3.12}$$

In a neighborhood of $U=0$, the boundary conditions (1.3b)–(1.3c) can be equivalently rewritten as

$$x=0:\quad v_3=p_2(t,v_1,v_2)+q_2(t), \tag{3.13}$$

$$x=0:\quad v_3=p_3(t,v_1,v_2)+q_3(t), \tag{3.14}$$

or

$$x=0:\quad v_1=\tilde p_2(t,v_2,v_3)+\tilde q_2(t), \tag{3.15}$$

$$x=0:\quad v_1=\tilde p_3(t,v_2,v_3)+\tilde q_3(t), \tag{3.16}$$

respectively. Similarly, in a neighborhood of $U=0$, the boundary condition (1.3d) can be rewritten as

$$x=0:\quad v_3=p_4(t,v_1,v_2)+q_4(t) \tag{3.17}$$

or, when

$$\beta\neq\frac{1}{c(t,0,0,0,0)},\quad \forall t\in[t_0,t_0+T_0], \tag{3.18}$$

$$x=0:\quad v_1=\tilde p_4(t,v_2,v_3)+\tilde q_4(t). \tag{3.19}$$

Moreover, we have

$$p_i(t,0,0)\equiv\tilde p_i(t,0,0)\equiv 0 \quad (i=2,3,4), \tag{3.20}$$

and

$$\|q_i\|_{C^1[t_0,t_0+T_0]},\ \|\tilde q_i\|_{C^1[t_0,t_0+T_0]}\to 0 \quad (i=2,3,4)$$

as $\|h\|_{C^1[t_0,t_0+T_0]}\to 0$.

Similarly, the boundary condition (1.4a) can be rewritten as

$$x=L:\quad v_1+v_3=2\bar h'(t) \tag{3.21}$$

together with

$$\bar h(t_0)=\varphi(L). \tag{3.22}$$

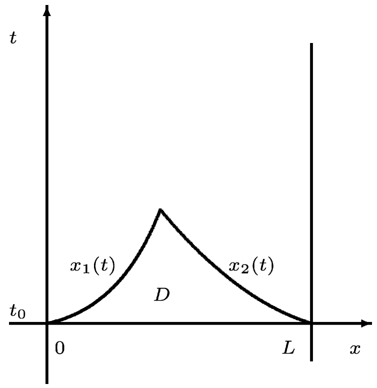

Fig. 1. Maximum determinate domain D of Cauchy problem.

(1.4b)–(1.4d) can be rewritten as

$$x = L: \quad v_1 = p(t, v_2, v_3) + q(t). \tag{3.23}$$

Moreover, when

$$\bar{\beta} \neq \frac{1}{c(t, L, 0, 0, 0)}, \quad \forall t \in [t_0, t_0 + T_0], \tag{3.24}$$

(3.23) can be equivalently rewritten as

$$x = L: \quad v_3 = \tilde{p}(t, v_1, v_2) + \tilde{q}(t), \tag{3.25}$$

where

$$p(t, 0, 0) \equiv \tilde{p}(t, 0, 0) \equiv 0, \tag{3.26}$$

and

$$\|q\|_{C^1[t_0, t_0+T_0]}, \|\tilde{q}\|_{C^1[t_0, t_0+T_0]} \to 0$$

as $\|\bar{h}\|_{C^1[t_0, t_0+T_0]} \to 0$.

Obviously, the conditions of C^2 compatibility at the points $(t_0, 0)$ and (t_0, L) for the mixed problem (1.1) and (1.3)–(1.5) guarantee the conditions of C^1 compatibility for the corresponding mixed problem of the first order quasilinear hyperbolic system (3.4)–(3.5), (3.11) (or (3.15) or (3.16) or (3.17)) and (3.21) (or (3.23)).

Applying the theory on the semiglobal C^1 solution to the mixed initial-boundary value problem of first order nonautonomous quasilinear hyperbolic systems (cf. [21]), we get

Lemma 3.1 *(Semiglobal C^2 solution). Suppose that $c, f \in C^1, c > 0$ and (1.2) holds. Suppose furthermore that $\varphi \in C^2$, $\psi \in C^1$, $h \in C^l, \bar{h} \in C^{\bar{l}}$ (see (1.6)–(1.7)) and the conditions of C^2 compatibility are supposed to be satisfied at the points $(t_0, 0)$ and (t_0, L) respectively. For any given $T_0 > 0$ (possibly quite large), if $\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]}$ and $\|(h, \bar{h})\|_{C^l[t_0, t_0+T_0] \times C^{\bar{l}}[t_0, t_0+T_0]}$ are sufficiently small (depending on t_0 and T_0), the mixed problem (1.1) and (1.3)–(1.5) admits a unique C^2 solution $u = u(t, x)$ (called semiglobal C^2 solution) on the domain $R(t_0, T_0) \triangleq \{(t, x) \mid t_0 \leqslant t \leqslant t_0 + T_0,\ 0 \leqslant x \leqslant L\}$, and the following estimate holds*

$$\|u\|_{C^2[R(t_0, T_0)]} \leqslant C\big(\big\|(\varphi, \psi)\big\|_{C^2[0,L] \times C^1[0,L]} + \big\|(h, \bar{h})\big\|_{C^l[t_0, t_0+T_0] \times C^{\bar{l}}[t_0, t_0+T_0]}\big). \tag{3.27}$$

Corollary 3.1. *Suppose that $c, f \in C^1$, $c > 0$ and (1.2) holds. If $\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]}$ is sufficiently small, then Cauchy problem (1.1) and (1.5) admits a unique global C^2 solution $u = u(t, x)$ on the whole maximum determinate domain $D = \{(t, x) \mid t \geqslant t_0,\ x_1(t) \leqslant x \leqslant x_2(t)\}$ (see [11]), where the two curves $x_1(t), x_2(t)$ are defined as follows:*

$$\begin{cases} \dfrac{dx_1}{dt} = c\big(t, x_1, u(t, x_1), u_x(t, x_1), u_t(t, x_1)\big), \\ t = t_0: \quad x_1 = 0 \end{cases} \tag{3.28}$$

and

$$\begin{cases} \dfrac{dx_2}{dt} = -c\big(t, x_2, u(t, x_2), u_x(t, x_2), u_t(t, x_2)\big), \\ t = t_0: \quad x_2 = L, \end{cases} \tag{3.29}$$

respectively (see Fig. 1). Moreover, we have the following estimate

$$\|u\|_{C^2[D]} \leqslant C\big\|(\varphi, \psi)\big\|_{C^2[0,L] \times C^1[0,L]}. \tag{3.30}$$

4. Exact boundary observability for 1-D nonautonomous quasilinear wave equations

Now we consider the exact boundary observability for system (1.1) and (1.3)–(1.5). Let t_0 be the initial time and let T be the observability time. Define the domain $R(t_0, T)$ similar to (3.1).

The principle of choosing the observed value is such that the observed values together with the boundary conditions can uniquely determine the value (u, u_x) on the boundary (cf. [10]). Hence, the observed value at $x = 0$ can be taken as

1. $u_x = k(t)$ for (1.3a), then
$$x = 0: \quad (u, u_x) = \big(h(t), k(t)\big), \tag{4.1a}$$
2. $u = k(t)$ for (1.3b), then
$$x = 0: \quad (u, u_x) = \big(k(t), h(t)\big), \tag{4.1b}$$
3. $u = k(t)$ for (1.3c), then
$$x = 0: \quad (u, u_x) = \big(k(t), \alpha k(t) + h(t)\big), \tag{4.1c}$$
4. $u = k(t)$ for (1.3d), then
$$x = 0: \quad (u, u_x) = \big(k(t), \beta k'(t) + h(t)\big). \tag{4.1d}$$

Then, by means of the observed value at $x = 0$, we get
$$x = 0: \quad (u, u_x) = \big(a(t), b(t)\big), \tag{4.2}$$
and for any given T,
$$\big\|(a, b)\big\|_{C^2[t_0,t_0+T]\times C^1[t_0,t_0+T]} \leqslant C\big(\|k\|_{C^d[t_0,t_0+T]} + \|h\|_{C^l[t_0,t_0+T]}\big), \tag{4.3}$$
where l is given by (1.6) and
$$d = \begin{cases} 1 & \text{for (1.3a)}, \\ 2 & \text{for (1.3b)–(1.3d)}. \end{cases} \tag{4.4}$$

The observed value $\bar{k}(t)$ at $x = L$ can be similarly taken, then we get
$$x = L: \quad (u, u_x) = \big(\bar{a}(t), \bar{b}(t)\big), \tag{4.5}$$
and for any given T,
$$\big\|(\bar{a}, \bar{b})\big\|_{C^2[t_0,t_0+T]\times C^1[t_0,t_0+T]} \leqslant C\big(\|\bar{k}\|_{C^{\bar{d}}[t_0,t_0+T]} + \|\bar{h}\|_{C^{\bar{l}}[t_0,t_0+T]}\big), \tag{4.6}$$
where $\bar{l}$ is given by (1.7) and
$$\bar{d} = \begin{cases} 1 & \text{for (1.4a)}, \\ 2 & \text{for (1.4b)–(1.4d)}. \end{cases} \tag{4.7}$$

Theorem 4.1 *(Two-sided observability). Suppose that $c, f \in C^1$, $c > 0$ and (1.2) holds. Suppose furthermore that there exists $T > 0$ such that*
$$\int_{t_0}^{t_0+T} \inf_{0\leqslant x\leqslant L} c(t, x, 0, 0, 0)\, dt > L. \tag{4.8}$$

For any given initial data (φ, ψ) with $\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]}$ to be suitably small, suppose finally that the conditions of C^2 compatibility are satisfied at the points $(t_0, 0)$ and (t_0, L), respectively. Then the initial data (φ, ψ) can be uniquely determined by the observed values $k(t)$ at $x = 0$ and $\bar{k}(t)$ at $x = L$ together with the known boundary functions $(h(t), \bar{h}(t))$ on the interval $[t_0, t_0 + T]$. Moreover, the following observability inequality holds:
$$\big\|(\varphi, \psi)\big\|_{C^2[0,L]\times C^1[0,L]} \leqslant C\Big(\big\|(k, \bar{k})\big\|_{C^d[t_0,t_0+T]\times C^{\bar{d}}[t_0,t_0+T]} + \big\|(h, \bar{h})\big\|_{C^l[t_0,t_0+T]\times C^{\bar{l}}[t_0,t_0+T]}\Big), \tag{4.9}$$
where d, $\bar{d}$, l and $\bar{l}$ are given by (4.4), (4.7), (1.6) and (1.7) respectively.

Proof. Noting (4.8), there exists $\varepsilon > 0$ such that

$$\int_{t_0}^{t_0+T} \inf_{\substack{0\leqslant x\leqslant L \\ |(u,v,w)|\leqslant\varepsilon}} c(t,x,u,v,w)\,dt > L, \tag{4.10}$$

in which $|(u,v,w)| = \sqrt{|u|^2+|v|^2+|w|^2}$.

By Lemma 3.1, when $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(h,\bar{h})\|_{C^l[t_0,t_0+T]\times C^{\bar{l}}[t_0,t_0+T]}$ are sufficiently small, the mixed problem (1.1) and (1.3)–(1.5) admits a unique semiglobal C^2 solution $u=u(t,x)$ with small C^2 norm on the domain $R(t_0,T)$. Hence, the C^d and $C^{\bar{d}}$ norm of the observed value $k(t)$ and $\bar{k}(t)$ are sufficiently small, respectively. In particular, we may suppose

$$\|u\|_{C^1[R(t_0,T)]} \leqslant \varepsilon. \tag{4.11}$$

Noting $c>0$, we can change the role of t and x in Eq. (1.1) in order to solve it in the x-direction.

By Corollary 3.1, the rightward Cauchy problem for equation (1.1) with the initial condition (4.2) admits a unique C^2 solution $u=\tilde{u}(t,x)$ on the whole maximum determinate domain D_r (see Fig. 2) and

$$\|\tilde{u}\|_{C^2[D_r]} \leqslant C\big(\|k\|_{C^d[t_0,t_0+T]} + \|h\|_{C^l[t_0,t_0+T]}\big). \tag{4.12}$$

Here $D_r = \{(t,x) \mid t_0 \leqslant t \leqslant t_0+T,\ 0 \leqslant x \leqslant \min\{x_1(t),x_2(t)\}\}$, in which the two curves $x_1(t), x_2(t)$ are defined as follows

$$\begin{cases} \dfrac{dx_1}{dt} = -c\big(t,x_1,u(t,x_1),u_x(t,x_1),u_t(t,x_1)\big), \\ t=t_0+T: \quad x_1=0 \end{cases} \tag{4.13}$$

and

$$\begin{cases} \dfrac{dx_2}{dt} = c\big(t,x_2,u(t,x_2),u_x(t,x_2),u_t(t,x_2)\big), \\ t=t_0: \quad x_2=0. \end{cases} \tag{4.14}$$

Similarly, the leftward Cauchy problem for Eq. (1.1) with the initial condition (4.5) admits a unique C^2 solution $u=\tilde{\tilde{u}}(t,x)$ on the whole maximum determinate domain D_l (see Fig. 2) and

$$\|\tilde{\tilde{u}}\|_{C^2[D_l]} \leqslant C\big(\|\bar{k}\|_{C^{\bar{d}}[t_0,t_0+T]} + \|\bar{h}\|_{C^{\bar{l}}[t_0,t_0+T]}\big). \tag{4.15}$$

Here $D_l = \{(t,x) \mid t_0 \leqslant t \leqslant t_0+T,\ \max\{x_3(t),x_4(t)\} \leqslant x \leqslant L\}$, in which the two curves $x_3(t), x_4(t)$ are defined as follows

$$\begin{cases} \dfrac{dx_3}{dt} = c\big(t,x_3,u(t,x_3),u_x(t,x_3),u_t(t,x_3)\big), \\ t=t_0+T: \quad x_3=L \end{cases} \tag{4.16}$$

and

$$\begin{cases} \dfrac{dx_4}{dt} = -c\big(t,x_4,u(t,x_4),u_x(t,x_4),u_t(t,x_4)\big), \\ t=t_0: \quad x_4=L. \end{cases} \tag{4.17}$$

We now claim that the domains D_r and D_l must intersect each other.

Since $x=x_1(t)$ passes through the point $(t_0+T,0)$, it follows from (4.13) that

$$x_1(t) = \int_t^{t_0+T} c\big(t,x_1,u(t,x_1),u_x(t,x_1),u_t(t,x_1)\big)\,dt. \tag{4.18}$$

Hence, noting (4.10)–(4.11), the intersection point of $x=x_1(t)$ with the line $x=L$ must be above the point (t_0,L), where $x=x_4(t)$ passes through. Noting that the ODE in (4.13) is the same as that in (4.17), we conclude by the uniqueness of C^1 solution that $x=x_1(t)$ stays above $x=x_4(t)$ all the time. Similarly, $x=x_3(t)$ always stays above $x=x_2(t)$. Thus D_r and D_l intersect each other (see Fig. 2).

Therefore, there exists $\widetilde{T} \in (t_0,t_0+T)$ such that the value $(u,u_t) = (\Phi(x),\Psi(x))$ on $t=\widetilde{T}$ can be completely determined by $u=\tilde{u}(t,x)$ and $u=\tilde{\tilde{u}}(t,x)$. Then we get from (4.12) and (4.15) that

$$\big\|(\Phi,\Psi)\big\|_{C^2[0,L]\times C^1[0,L]} \leqslant C\Big(\big\|(k,\bar{k})\big\|_{C^d[t_0,t_0+T]\times C^{\bar{d}}[t_0,t_0+T]} + \big\|(h,\bar{h})\big\|_{C^l[t_0,t_0+T]\times C^{\bar{l}}[t_0,t_0+T]}\Big). \tag{4.19}$$

Since both $u=\tilde{u}(t,x)$ and $u=\tilde{\tilde{u}}(t,x)$ are the restriction of the C^2 solution $u=u(t,x)$ to the original mixed problem (1.1) and (1.3)–(1.5) on the corresponding maximum determinate domains respectively, we have

$$t=\widetilde{T}: \quad u=\Phi(x), \qquad u_t=\Psi(x), \quad 0\leqslant x\leqslant L. \tag{4.20}$$

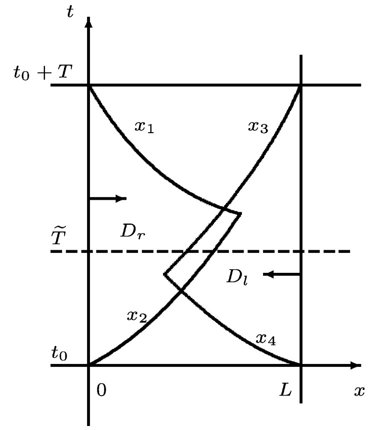

Fig. 2. D_r and D_l intersect each other.

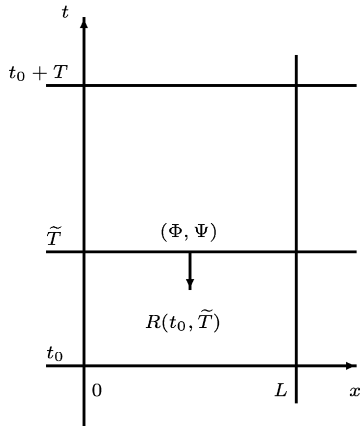

Fig. 3. Solve the backward problem on $R(t_0, \widetilde{T})$.

By Lemma 3.1, the backward mixed initial-boundary value problem (1.1) with the initial condition (4.20) and the boundary conditions

$$x = 0:\quad u = a(t), \tag{4.21}$$

$$x = L:\quad u = \bar{a}(t) \tag{4.22}$$

admits a unique semiglobal C^2 solution $u = \hat{u}(t,x)$ on

$$R(t_0, \widetilde{T}) = \{(t,x) \mid t_0 \leqslant t \leqslant t_0 + \widetilde{T},\ 0 \leqslant x \leqslant L\} \tag{4.23}$$

(see Fig. 3), since the conditions of C^2 compatibility at the points $(t,x) = (\widetilde{T}, 0)$ and $(\widetilde{T}, L)$ are obviously satisfied respectively. By the uniqueness of C^2 solution, $u = \hat{u}(t,x)$ must be the restriction of the original C^2 solution $u = u(t,x)$ on $R(t_0, \widetilde{T})$, and the following estimate holds:

$$\|u\|_{C^2[R(t_0,\widetilde{T})]} \leqslant C\big(\big\|(\Phi, \Psi)\big\|_{C^2[0,L]\times C^1[0,L]} + \big\|(a, \bar{a})\big\|_{C^2[t_0,t_0+T]\times C^2[t_0,t_0+T]}\big). \tag{4.24}$$

Finally, (4.9) follows immediately from (1.5), (4.3), (4.6) and (4.19). This concludes the proof of Theorem 4.1. □

Theorem 4.2 *(One-sided observability). Under the assumptions of Theorem* 4.1 (*except* (4.8)), *suppose furthermore that $\bar{\beta}$ in the boundary condition* (1.4d) *satisfies*

$$\bar{\beta} \neq \frac{1}{c(t, L, 0, 0, 0)}, \quad \forall t \in [t_0, t_0 + T] \tag{4.25}$$

and there exists $T > 0$ such that

$$\int_{t_0}^{t_0+T} \inf_{0 \leqslant x \leqslant L} c(t, x, 0, 0, 0)\, dt > 2L. \tag{4.26}$$

Then, the initial data (φ, ψ) can be uniquely determined by the observed value $k(t)$ at $x = 0$ together with the known boundary functions $(h(t), \bar{h}(t))$ on the interval $[t_0, t_0 + T]$. Moreover, the following observability inequality holds:

$$\big\|(\varphi, \psi)\big\|_{C^2[0,L]\times C^1[0,L]} \leqslant C\big(\|k\|_{C^d[t_0,t_0+T]} + \big\|(h, \bar{h})\big\|_{C^l[t_0,t_0+T]\times C^{\bar{l}}[t_0,t_0+T]}\big). \tag{4.27}$$

Proof. Similarly to the proof of Theorem 4.1, the rightward Cauchy problem for Eq. (1.1) with the initial condition (4.2) admits a unique C^2 solution $u = \tilde{u}(t,x)$ on the whole maximum determinate domain D_r and estimate (4.12) holds. Under assumption (4.26), D_r must intersect the line $x = L$ (see Fig. 4).

Thus, there exists $\widetilde{T} \in (t_0, t_0 + T)$ such that the value $(u, u_t) = (\Phi(x), \Psi(x))$ on $t = \widetilde{T}$ can be completely determined by $u = \tilde{u}(t,x)$. Then, we get from (4.12) that

$$\big\|(\Phi, \Psi)\big\|_{C^2[0,L]\times C^1[0,L]} \leqslant C\big(\|k\|_{C^d[t_0,t_0+T]} + \|h\|_{C^l[t_0,t_0+T]}\big). \tag{4.28}$$

Since the conditions of C^2 compatibility at the points $(t,x) = (\widetilde{T}, 0)$ and $(\widetilde{T}, L)$ are obviously satisfied respectively, by Lemma 3.1, the backward mixed problem (1.1) with the initial condition (4.20) and the boundary conditions (1.4) and

$$x = 0:\quad u = a(t) \tag{4.29}$$

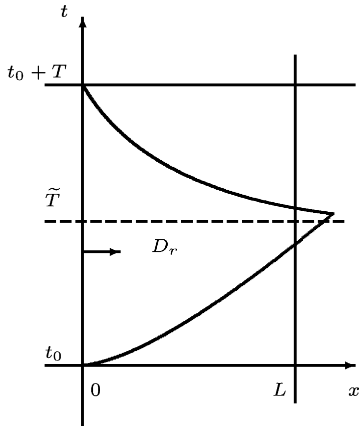

Fig. 4. D_r intersects $x = L$.

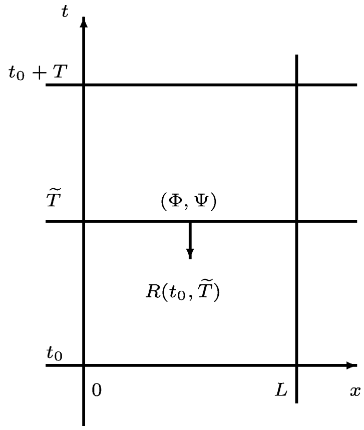

Fig. 5. Solve backward problem on $R(t_0, \widetilde{T})$.

admits a unique C^2 solution $u = \hat{u}(t, x)$ on $R(t_0, \widetilde{T})$ (see Fig. 5). By the uniqueness of solution, $u = \hat{u}(t, x)$ must be the restriction of the original C^2 solution $u = u(t, x)$ on $R(t_0, \widetilde{T})$, and the following estimate holds:

$$\|u\|_{C^2[R(t_0,\widetilde{T})]} \leqslant C\big(\big\|(\Phi, \Psi)\big\|_{C^2[0,L]\times C^1[0,L]} + \big\|(a, \bar{h})\big\|_{C^2[t_0,t_0+T]\times C^{\bar{l}}[t_0,t_0+T]}\big). \tag{4.30}$$

Noting (1.5), (4.27) follows immediately from (4.3) and (4.28). This finishes the proof of Theorem 4.2. □

5. Remarks

Remark 5.1. In Theorem 4.1, (4.8) is a sharp estimate on the observability time T, which guarantees that two maximum determinate domains D_r and D_l intersect each other. In Theorem 4.2, (4.26) is a sharp estimate on the observability time T, which guarantees that the maximum determinate domain D_r of the rightward Cauchy problem must intersect the line $x = L$. The assumptions (4.8) and (4.26) on the observation time allow the propagation speed c to be close to zero, which is not the case in [3] even if Eq. (1.1) is linear, i.e., $c = c(t, x)$.

Remark 5.2. In Theorem 4.2, if the observed value $\bar{k}(t)$ is chosen at $x = L$ and we assume

$$\beta \neq \frac{1}{c(t, 0, 0, 0, 0)}, \quad \forall t \in [t_0, t_0 + T] \tag{5.1}$$

instead of (4.25), a similar result can be obtained.

Remark 5.3. Consider the n-dimensional quasilinear wave equation with rotation invariance

$$u_{tt} - c^2\big(t, |x|, u, u_t, x \cdot \nabla u\big)\Delta u = f\big(t, |x|, u, u_t, x \cdot \nabla u\big) \tag{5.2}$$

on the hollow ball

$$D = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; r_1 \leqslant |x| \leqslant r_2,\ |x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \right\} \quad (0 < r_1 < r_2). \tag{5.3}$$

Under the assumption of spherical symmetry, (5.2) can be reduced to the following 1-D nonautonomous wave equation

$$u_{tt} - c^2(t, r, u, u_t, ru_r)u_{rr} = f(t, r, u, u_t, ru_r) + \left(\frac{n-1}{r}\right)c^2(t, r, u, u_t, ru_r)u_r, \tag{5.4}$$

where $r = |x|$, then we can apply Theorems 4.1–4.2 directly to obtain the corresponding exact boundary observability with spherical symmetry data.

Remark 5.4. Different from the nonautonomous case, the exact boundary observability can be always realized for the 1-D essential autonomous quasilinear wave equation

$$u_{tt} - c^2(x, u, u_x, u_t)u_{xx} = f(t, x, u, u_x, u_t), \tag{5.5}$$

provided that the observability time T is large enough. In fact, by Theorems 4.1–4.2, two-sided (resp., one-sided) exact boundary observability for (5.5) can be realized on the interval $[t_0, t_0 + T]$ if

$$T > \sup_{0 \leqslant x \leqslant L} \frac{L}{c(x, 0, 0, 0)} \quad \left(\text{resp., } T > \sup_{0 \leqslant x \leqslant L} \frac{2L}{c(x, 0, 0, 0)}\right). \tag{5.6}$$

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