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Weighted norm inequalities, Gaussian bounds and sharp spectral multipliers

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Abstract

Let L be a non-negative self-adjoint operator acting on $L^2(X)$ where X is a space of homogeneous type. Assume that L generates a holomorphic semigroup e^{-tL} whose kernels $p_t(x, y)$ have Gaussian upper bounds but there is no assumption on the regularity in variables x and y. In this article, we study weighted L^p-norm inequalities for spectral multipliers of L. We show that sharp weighted Hörmander-type spectral multiplier theorems follow from Gaussian heat kernel bounds and appropriate L^2 estimates of the kernels of the spectral multipliers. These results are applicable to spectral multipliers for large classes of operators including Laplace operators acting on Lie groups of polynomial growth or irregular non-doubling domains of Euclidean spaces, elliptic operators on compact manifolds and Schrödinger operators with non-negative potentials.

Keywords: Hörmander-type spectral multiplier theorems; Non-negative self-adjoint operator; Weights; Heat semigroup; Plancherel-type estimate; Space of homogeneous type

Contents

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1. Introduction

Suppose that L is a non-negative self-adjoint operator acting on $L^2(X)$. Let $E(\lambda)$ be the spectral resolution of L. By the spectral theorem, for any bounded Borel function $F : [0,\infty) \to \mathbb{C}$, one can define the operator

$$F(L) = \int_0^\infty F(\lambda)\, dE(\lambda), \tag{1.1}$$

which is bounded on $L^2(X)$. A natural problem considered in the spectral multipliers theory is to give sufficient conditions on F and L which imply the boundedness of $F(L)$ on various functional spaces defined on X. This topic has attracted a lot of attention and has been studied extensively by many authors: for example, for sub-Laplacian on nilpotent groups in [5,12], for sub-Laplacian on Lie groups of polynomial growth in [1], for Schrödinger operator on Euclidean space $\mathbb{R}^n$ in [17], for sub-Laplacian on Heisenberg groups in [26] and many others. For more information about the background of this topic, the reader is referred to [1,2,4,5,10,12,14,15,23] and the references therein. We also refer the reader to [37] and the references therein for examples of potential applications of the spectral multiplier results.

We wish to point out [14], which is closely related to this paper. In [14], a sharp spectral multiplier for a non-negative self-adjoint operator L was obtained under the assumption of the kernel $p_t(x,y)$ of the analytic semigroup e^{-tL} having a Gaussian upper bound. As there was no assumption on smoothness of the space variables of $p_t(x,y)$, the singular integral $F(L)$ does not satisfy the standard kernel regularity condition of a so-called Calderón–Zygmund operator, thus standard techniques of Calderón–Zygmund theory are not applicable. The lacking of smoothness of the kernel was indeed the main obstacle in [14] and it was overcome by shrewd exploitation of the analyticity of the kernel $p_t(x,y)$ in variable t, together with a so-called Plancherel estimate, see Remark 2 after Corollary 3.4.

We will now recall some of main features of the spectral multipliers theory. An interesting example of a spectral multiplier result comes from the paper [1] where Alexopoulos considers the operators acting on Lie groups of polynomial growth. He proved that if L is a group invariant Laplacian and n is the maximum of the local and global dimension of the group then $F(L)$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if the function F is differentiable s times where $s = [\frac{n}{2}] + 1$ and satisfies

$$\left|\lambda^k F^{(k)}(\lambda)\right| \leqslant C$$

for some constant C and $k = 0, 1, \ldots, s$, see also Section 6.1 and Proposition 6.2 below. The philosophy is that we need function F to possess just more than $n/2$ derivatives (with suitable bounds) for $F(L)$ to be bounded on all L^p spaces, $1 < p < \infty$.

When s is an even number the above condition can be written in the following way

$$\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty, \tag{1.2}$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^p} = \|(I - d^2/dx^2)^{s/2} F\|_{L^p}$ and η is an auxiliary non-zero cut-off function such that $\eta \in C_c^\infty(\mathbb{R}_+)$. We note that condition (1.2) is actually independent of the choice of η. It is well known that condition (1.2) can be generalized with positive numbers $s > 0$ and it is sufficient to take real value $s > n/2$, see [2,14]. It is an interesting question when condition (1.2) can be replaced by the following weaker condition

$$\sup_{t>0} \|\eta\delta_t F\|_{W_s^2} < \infty \tag{1.3}$$

for some $s > n/2$. Already for the standard Laplace operator on the Euclidean space $\mathbb{R}^n$, the classical Fourier multiplier result of Hörmander [18] applied to radial functions says that the weaker W_s^2 condition for any $s > n/2$ is enough to guarantee L^p boundedness of $F(\Delta)$ for all $1 < p < \infty$, see also [5] for further discussion. Actually, replacing the W_s^∞ norm in condition (1.2) by the W_s^2 norm in condition (1.3) is essentially the same problem which one encounters in sharp Bochner–Riesz summability analysis, see [6,14,30,33,34]. Discussion of possibility of replacing condition (1.2) by (1.3) is one of the main themes of [14].

The aim of this paper is to extend the study of sharp spectral multipliers in [14] to the setting of weighted L^p spaces. It turns out that for a function F having more than $n/2$ suitable derivatives, the range of p that we can obtain for $F(L)$ to be bounded depends also on the weight w. Most of the results of [14] follow from Theorems 3.1, 3.2 and 3.3 which are the main results of this paper; see Remark 1 after Corollary 3.4. We use the techniques developed in [14] to estimate the kernels of spectral multipliers. The new contribution of this paper is a development of an original technique to deal with singular integral nature of the considered spectral multipliers to obtain generalization of unweighted results described in [14] to weighted L^p spaces.

This paper is organized as follows. In Section 2, we recall basic properties of spaces of homogeneous type, the class of Muckenhoupt weights and a sufficient condition for boundedness of weighted singular integrals from [3]. We state the main results on weighted spectral multipliers, Theorems 3.1 and 3.3 in Section 3. Section 4 is devoted to the proofs of these theorems. In Section 5, we use complex interpolation to obtain boundedness for spectral multipliers on weighed L^p spaces. In Section 6, we give applications of our results to various operators in different settings, including Laplace operators on homogeneous groups and on irregular domains of Euclidean spaces, elliptic pseudo-differential operators on compact manifolds, Schrödinger operators with positive potentials and holomorphic functional calculi of non-negative self-adjoint operators.

2. Singular integrals and weights

Let (X, d, μ) be a space endowed with a distance d and a non-negative Borel measure μ on X. Set $B(x,r) = \{y \in X\colon d(x,y) < r\}$ and $V(x,r) = \mu(B(x,r))$. We shall often just use B instead of $B(x,r)$. Recall that (X, d, μ) satisfies the doubling volume property provided that there exists a constant $C > 0$ such that

$$V(x, 2r) \leqslant CV(x,r) \quad \forall r > 0,\ x \in X, \tag{2.1}$$

more precisely if there exist $n, C_n > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \leqslant C_n\left(\frac{r}{s}\right)^n, \quad \forall r \geqslant s > 0,\ x \in X. \tag{2.2}$$

The parameter n is a measure of the doubling dimension of the space. It also follows from the doubling condition that there exist C and D, $0 \leqslant D \leqslant n$ so that

$$V(y,r) \leqslant C\left(1 + \frac{d(x,y)}{r}\right)^D V(x,r) \quad \forall r > 0,\ x, y \in X \tag{2.3}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (2.3) with $D = n$ is a direct consequence of the triangle inequality for the metric d and (2.2). In many cases like the Euclidean space $\mathbb{R}^n$ or Lie groups of polynomial growth, D can be chosen to be 0.

Muckenhoupt weights. Next we review the definitions of Muckenhoupt classes of weights. We use the notation

$$\fint_E h = \frac{1}{V(E)} \int_E h(x)\, d\mu(x)$$

and we often forget the measure and variable of the integrand in writing integrals.

In what follows for any number or symbol s with value in $[1, \infty]$ by s' we denote it's conjugate, that is $\frac{1}{s} + \frac{1}{s'} = 1$.

A weight w is a non-negative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset X$,

$$\left(\fint_B w\right)\left(\fint_B w^{1-p'}\right)^{p-1} \leqslant C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such $\mathcal{M}w \leqslant Cw$ a.e. where $\mathcal{M}$ denotes the uncentered maximal operator over balls in X, that is

$$\mathcal{M}w(x) = \sup_{B \ni x} \fint_B w.$$

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for every ball $B \subset X$,

$$\left(\fint_B w^q\right)^{1/q} \leqslant C\left(\fint_B w\right).$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, for any ball B,

$$w(x) \leqslant C \fint_B w, \quad \text{for a.e. } x \in B.$$

Note that we have excluded the case $q = 1$ since the class RH_1 consists of all weights, and that is the way RH_1 is understood in what follows.

We sum up some properties of the A_p and RH_q classes in the following lemmas.

Lemma 2.1. *Suppose that* (X, d, μ) *is a metric, measure space, which satisfies doubling condition* (2.1)*. Then the following properties hold for the weights classes* A_p *and* RH_q *defined on* (X, d, μ)*:*

(i) $A_1 \subset A_p \subset A_q$ *for* $1 < p \leqslant q < \infty$.
(ii) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 \leqslant p \leqslant q < \infty$.
(iii) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.
(iv) *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.
(v) $A_\infty = \bigcup_{1 \leqslant p < \infty} A_p \subseteq \bigcup_{1 < q \leqslant \infty} RH_q$.
(vi) *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.
(vii) *If* $1 \leqslant q \leqslant \infty$ *and* $1 \leqslant s < \infty$, *then* $w \in A_q \cap RH_s$ *if and only if* $w^s \in A_{s(q-1)+1}$.

Proof. Properties (i)–(vi) are standard, see for instance, [38,16] and [11]. For (vii), see [21]. □

Note that under additional assumption on the measure μ that the function $\mu(B(x, r))$ increases continuously with r for each $x \in X$, it is shown that $A_\infty = \bigcup_{1 \leqslant p < \infty} A_p = \bigcup_{1 < q \leqslant \infty} RH_q$ (see Theorem 18, Chapter 1 [38]). However, we do not need this property in the sequel.

Lemma 2.2. *Let* $1 < p < r'$. *Then* $w \in A_p \cap RH_{(\frac{r'}{p})'}$ *if and only if* $w^{1-p'} = w^{-\frac{1}{p-1}} \in A_{\frac{p'}{r}}$.

Proof. Lemma 2.2 is a special case of [3, Lemma 4.4] (with $p_0 = 1$ and $q_0' = r$ in the notation of [3]). □

Singular integrals on weighted spaces. The following result, see [3, Theorem 3.7] is the main technical tool to extend unweighted L^p boundedness of spectral multipliers in [14] to weighted L^p results.

Theorem 2.3. *Let $1 \leqslant p_0 < \infty$. Let T be a sublinear operator acting on $L^{p_0}(X)$, Let $\{A_r\}_{r>0}$ be a family of operators acting on $L^{p_0}(X)$. Assume that*

$$\left(\oint_B \left|T(I - A_{r_B})f\right|^{p_0} d\mu\right)^{1/p_0} \leqslant C\mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \tag{2.4}$$

and

$$\|TA_{r_B}f\|_{L^\infty(B)} \leqslant C\mathcal{M}\left(|Tf|^{p_0}\right)^{\frac{1}{p_0}}(x) \tag{2.5}$$

for all $f \in L^{p_0}(X)$, and all ball B with radius r_B and all $B \ni x$. Then for all $p_0 < p < \infty$ and $w \in A_{p/p_0} = A_{p/p_0} \cap RH_1$, there exists a constant C such that

$$\|Tf\|_{L^p(X,w)} \leqslant C\|f\|_{L^p(X,w)}. \tag{2.6}$$

Proof. Theorem 2.3 is a special case of [3, Theorem 3.7] (with $q_0 = \infty$ in the notation of [3]). □

Given $1 \leqslant p_0 < p < q_0$, we observe that if w is any given weight so that $w, w^{1-p'} \in L^1_{\mathrm{loc}}(X)$, then a given linear operator T is bounded on $L^p(X, w)$ if and only if its adjoint (with respect to $d\mu$) T^* is bounded on $L^{p'}(w^{1-p'})$. Therefore,

$$T : L^p(X, w) \to L^p(X, w) \quad \text{for all } w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})'} \tag{2.7}$$

if and only if

$$T^* : L^{p'}(X, w) \to L^{p'}(X, w) \quad \text{for all } w \in A_{\frac{p'}{q_0'}} \cap RH_{(\frac{p_0'}{p'})'}. \tag{2.8}$$

The following result is a special case of interpolation with change of measures. It was proved in [35] and [36] when $X = \mathbb{R}^n$ is the Euclidean space.

Proposition 2.4. *Let $1 < r \leqslant q < \infty$ and let w_0 and w_1 be two positive weights. If T is a bounded linear operator acting on $L^r(X, w_0)$ and $L^q(X, w_1)$, then T is bounded on $L^p(X, w)$ for $r \leqslant p \leqslant q$ and $w = w_0^t w_1^{1-t}$, provided $t = \frac{q-p}{q-r}$ for $r \neq q$ and $0 \leqslant t \leqslant 1$ for $r = q$.*

Note that $w^r \in A_p$, $r \geqslant 1$, if and only if $w \in A_p$ and w satisfies $w \in RH_r$ and $w^{1-p'} = w^{-1/(p-1)} \in RH_r$ for $p > 1$; when $p = 1$, we only need $w \in RH_r$ (see pp. 351–352 of [23]).

3. General spectral multiplier theorems on weighted spaces

Let (X, d, μ) be a space of homogeneous type. Recall that D is the power that appeared in property (2.3) and n the dimension entering doubling volume condition (2.2).

Unless otherwise specified in the sequel we always assume that L is a non-negative self-adjoint operator on $L^2(X)$ and that the semigroup e^{-tL}, generated by $-L$ on $L^2(X)$, has the kernel $p_t(x, y)$ which satisfies the following Gaussian upper bound

$$\left|p_t(x, y)\right| \leqslant \frac{C}{V(x, t^{1/m})} \exp\left(-\frac{d(x, y)^{m/(m-1)}}{ct^{1/(m-1)}}\right) \tag{GE}$$

for all $t > 0$, and $x, y \in X$, where C, c and m are positive constants and $m \geqslant 2$.

Such estimates are typical for elliptic or sub-elliptic differential operators of order m (see for instance, [9,14,27,28] and [39]).

Theorems 3.1, 3.2 and 3.3 below are the main new results obtained in this paper.

Theorem 3.1. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound (GE). Let $s > \frac{n}{2}$ and let $r_0 = \max(1, \frac{2(n+D)}{2s+D})$. Assume that for any $R > 0$ and all Borel functions F such that $\operatorname{supp} F \subseteq [0, R]$,*

$$\int_X \left|K_{F(\sqrt[m]{L})}(x, y)\right|^2 d\mu(x) \leqslant \frac{C}{V(y, R^{-1})} \|\delta_R F\|_{L^q}^2 \tag{3.1}$$

for some $q \in [2, \infty]$. Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\delta_t F\|_{W_s^q} < \infty$, the operator $F(L)$ is bounded on $L^p(X, w)$ for all p and w satisfying $r_0 < p < \infty$ and $w \in A_{\frac{p}{r_0}}$. In addition,

$$\left\|F(L)\right\|_{L^p(X,w)\to L^p(X,w)} \leqslant C_s\Big(\sup_{t>0} \|\eta\delta_t F\|_{W_s^q} + \left|F(0)\right|\Big).$$

Note that Gaussian bound (GE) implies estimates (3.1) for $q = \infty$. This means that one can omit condition (3.1) if the case $q = \infty$ is consider. We describe the details in Theorem 3.2 below.

Theorem 3.2. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound (GE). Let $s > \frac{n}{2}$ and let $r_0 = \max(1, \frac{2(n+D)}{2s+D})$. Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty$, the operator $F(L)$ is bounded on $L^p(X, w)$ for all p and w satisfying $r_0 < p < \infty$ and $w \in A_{\frac{p}{r_0}}$. In addition,*

$$\left\|F(L)\right\|_{L^p(X,w)\to L^p(X,w)} \leqslant C_s\Big(\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} + \left|F(0)\right|\Big).$$

Proof. Note that it was proved in Lemma 2.2 of [14], that for any Borel function F such that $\operatorname{supp} F \subset [0, R]$,

$$\begin{aligned}\left\|K_{F(\sqrt[m]{L})}(\cdot, y)\right\|_{L^2(X)}^2 &= \left\|K_{\overline{F}(\sqrt[m]{L})}(y, \cdot)\right\|_{L^2(X)}^2 \\ &\leqslant \frac{C}{V(y, R^{-1})} \|F\|_{L^\infty}^2\end{aligned} \tag{3.2}$$

where $\overline{F}$ denotes the complex conjugate of F.

This shows that estimate (3.1) always holds for $q=\infty$, and Theorem 3.2 follows from Theorem 3.1. □

From the point of view of some applications of spectral multipliers the sharp results and the required number of derivatives are not essential for the final outcome, see for example [37]. For this kind of applications Theorem 3.2 is the best solution because using it one does not have to consider or prove condition (3.1). Nevertheless, Theorem 3.1 and condition (3.1) are of significant interest independent of their applications. In the case of standard Laplace operator condition (3.1) is equivalent with $(1,2)$ restriction theorem and both Theorem 3.1 and condition (3.1) are a new part of Bochner–Riesz analysis. Estimates (3.1) are also closely related to Strichartz and other dissipative type estimates. For further discussion of condition (3.1), see also [14].

It is not difficult to see that condition (3.1) with some $q<\infty$ implies that the set of point spectrum of the considered operator is empty because the L^q norm of characteristic function of any singleton subset of $\mathbb{R}$ is zero. Hence if $q<\infty$ then $F(\sqrt[m]{L})$ does not depend on the value of $F(0)$ because then the point spectrum is empty and the spectral projection on zero eigenvalue $E(\{0\})=0$. Therefore if $q<\infty$ then one can skip $|F(0)|$ in the concluding estimates of Theorem 3.1. See [14, (3.3)] for more detailed explanation.

The fact that the point spectrum of the considered operator is empty implies also that for elliptic operators on compact manifolds condition (3.1) cannot hold for any $q<\infty$. To be able to study these operators as well, similarly as in [7,14] we introduce some variation of condition (3.1). For a Borel function F such that $\operatorname{supp} F\subseteq[-1,2]$ we define the norm $\|F\|_{N,q}$ by the formula

$$\|F\|_{N,q}=\left(\frac{1}{3N}\sum_{\ell=1-N}^{2N}\sup_{\lambda\in[\frac{\ell-1}{N},\frac{\ell}{N})}\left|F(\lambda)\right|^q\right)^{1/q},$$

where $q\in[1,\infty)$ and $N\in\mathbb{Z}_+$. For $q=\infty$, we put $\|F\|_{N,\infty}=\|F\|_{L^\infty}$. It is obvious that $\|F\|_{N,q}$ increases monotonically in q.

The next theorem is a variation of Theorem 3.1. This variation can be used in case of operators with nonempty point spectrum, see also [7, Theorem 3.6] and [14, Theorem 3.2].

Theorem 3.3. *Assume that $\mu(X)<\infty$. Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound (GE). Let $s>\frac{n}{2}$ and let $r_0=\max(1,\frac{2(n+D)}{2s+D})$. Suppose that for any $N\in\mathbb{Z}_+$ and for all Borel functions F such that* $\operatorname{supp} F\subseteq[-1,N+1]$,

$$\int_X\left|K_{F(\sqrt[m]{L})}(x,y)\right|^2d\mu(x)\leqslant\frac{C}{V(y,N^{-1})}\|\delta_N F\|_{N,q}^2 \tag{3.3}$$

for some $q\geqslant 2$. Then for any bounded Borel function F such that $\sup_{t>1}\|\eta\delta_t F\|_{W_s^q}<\infty$, *the operator $F(L)$ is bounded on $L^p(X,w)$ for all p and w satisfying $r_0<p<\infty$ and $w\in A_{\frac{p}{r_0}}$. In addition,*

$$\|F(L)\|_{L^p(X,w)\to L^p(X,w)}\leqslant C_s\Big(\sup_{t>1}\|\eta\delta_t F\|_{W_s^q}+\|F\|_{L^\infty}\Big).$$

We will discuss the proofs of Theorems 3.1 and 3.3 in Section 4. These results have the following corollary.

Corollary 3.4. *Let* $s > \frac{n}{2}$ *and let* $r_0 = \max(1, \frac{2(n+D)}{2s+D})$ *and* $\frac{1}{r_0} + \frac{1}{r_0'} = 1$. *Suppose in addition that* $1 < p < r_0'$ *and* $w \in A_p \cap RH_{(\frac{r_0'}{p})'}$.

(a) *Assume also that the operator* L *satisfies the assumptions of Theorem* 3.1 *for some* $2 \leqslant q \leqslant \infty$, *then*

$$\|F(L)\|_{L^p(X,w)\to L^p(X,w)} \leqslant C_s\Big(\sup_{t>0}\|\eta\delta_t F\|_{W^q_s} + |F(0)|\Big).$$

(b) *Alternatively assume in addition that the operator* L *satisfies the assumptions of Theorem* 3.3 *for some* $2 \leqslant q \leqslant \infty$, *then*

$$\|F(L)\|_{L^p(X,w)\to L^p(X,w)} \leqslant C_s\Big(\sup_{t>1}\|\eta\delta_t F\|_{W^q_s} + \|F\|_{L^\infty}\Big).$$

Proof. Suppose $1 < p < r_0'$ and $w \in A_p \cap RH_{(\frac{r_0'}{p})'}$. We have that $w^{-\frac{1}{p-1}} \in A_{\frac{p'}{r_0}}$. Then for $f \in L^\infty_c(X)$ (i.e. bounded with compact support),

$$\|F(L)f\|_{L^p(X,w)} = \left|\int_X F(L)f(x)\overline{g(x)}\,d\mu(x)\right|,$$

where the supremum is taken over all functions $g \in L^\infty_c(X)$ such that $\|g\|_{L^{p'}(X,w^{-\frac{1}{p-1}})} = 1$. Let $\overline{F}(L)$ be the operator with multiplier $\overline{F}$, the complex conjugate of F. Then $\overline{F}$ satisfies the same estimates as F, and we have

$$\begin{aligned}\|F(L)f\|_{L^p(X,w)} &= \sup\left|\int_X f(x)\overline{F}(L)g(x)\,d\mu(x)\right| \\ &\leqslant \sup\|f\|_{L^p(X,w)}\big\|\overline{F}(L)g\big\|_{L^{p'}(X,w^{-\frac{1}{p-1}})} \\ &\leqslant C\|f\|_{L^p(X,w)}\end{aligned}$$

since $p' > r_0$, and we can apply Theorems 3.1 or 3.3 to the weight $w^{-\frac{1}{p-1}} \in A_{\frac{p'}{r_0}}$. □

Remarks. 1) Note that Theorems 3.1 and 3.3 imply the main results obtained in [14]. Indeed the trivial weight $w = 1$ is in all A_p classes, so under the assumptions of Theorems 3.1 and 3.3 the operator $F(L)$ is bounded on all L^p spaces $1 < p < \infty$. Note that for $p < 2$, L^p boundedness of $F(L)$ follows by considering the adjoint operator $F(L)^* = \overline{F}(L)$. Similarly to the results in [14] the important point of this paper is that if one can obtain (3.1) or (3.3) for some $q < \infty$ then one can prove stronger multiplier results than in case $q = \infty$. The estimates (3.1) for $q = \infty$ are not necessary because estimates (3.1) with $q = \infty$ follow from Gaussian bound assumption (*GE*), see Theorem 3.2. If one has (3.1) or (3.3) for $q = 2$, then this implies the sharp weighted Hörmander-type multiplier result. Actually, we believe that to obtain any sharp weighted Hörmander-type multiplier theorem one has to investigate conditions of the same type as (3.1) or (3.3), i.e. conditions which allow us to estimate the norm $\|K_{F\sqrt[m]{L}}(\cdot,y)\|^2_{L^2(X,\mu)}$ in terms of some kind of L^q norm of the function F.

2) We call hypothesis (3.1) or (3.3) the Plancherel estimates or the Plancherel conditions. For the standard Laplace operator on Euclidean spaces $\mathbb{R}^n$, this is equivalent to $(1,2)$ Stein–Tomas restriction theorem (which is also the Plancherel estimate of the Fourier transform). Assumption that $q \geqslant 2$ is not necessary in the proofs of Theorems 3.1 and 3.3. However we do not expect that there are any examples where estimates (3.1) or (3.3) hold with $q<2$ because this would imply the Riesz summability for the index $\alpha<(n-1)/2$ which is false for the standard Laplace operator.

3) If we take $s>n/2$ in Theorems 3.1 and 3.3, then for every $w \in A_1 \cap RH_2$, the operator $F(L)$ maps $L^1(X,w)$ into $L^{1,\infty}(X,w)$, that is, there is a constant $C>0$, independent of f and λ, such that

$$w\left\{x \in X\colon \left|F(L)f(x)\right| > \lambda\right\} \leqslant \frac{C}{\lambda}\|f\|_{L^1(X,w)}, \quad \lambda > 0.$$

The proof follows from the line of Theorem 5.8 in [24], together with the proofs of Theorems 3.1 and 3.2 in [14], respectively. The details are left to the reader.

4. Proofs of Theorems 3.1 and 3.3

Recall that $B=B(x_B,r_B)$ is the ball of radius r_B centered at x_B. Given $\lambda>0$, we will write λB for the ball with the same center as B and with radius $r_{\lambda B}=\lambda r_B$. We set

$$U_0(B)=B, \quad \text{and} \quad U_j(B)=2^jB\backslash 2^{j-1}B \quad \text{for } j=1,2,\ldots. \tag{4.1}$$

As a preamble to the proof of Theorem 3.1, we record a useful auxiliary result. For a proof, see pp. 453–454, Lemma 4.3 of [14].

Lemma 4.1.

(a) *Suppose that L satisfies* (3.1) *for some $q \in [2,\infty]$ and that $R>0$, $s>0$. Then for any $\epsilon>0$, there exists a constant $C=C(s,\epsilon)$ such that*

$$\int_X \left|K_{F(\sqrt[m]{L})}(x,y)\right|^2\left(1+Rd(x,y)\right)^s d\mu(x) \leqslant \frac{C}{V(y,R^{-1})}\|\delta_R F\|^2_{W^q_{\frac{s}{2}+\epsilon}} \tag{4.2}$$

for all Borel functions F such that $\operatorname{supp} F \subseteq [R/4,R]$.

(b) *Suppose that L satisfies* (3.3) *for some $q \in [2,\infty]$ and that $N>8$ is a natural number. Then for any $s>0$, $\epsilon>0$ and function $\xi \in C_c^\infty([-1,1])$ there exists a constant $C=C(s,\epsilon,\xi)$ such that*

$$\int_X \left|K_{F*\xi(\sqrt[m]{L})}(x,y)\right|^2\left(1+Nd(x,y)\right)^s d\mu(x) \leqslant \frac{C}{V(y,N^{-1})}\|\delta_N F\|^2_{W^q_{\frac{s}{2}+\epsilon}} \tag{4.3}$$

for all Borel functions F such that $\operatorname{supp} F \subseteq [N/4,N]$.

Proof of Theorem 3.1. We fix s such that $s > \frac{n}{2}$, and thus $\frac{2(n+D)}{2s+D} < 2$. In this case, we take one parameter p_0 in the sequel such that p_0 belongs to the interval $(\max\{\frac{2(n+D)}{2s+D}, 1\}, 2)$. Let $M \in \mathbb{N}$ such that $M > s/m$, where m is the constant in (GE). We will show that for all balls $B \ni x$,

$$\left(\fint_B \left|F(L)\left(I - e^{-r_B^m L}\right)^M f\right|^{p_0} d\mu\right)^{1/p_0} \leqslant C\mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \tag{4.4}$$

for all $f \in L_c^\infty(X)$.

Let us prove (4.4). Observe that $\sup_{t>0} \|\eta\delta_t F\|_{W_s^p} \sim \sup_{t>0} \|\eta\,\delta_t G\|_{W_s^p}$ where $G(\lambda) = F(\sqrt[m]{\lambda})$. For this reason, we can replace $F(L)$ by $F(\sqrt[m]{L})$ in the proof. Notice that $F(\lambda) = F(\lambda) - F(0) + F(0)$ and hence

$$F\left(\sqrt[m]{L}\right) = \left(F(\cdot) - F(0)\right)\left(\sqrt[m]{L}\right) + F(0)I.$$

Replacing F by $F - F(0)$, we may assume in the sequel that $F(0) = 0$. Let $\varphi \in C_c^\infty(0,\infty)$ be a non-negative function satisfying $\operatorname{supp}\varphi \subseteq [\frac{1}{4}, 1]$ and $\sum_{\ell=-\infty}^{\infty} \varphi(2^{-\ell}\lambda) = 1$ for any $\lambda > 0$, and let φ_ℓ denote the function $\varphi(2^{-\ell}\cdot)$. Then

$$F(\lambda) = \sum_{\ell=-\infty}^{\infty} \varphi\left(2^{-\ell}\lambda\right)F(\lambda) = \sum_{\ell=-\infty}^{\infty} F^\ell(\lambda), \quad \forall\lambda \geqslant 0. \tag{4.5}$$

This decomposition implies that the sequence $\sum_{\ell=-N}^{N} F^\ell(\sqrt[m]{L})$ converges strongly in $L^2(X)$ to $F(\sqrt[m]{L})$ (see for instance, Reed and Simon [29, Theorem VIII.5]). For every $\ell \in \mathbb{Z}$, $r > 0$ and $\lambda > 0$, we set

$$F_{r,M}(\lambda) = F(\lambda)\left(1 - e^{-(r\lambda)^m}\right)^M, \tag{4.6}$$

$$F_{r,M}^\ell(\lambda) = F^\ell(\lambda)\left(1 - e^{-(r\lambda)^m}\right)^M. \tag{4.7}$$

Given a ball $B \subset X$, we use the decomposition $f = \sum_{j=0}^{\infty} f_j$ in which $f_j = f\chi_{U_j(B)}$, and $U_j(B)$ were defined in (4.1). We may write

$$\begin{aligned} F\left(\sqrt[m]{L}\right)\left(1 - e^{-r_B^m L}\right)^M f &= F_{r_B,M}\left(\sqrt[m]{L}\right)f \\ &= \sum_{j=1}^{2} F_{r_B,M}\left(\sqrt[m]{L}\right)f_j + \lim_{N\to\infty} \sum_{\ell=-N}^{N} \sum_{j=3}^{\infty} F_{r_B,M}^\ell\left(\sqrt[m]{L}\right)f_j, \end{aligned} \tag{4.8}$$

where the sequence converges strongly in $L^2(X)$.

From Gaussian condition (GE), we have that for any $t > 0$, $\|e^{-tL}f\|_{L^p(X)} \leqslant C\|f\|_{L^p(X)}$. This, in combination with L^p-boundedness of the operator $F(\sqrt[m]{L})$ (see Theorem 3.1 [14]), gives that for all balls $B \ni x$,

$$\left(\fint_B \left|F_{r_B,M}\big(\sqrt[m]{L}\big)f_j\right|^{p_0} d\mu\right)^{1/p_0} \leqslant V(B)^{-1/p_0}\left\|F_{r_B,M}\big(\sqrt[m]{L}\big)f_j\right\|_{L^{p_0}(X)}$$
$$\leqslant CV(B)^{-1/p_0}\|f_j\|_{L^{p_0}(X)}$$
$$\leqslant C\mathcal{M}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x) \tag{4.9}$$

for $j = 1, 2$.

Fix $j \geqslant 3$. Let $p_1 \geqslant 2$ and $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$. By Hölder's inequality, we have that for all balls $B \ni x$,

$$\left(\fint_B \left|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)f_j\right|^{p_0} d\mu\right)^{1/p_0}$$
$$\leqslant V(B)^{-\frac{1}{p_1}}\left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)f_j\right\|_{L^{p_1}(B)}$$
$$\leqslant V(B)^{-\frac{1}{p_1}}\left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|_{L^{p_0}(U_j(B))\to L^{p_1}(B)}\|f_j\|_{L^{p_0}(X)}$$
$$\leqslant C2^{\frac{jn}{p_0}}V(B)^{\frac{1}{2}}\left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|_{L^{p_0}(U_j(B))\to L^{p_1}(B)}\mathcal{M}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x). \tag{4.10}$$

Let $\frac{1}{p_0} = \frac{\theta}{1} + \frac{1-\theta}{2}$ and $\frac{1}{p_1} = \frac{\theta}{2}$, that is $\theta = 2(\frac{1}{p_0} - \frac{1}{2})$. By interpolation,

$$\left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|_{L^{p_0}(U_j(B))\to L^{p_1}(B)}$$
$$\leqslant \left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|^{1-\theta}_{L^2(U_j(B))\to L^\infty(B)}\left\|\overline{F}^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|^{\theta}_{L^2(B)\to L^\infty(U_j(B))}. \tag{4.11}$$

Next we estimate $\|F^{\ell}_{r_B,M}(\sqrt[m]{L})\|_{L^2(U_j(B))\to L^\infty(B)}$. For every $\ell \in \mathbb{Z}$, let $K_{F^{\ell}_{r_B,M}(\sqrt[m]{L})}(y,z)$ be the Schwartz kernel of operator $F^{\ell}_{r_B,M}(\sqrt[m]{L})$. Then we have

$$\left\|F^{\ell}_{r_B,M}\big(\sqrt[m]{L}\big)\right\|^2_{L^2(U_j(B))\to L^\infty(B)}$$
$$= \sup_{y\in B}\int_{U_j(B)} \left|K_{F^{\ell}_{r_B,M}(\sqrt[m]{L})}(y,z)\right|^2 d\mu(z)$$
$$\leqslant C2^{-2sj}\big(2^{\ell}r_B\big)^{-2s}\sup_{y\in B}\int_X \left|K_{F^{\ell}_{r_B,M}(\sqrt[m]{L})}(y,z)\right|^2\big(1+2^{\ell}d(y,z)\big)^{2s}d\mu(z). \tag{4.12}$$

We then apply Lemma 4.1 with $F = F^{\ell}_{r_B,M}$ and $R = 2^{\ell}$ to obtain

$$\int_X \left|K_{F^{\ell}_{r_B,M}(\sqrt[m]{L})}(y,z)\right|^2\big(1+2^{\ell}d(y,z)\big)^{2s}d\mu(z) \leqslant \frac{C_s}{V(y,2^{-\ell})}\left\|\delta_{2^{\ell}}\big(F^{\ell}_{r_B,M}\big)\right\|^2_{W^q_s}. \tag{4.13}$$

Now for any Sobolev space $W_s^q(\mathbb{R})$, if k is an integer greater than s, then

$$\begin{aligned}\left\|\delta_{2^\ell}\left(F_{r_B,M}^\ell\right)\right\|_{W_s^q} &= \left\|\varphi(t)F\left(2^\ell t\right)\left(1-e^{-(2^\ell r_B t)^m}\right)^M\right\|_{W_s^q}\\ &\leqslant C\left\|\left(1-e^{-(2^\ell r_B t)^m}\right)^M\right\|_{C^k([\frac14,1])}\left\|\delta_{2^\ell}[\varphi_\ell F]\right\|_{W_s^q}\\ &\leqslant C\min\left\{1,\left(2^\ell r_B\right)^{mM}\right\}\left\|\delta_{2^\ell}[\varphi_\ell F]\right\|_{W_s^q}.\end{aligned}\tag{4.14}$$

Note that for all $y\in B$, $B\subset B(y,2r_B)$ so by (2.2)

$$\frac{1}{V(y,2^{-\ell})}\leqslant C\sup_{y\in B}\frac{V(y,2r_B)}{V(y,2^{-\ell})V(B)}\leqslant\frac{C}{V(B)}\max\left\{1,\left(2^\ell r_B\right)^n\right\}.\tag{4.15}$$

Hence by (4.14) and (4.15),

$$\begin{aligned}&\left\|F_{r_B,M}^\ell\left(\sqrt[m]{L}\right)\right\|_{L^2(U_j(B))\to L^\infty(B)}\\ &\quad\leqslant C\left(2^{-2sj}\left(2^\ell r_B\right)^{-2s}\min\left\{1,\left(2^\ell r_B\right)^{2mM}\right\}\max\left\{1,\left(2^\ell r_B\right)^n\right\}\frac{1}{V(B)}\right)^{1/2}\left\|\delta_{2^\ell}[\varphi_\ell F]\right\|_{W_s^q}.\end{aligned}\tag{4.16}$$

We now turn to estimate the term $\|\overline{F}_{r_B,M}^\ell(\sqrt[m]{L})\|_{L^2(B)\to L^\infty(U_j(B))}$. The calculations symmetric to (4.12), (4.13) and (4.14) with $\sup_{y\in B}$ replaced by $\sup_{z\in U_j(B)}$ yield,

$$\begin{aligned}&\left\|\overline{F}_{r_B,M}^\ell\left(\sqrt[m]{L}\right)\right\|_{L^2(B)\to L^\infty(U_j(B))}\\ &\quad\leqslant C\left(2^{-2sj}\left(2^\ell r_B\right)^{-2s}\min\left\{1,\left(2^\ell r_B\right)^{2mM}\right\}\sup_{z\in U_j(B)}\frac{1}{V(z,2^{-\ell})}\right)^{1/2}\left\|\delta_{2^\ell}[\varphi_\ell F]\right\|_{W_s^q}.\end{aligned}$$

Next by (2.2) and (2.3)

$$\begin{aligned}\sup_{z\in U_j(B)}\frac{1}{V(z,2^{-\ell})}&\leqslant C\sup_{z\in U_j(B)}\left(\frac{V(z,r_B)}{V(z,2^{-\ell})}\times\left(1+\frac{d(z,x_B)}{r_B}\right)^D\right)\frac{1}{V(x_B,r_B)}\\ &\leqslant C\frac{2^{jD}}{V(B)}\max\left\{1,\left(2^\ell r_B\right)^n\right\}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\overline{F}_{r_B,M}^\ell\left(\sqrt[m]{L}\right)\right\|_{L^2(B)\to L^\infty(U_j(B))}\\ &\quad\leqslant C\left(2^{-2sj}\left(2^\ell r_B\right)^{-2s}2^{jD}\min\left\{1,\left(2^\ell r_B\right)^{2mM}\right\}\max\left\{1,\left(2^\ell r_B\right)^n\right\}\frac{1}{V(B)}\right)^{1/2}\left\|\delta_{2^\ell}[\varphi_\ell F]\right\|_{W_s^q}.\end{aligned}\tag{4.17}$$

It then follows from estimates (4.16) and (4.17), in combination with (4.11) and (4.10) that

$$\left(\oint_B \left|F_{r,M}^{\ell}\left(\sqrt[m]{L}\right) f_j\right|^{p_0} d\mu\right)^{1/p_0}$$
$$\leqslant C 2^{-js+\frac{jn}{p_0}+\frac{jD\theta}{2}} \left(\left(2^{\ell} r_B\right)^{-s} \min\left\{1, \left(2^{\ell} r_B\right)^{mM}\right\} \max\left\{1, \left(2^{\ell} r_B\right)^{\frac{n}{2}}\right\}\right)$$
$$\times \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \sup_{\ell\in\mathbb{Z}} \left\|\delta_{2^{\ell}}[\varphi_{\ell} F]\right\|_{W_s^q}. \tag{4.18}$$

Therefore,

$$\sum_{j=3}^{\infty} \sum_{\ell=-\infty}^{\infty} \left(\oint_B \left|F_{r,M}^{\ell}\left(\sqrt[m]{L}\right) f_j\right|^{p_0} d\mu\right)^{1/p_0}$$
$$\leqslant C \sum_{j=3}^{\infty} 2^{-js+\frac{jn}{p_0}+\frac{jD\theta}{2}} \left(\sum_{\ell=-\infty}^{\infty} \left(2^{\ell} r_B\right)^{-s} \min\left\{1, \left(2^{\ell} r_B\right)^{mM}\right\} \max\left\{1, \left(2^{\ell} r_B\right)^{\frac{n}{2}}\right\}\right)$$
$$\times \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \sup_{\ell\in\mathbb{Z}} \left\|\delta_{2^{\ell}}[\varphi_{\ell} F]\right\|_{W_s^q}$$
$$\leqslant C \sum_{j=3}^{\infty} 2^{(\frac{n+D}{p_0}-(s+\frac{D}{2}))j} \left(\sum_{\ell:\, 2^{\ell} r_B>1} \left(2^{\ell} r_B\right)^{-s+\frac{n}{2}} + \sum_{\ell:\, 2^{\ell} r_B\leqslant 1} \left(2^{\ell} r_B\right)^{mM-s}\right)$$
$$\times \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \sup_{\ell\in\mathbb{Z}} \left\|\delta_{2^{\ell}}[\varphi_{\ell} F]\right\|_{W_s^q}$$
$$\leqslant C \sum_{j=3}^{\infty} 2^{(\frac{n+D}{p_0}-(s+\frac{D}{2}))j} \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \sup_{\ell\in\mathbb{Z}} \left\|\delta_{2^{\ell}}[\varphi_{\ell} F]\right\|_{W_s^q}$$
$$\leqslant C \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x) \sup_{\ell\in\mathbb{Z}} \left\|\delta_{2^{\ell}}[\varphi_{\ell} F]\right\|_{W_s^q}. \tag{4.19}$$

Here, the second inequality is obtained by using condition $\theta = 2(\frac{1}{p_0} - \frac{1}{2})$, and the third inequality follows from the convergence of power series with common ratio $1/2$. In the last inequality we have used the fact that $p_0 > \frac{2(n+D)}{2s+D}$.

Combining estimates (4.9) and (4.19), we have therefore proved (4.4), and then estimate (2.4) holds for $T = F(L)$ and $A_{r_B} = I - (I - e^{-r_B^m L})^M$. Note also that estimate (2.5) always holds for $A_{r_B} = I - (I - e^{-r_B^m L})^M$. Indeed note that $T = F(L)$ and $A_{r_B} = I - (I - e^{-r_B^m L})^M$ commutes so it is enough to show that

$$\|A_{r_B} f\|_{L^\infty(B)} \leqslant C \mathcal{M}\left(|f|^{p_0}\right)^{\frac{1}{p_0}}(x).$$

It is not difficult to see that it is enough to prove the above inequality for $p_0 = 1$. However $A_{r_B} = I - (I - e^{-r_B^m L})^M$ is a finite linear combination of the terms $e^{-jr_B^m L}$, $j = 1, \dots, M$, which all satisfy Gaussian bounds and the above inequality and in turn (2.5) follows from that observation.

It then follows from Theorem 2.3 that for all $p > p_0 > r_0 = \frac{2(n+D)}{2s+D}$, the operator $F(L)$ is bounded on $L^p(X,w)$ provided that $w \in A_{\frac{p}{p_0}}$. On the other hand, we note that

$$A_{\frac{p}{r_0}} = \bigcup_{p_0 > r_0} A_{\frac{p}{p_0}}.$$

This implies for all $p > r_0$ and all $w \in A_{\frac{p}{r_0}}$, that the operator $F(L)$ is bounded on $L^p(X,w)$. □

Proof of Theorem 3.3. Note that the condition $\mu(X) < \infty$ implies that X is bounded. Hence $X = B(x_0, r_0)$ for some $x_0 \in X$ and $0 < r_0 < \infty$ [24]. It follows from condition (2.3) that for any $x \in X$, $V(x_0, 1) \leqslant C(1 + d(x, x_0))^D V(x, 1) \leqslant CV(x, 1)$. This shows that for any $x, y \in X$, $|K_{e^{-L}}(x, y)| \leqslant CV(x_0, 1)^{-1}$. As a consequence,

$$\max\{\|e^{-L}\|_{L^1(X)\to L^2(X)}, \|e^{-L}\|_{L^2(X)\to L^\infty(X)}\} \leqslant C. \tag{4.20}$$

On the other hand, for any bounded Borel function F such that $\operatorname{supp} F \subseteq [0, 16]$, the operator $F(\sqrt[m]{L})e^{2L}$ is bounded on $L^2(X)$. This, together with (4.20), yields

$$\begin{aligned}\|F(\sqrt[m]{L})\|_{L^1(X)\to L^\infty(X)} &= \|e^{-L}(F(\sqrt[m]{L})e^{2L})e^{-L}\|_{L^1(X)\to L^\infty(X)} \\ &\leqslant \|e^{-L}\|_{L^1(X)\to L^2(X)}\|F(\sqrt[m]{L})e^{2L}\|_{L^2(X)\to L^2(X)}\|e^{-L}\|_{L^2(X)\to L^\infty(X)} \\ &\leqslant C\|F\|_{L^\infty} < \infty.\end{aligned}$$

This implies that the kernel $K_{F(\sqrt[m]{L})}(x, y)$ of the operator $F(\sqrt[m]{L})$ satisfies

$$\sup_{y\in X}|K_{F(\sqrt[m]{L})}(x, y)| \leqslant C < \infty.$$

Hence, for any $x \in X$,

$$\begin{aligned}|F(\sqrt[m]{L})f(x)| &= \left|\int_X K_{F(\sqrt[m]{L})}(x, y)f(y)\,d\mu(y)\right| \\ &\leqslant C\int_X |f(y)|\,d\mu(y) \\ &\leqslant C\mathcal{M}(f)(x),\end{aligned}$$

and for any $1 < p < \infty$ and $w \in A_p$,

$$\|F(\sqrt[m]{L})f\|_{L^p(X,w)} \leqslant C\|\mathcal{M}(f)\|_{L^p(X,w)} \leqslant C\|f\|_{L^p(X,w)}.$$

Therefore, in order to prove Theorem 3.3, we can assume that $\operatorname{supp} F \subset [8, \infty]$. Following the proof of Theorem 3.1, we set $F^\ell(\lambda) = \varphi(2^{-\ell}\lambda)F(\lambda)$, and

$$\tilde{F} = \sum_{\ell=3}^{\infty} F^\ell * \xi,$$

where ξ is a function defined in (b) of Lemma 4.1.

By repeating the proof of Theorem 3.1 and using (4.3) in place of (4.2) we can prove that the operator $\tilde{F}(\sqrt[m]{L})$ is bounded on $L^p(X,w)$ for all p and w satisfying (i) and (ii) in Theorem 3.3. To prove Theorem 3.3, it follows by Theorem 2.3 again that it suffices to show that for all balls $B \ni x$,

$$\left(\oint_B \big|\big(F(\sqrt[m]{L}) - \tilde{F}(\sqrt[m]{L})\big)\big(I - e^{-r_B^m L}\big)^M f\big|^{p_0}\, d\mu\right)^{1/p_0} \leqslant C\mathcal{M}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x) \tag{4.21}$$

for all $f \in L_c^\infty(X)$.

Let us prove (4.21). For every $\ell \geqslant 3$ and $r > 0$, we set $H_{r,M}^\ell(\lambda) = (F^\ell(\lambda) - F^\ell * \xi(\lambda))(1 - e^{-(r\lambda)^m})^M$, $\lambda > 0$. (Note that $\operatorname{supp} H_{r_B,M}^\ell \subseteq [0, 2^\ell + 1]$.) Now for a given ball $B \subset X$, we put $f = \sum_{j=0}^\infty f_j$, where $f_j = f\chi_{U_j(B)}$, and $U_j(B)$ were defined in (4.1). We may write

$$\begin{aligned}\big(F(\sqrt[m]{L}) - \tilde{F}(\sqrt[m]{L})\big)\big(I - e^{-r_B^m L}\big)^M f &= \sum_{j=1}^{2}\big(F(\sqrt[m]{L}) - \tilde{F}(\sqrt[m]{L})\big)\big(I - e^{-r_B^m L}\big)^M f_j \\ &\quad + \lim_{N\to\infty}\sum_{\ell=3}^{N}\sum_{j=3}^{\infty} H_{r_B,M}^\ell(\sqrt[m]{L}) f_j. \end{aligned}\tag{4.22}$$

A similar argument as in the proof of Theorem 3.1 gives the desired estimates for $j = 1, 2$. Next, fix $j \geqslant 3$. For every $\ell \geqslant 3$, let $K_{H_{r_B,M}^\ell(\sqrt[m]{L})}(y,z)$ be the Schwartz kernel of operator $H_{r_B,M}^\ell(\sqrt[m]{L})$. Let $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$, and denote by $\frac{1}{p_0} = \frac{\theta}{1} + \frac{1-\theta}{2}$ and $\frac{1}{p_1} = \frac{\theta}{2}$, that is $\theta = 2(\frac{1}{p_0} - \frac{1}{2})$. Following (4.10) and (4.11), we use Hölder's inequality and interpolation again to obtain that for all balls $B \ni x$,

$$\begin{aligned}&\left(\oint_B \big|H_{r_B,M}^\ell(\sqrt[m]{L}) f_j\big|^{p_0}\, d\mu\right)^{1/p_0} \\ &\quad \leqslant C 2^{\frac{jn}{p_0}} V(B)^{\frac{1}{2}}\mathcal{M}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x)\big\|H_{r_B,M}^\ell(\sqrt[m]{L})\big\|_{L^2(U_j(B))\to L^\infty(B)}^{1-\theta} \\ &\qquad \times \big\|\overline{H}_{r_B,M}^\ell(\sqrt[m]{L})\big\|_{L^2(B)\to L^\infty(U_j(B))}^{\theta}. \end{aligned}\tag{4.23}$$

The Hölder inequality, together with condition that X is bounded give

$$\begin{aligned}&\big\|H_{r_B,M}^\ell(\sqrt[m]{L})\big\|_{L^2(U_j(B))\to L^\infty(B)}^2 \\ &\quad = \sup_{y\in B}\int_{U_j(B)} \big|K_{H_{r_B,M}^\ell(\sqrt[m]{L})}(y,z)\big|^2\, d\mu(z) \\ &\quad \leqslant C\big(2^j r_B\big)^{-2s}\sup_{y\in B}\int_X \big|K_{H_{r_B,M}^\ell(\sqrt[m]{L})}(y,z)\big|^2 d(y,z)^{2s}\, d\mu(z) \end{aligned}$$

$$\leqslant C_X\big(2^j r_B\big)^{-2s}\sup_{y\in B}\int_X \big|K_{H^\ell_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2\,d\mu(z)$$

$$\leqslant \sup_{y\in B}\frac{C_X}{V(y,2^{-\ell})}\big(2^j r_B\big)^{-2s}\big\|\delta_{2^\ell}\big(H^\ell_{r_B,M}\big)\big\|^2_{2^\ell,q}, \tag{4.24}$$

where the last inequality follows from the fact that $\operatorname{supp} H^\ell_{r_B,M}\subseteq[0,2^\ell+1]$, and then from (3.3) with $N=2^\ell$, we have that

$$\int_X \big|K_{H^\ell_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2\,d\mu(z)\leqslant\frac{C}{V(y,2^{-\ell})}\big\|\delta_{2^\ell}\big(H^\ell_{r_B,M}\big)\big\|^2_{2^\ell,q}.$$

From the expression $H^\ell_{r_B,M}(\lambda)=(F^\ell(\lambda)-F^\ell*\xi(\lambda))(1-e^{-(r_B\lambda)^m})^M$, one obtains

$$\begin{aligned}\big\|\delta_{2^\ell}\big(H^\ell_{r_B,M}\big)\big\|_{2^\ell,q}&=\big\|\delta_{2^\ell}\big[F^\ell(\lambda)-F^\ell*\xi(\lambda)\big]\big(1-e^{-(2^\ell r_B\lambda)^m}\big)^M\big\|_{2^\ell,q}\\&\leqslant C\min\big\{1,\big(2^\ell r_B\big)^{mM}\big\}\big\|\delta_{2^\ell}\big[F^\ell(\lambda)-F^\ell*\xi(\lambda)\big]\big\|_{2^\ell,q}.\end{aligned}\tag{4.25}$$

Everything then boils down to estimating $\|\cdot\|_{2^\ell,q}$ norm of $\delta_{2^\ell}[F^\ell(\lambda)-F^\ell*\xi(\lambda)]$. We make the following claim. For its proof, we refer to p. 26, claim (3.29) of [7] or p. 459, Proposition 4.6 of [14].

Proposition 4.2. *Suppose that $\xi\in C_c^\infty$ is a function such that $\operatorname{supp}\xi\subset[-1,1]$, $\xi\geqslant 0$, $\hat{\xi}(0)=1$, $\hat{\xi}^{(\kappa)}(0)=0$ for all $1\leqslant\kappa\leqslant[s]+2$ and set $\xi_N(t)=N\xi(Nt)$. Assume also that $\operatorname{supp} G\subset[0,1]$. Then*

$$\|G-G*\xi_N\|_{N,q}\leqslant CN^{-s}\|G\|_{W^q_s}$$

for all $s>1/q$.

By Proposition 4.2

$$\big\|\delta_{2^\ell}\big[F^\ell(\lambda)-F^\ell*\xi(\lambda)\big]\big\|_{2^\ell,q}=\big\|\delta_{2^\ell}[\varphi_\ell F]-\xi_{2^\ell}*\delta_{2^\ell}[\varphi_\ell F]\big\|_{2^\ell,q}\leqslant C2^{-\ell s}\big\|\delta_{2^\ell}[\varphi_\ell F]\big\|_{W^q_s},$$

and thus

$$\big\|\delta_{2^\ell}\big(H^\ell_{r_B,M}\big)\big\|_{2^\ell,q}\leqslant C2^{-\ell s}\min\big\{1,\big(2^\ell r_B\big)^{mM}\big\}\big\|\delta_{2^\ell}[\varphi_\ell F]\big\|_{W^q_s}. \tag{4.26}$$

Substituting (4.26) back into (4.24), we then use the doubling property (2.2) to obtain

$$\begin{aligned}&\big\|H^\ell_{r_B,M}\big(\sqrt[m]{L}\big)\big\|^2_{L^2(U_j(B))\to L^\infty(B)}\\&\quad\leqslant\frac{C}{V(B)}2^{-2sj}\big(2^\ell r_B\big)^{-2s}\min\big\{1,\big(2^\ell r_B\big)^{2mM}\big\}\max\big\{1,\big(2^\ell r_B\big)^n\big\}\big\|\delta_{2^\ell}[\varphi_\ell F]\big\|^2_{W^q_s},\end{aligned}$$

which is exactly the same estimate as in (4.16).

Following the proof of Theorem 3.1, an argument as above shows the same estimate (4.17) for the term $\|\overline{H}^{\ell}_{r_B,M}(\sqrt[m]{L})\|_{L^2(B)\to L^\infty(U_j(B))}$. The rest of the proof of (4.21) is just a repetition of the proof of Theorem 3.1, so we skip it. Hence, we complete the proof of Theorem 3.3 when X has a finite measure, i.e., $\mu(X)<\infty$. □

5. Two interpolation results

In this section we continue to assume that L is a non-negative self-adjoint operator on $L^2(X)$, which has a kernel $p_t(x,y)$ satisfying a Gaussian upper bound (GE). Using interpolation, other conditions on the weight can be found which guarantee that $F(L)$ is a bounded operator. We first prove the following result.

Theorem 5.1. *Let $s>\frac{n}{2}$ and let $r_0=\max\{\frac{2(n+D)}{2s+D},1\}$. Suppose that the operator L satisfies condition* (3.1) *with some $q\in[2,\infty]$. If $1<p<\infty$ and $w^{r_0}\in A_p$, then for any bounded Borel function F such that $\sup_{t>0}\|\eta\delta_t F\|_{W^q_s}<\infty$, the operator $F(L)$ is bounded on $L^p(X,w)$. Moreover,*

$$\|F(L)\|_{L^p(X,w)\to L^p(X,w)}\leqslant C_s\Big(\sup_{t>0}\|\eta\delta_t F\|_{W^q_s}+|F(0)|\Big).$$

Proof. We will derive Theorem 5.1 from Theorem 3.1 by using Proposition 2.4 and the characterization of A_p functions that if $w\in A_p$, then there are A_1 weights u and v such that $w=uv^{1-p}$ [22].

Following the proof of Theorem 2 of [23], we fix p, $1<p<\infty$, and w so that $w^{r_0}\in A_p$ where $r_0=\frac{2(n+D)}{2s+D}$. We have that $w^{r_0}=uv^{1-p}$, $u,v\in A_1$, or $w=u^{r_0^{-1}}v^{\frac{1-p}{r_0}}$. Next, write this as

$$w=u^{r_0^{-1}}v^{\frac{1-p}{r_0}}=\big(u^\alpha v^\beta\big)^t\big(u^\gamma v^\delta\big)^{1-t}=w_0^t w_1^{1-t},$$

in which

$$\alpha t+\gamma(1-t)=r_0^{-1}, \tag{5.1}$$

$$\beta t+\delta(1-t)=r_0^{-1}(1-p). \tag{5.2}$$

Then in order to use Proposition 2.4 for weights which satisfy Theorem 3.1, we require

$$w_0^{-\frac{1}{r-1}}\in A_{\frac{r'}{r_0}},\quad 1<r<\min\{r_0',p\}, \tag{5.3}$$

$$w_1\in A_{\frac{q}{r_0}},\quad q>\max\{r_0,p\}, \tag{5.4}$$

$$t=\frac{q-p}{q-r}. \tag{5.5}$$

Recall that $u\in A_1$ (similarly $v\in A_1$) implies

$$\fint_B u\leqslant Cu(x)\quad\text{for almost all } x\in B.$$

Therefore, if $\alpha > 0$ and $\beta < 0$, letting $s = \frac{r'}{r_0}$, we have

$$\begin{aligned}
&\left(\fint_B w_0^{-\frac{1}{r-1}}\right)\left(\fint_B w_0^{\frac{1}{(r-1)(s-1)}}\right)^{s-1} \\
&\quad\leqslant \left(\fint_B u^{-\frac{\alpha}{r-1}} v^{-\frac{\beta}{r-1}}\right)\left(\fint_B u^{\frac{\alpha}{(r-1)(s-1)}} v^{\frac{\beta}{(r-1)(s-1)}}\right)^{s-1} \\
&\quad\leqslant \left(\fint_B u\right)^{-\frac{\alpha}{r-1}}\left(\fint_B v^{-\frac{\beta}{r-1}}\right)\left(\fint_B v\right)^{\frac{\beta}{r-1}}\left(\int_B u^{\frac{\alpha}{(r-1)(s-1)}}\right)^{s-1} \\
&\quad\leqslant C,
\end{aligned}$$

if

$$\alpha = (r-1)\left(\frac{r'}{r_0} - 1\right) = \frac{r}{r_0} - r + 1 \quad \text{and} \quad \beta = -(r-1);$$

this is $w_0^{-\frac{1}{r-1}} \in A_{\frac{r'}{r_0}}$ for these values of α and β. Similarly, we can show $w_1 \in A_{\frac{q}{r_0}}$ if $\gamma = 1$ and $\delta = -(\frac{q}{r_0} - 1)$. Using these values of α and γ, we have (5.1) if $t = \frac{1}{r.}$ Next, solving (5.2) for q, we get $q = r'(p-1)$. This value of q also satisfies (5.5). Therefore, if we choose $r < \min\{r_0', p\}$ close enough to 1 so that $q = r'(p-1) > \max\{r_0, p\}$, then (5.1)–(5.5) hold. This proves Theorem 5.1. □

If $X = \mathbb{R}^n$ then Theorem 5.1 can be strengthen for the following polynomial weights. When $w(x) = |x|^\beta$, we have $w \in A_p$ if $-n < \beta < n(p-1)$. Applying Theorem 3.1 and Theorem 3 of [23] to such w and using interpolation with change of measures, we have the following theorem.

Theorem 5.2. *Let $s > \frac{n}{2}$. Suppose that the operator L satisfies condition* (3.1) *with some $q \in [2, \infty]$. If $1 < p < \infty$ and $\max\{-n, -sp\} < \beta < \min\{n(p-1), sp\}$, then for any bounded Borel function F such that $\sup_{t>0} \|\eta\delta_t F\|_{W_s^q} < \infty$, the operator $F(L)$ is bounded on $L^p(\mathbb{R}^n, |x|^\beta)$. In addition,*

$$\|F(L)\|_{L^p(\mathbb{R}^n,|x|^\beta)\to L^p(\mathbb{R}^n,|x|^\beta)} \leqslant C_s\Big(\sup_{t>0}\|\eta\delta_t F\|_{W_s^q} + |F(0)|\Big).$$

In particular, if $s < n$ and $\frac{n}{s} < p < (\frac{n}{s})'$, we get $-n < \beta < n(p-1)$; we may also take $p = \frac{n}{s}$ and $p = (\frac{n}{s})'$.

Proof. The proof of Theorem 5.2 can be obtained by making minor modifications with the proof of Theorem 3 in [23] and using Theorem 3.1. We give a brief argument of this proof for completeness and convenience of the reader.

Notice that $-n \geqslant -sp$ if $n/s \leqslant p$, and $n(p-1) \leqslant sp$ if $p \leqslant (n/s)'$. Therefore, for $s < n$ the conclusion of Theorem 5.2 can be divided into three cases:

$$1 < p < \frac{n}{s} \quad \text{and} \quad -sp < \beta < n(p-1), \tag{5.6}$$

$$\frac{n}{s} \leqslant p \leqslant \left(\frac{n}{s}\right)' \quad \text{and} \quad -n < \beta < n(p-1), \tag{5.7}$$

$$\left(\frac{n}{s}\right)' < p < \infty \quad \text{and} \quad -n < \beta < sp. \tag{5.8}$$

Since (5.8) is the dual of (5.6), we need only to concern ourselves with (5.6) and (5.7).

Because $|x|^\beta \in A_p$ if and only if $-n < \beta < n(p-1)$, it follows from Theorem 3.1 that for $s < n$, $F(L)$ is bounded on $L^p(\mathbb{R}^n, |x|^\beta)$ if

$$\frac{n}{s} \leqslant p < \infty \quad \text{and} \quad -n < \beta < ps - n, \tag{5.9}$$

$$1 < p \leqslant \left(\frac{n}{s}\right)' \quad \text{and} \quad -n + p(n-s) < \beta < n(p-1). \tag{5.10}$$

However, combining (5.9) and (5.10), we have (5.7) and are left with only proving (5.6).

Let $q = n/s$ and $r < n/s$; then also $r < (n/s)'$. By (5.10) and (5.7), $F(L)$ is bounded on $L^p(\mathbb{R}^n, |x|^{\beta_0})$ and $L^p(\mathbb{R}^n, |x|^{\beta_1})$ for $-n + r(n-s) < \beta_0 < n(r-1)$ and $-n < \beta_1 < n(q-1)$. Using Proposition 2.4, if $r < p < q$ we see that $F(L)$ is bounded on $L^p(\mathbb{R}^n, |x|^\beta)$ for

$$\beta = \beta_0\left(\frac{q-p}{q-r}\right) + \beta_1\left(\frac{p-r}{q-r}\right).$$

Thus β satisfies

$$\{-n + r(n-s)\}\left(\frac{q-p}{q-r}\right) - n\left(\frac{p-r}{q-r}\right) < \beta < n(r-1)\left(\frac{q-p}{q-r}\right) + n(q-1)\left(\frac{p-r}{q-r}\right).$$

Simplifying and using the fact that $q = n/s$, we get

$$\frac{n^2(r-1)}{n-sr} + \frac{psr(s-n)}{n-sr} < \beta < n(p-1). \tag{5.11}$$

But, as $r \to 1$, the left-hand side of (5.11) tends to $-sp$. So, taking r sufficiently close to 1 allows us to choose any β satisfying $-sp < \beta < n(p-1)$.

When $s = n$, the restriction in Theorem 5.2 is $-n < \beta < n(p-1)$ for $1 < p < \infty$. But, when $s = n$ in Theorem 3.1, we require $w \in A_p$, and $|x|^\beta \in A_p$ if $-n < \beta < n(p-1)$. This completes the proof of Theorem 5.2. □

Note that in the case of Fourier spectral multipliers Theorem 5.2 is best possible, except for endpoint equalities for β see [23, pp. 360–361].

6. Applications

6.1. Homogeneous groups

Let $\mathbf{G}$ be a Lie group of polynomial growth and let $X_1, \ldots, X_k$ be a system of left-invariant vector fields on $\mathbf{G}$ satisfying the Hörmander condition. We define the Laplace operator L acting on $L^2(\mathbf{G})$ by the formula

$$L = -\sum_{i=1}^{k} X_i^2. \tag{6.1}$$

If $B(x,r)$ is the ball defined by the distance associated with system $X_1, \ldots, X_k$ (see e.g. Chapter III.4 [39]), then there exist natural numbers $n_0, n_\infty \geqslant 0$ such that $V(x,r) \sim r^{n_0}$ for $r \leqslant 1$ and $V(x,r) \sim r^{n_\infty}$ for $r > 1$ (see e.g. Chapter III.2 [39]). Note that this implies that doubling condition (2.2) holds with the doubling dimension $n = \max\{n_0, n_\infty\}$. Note also that one can take $D = 0$ in the estimates (2.3). We call $\mathbf{G}$ a homogeneous group if there exists a family of dilations on $\mathbf{G}$. A family of dilations on a Lie group $\mathbf{G}$ is a one-parameter group $(\tilde{\delta}_t)_{t>0}$ $(\tilde{\delta}_t \circ \tilde{\delta}_t = \tilde{\delta}_{ts})$ of automorphisms of $\mathbf{G}$ determined by

$$\tilde{\delta}_t Y_j = t^{n_j} Y_j, \tag{6.2}$$

where $Y_1, \ldots, Y_\ell$ is a linear basis of Lie algebra of $\mathbf{G}$ and $n_j \geqslant 1$ for $1 \leqslant j \leqslant \ell$ (see [15]). We say that an operator L defined by (6.1) is homogeneous if $\tilde{\delta}_t X_i = t X_i$ for $1 \leqslant i \leqslant k$ and the system $X_1, \ldots, X_k$ satisfies the Hörmander condition. Then for the sub-Riemannian geometry corresponding to the system $X_1, \ldots, X_k$ one has $n_0 = n_\infty = \sum_{j=1}^{\ell} n_j$ (see [15]). Hence the doubling dimension is equal to $n = n_0 = n_\infty$.

Spectral multiplier theorems for the homogeneous Laplace operators acting on homogeneous groups were investigated by Hulanicki and Stein [20], Folland and Stein [15, Theorem 6.25], and De Michele and Mauceri [10]. See also [5] and [25]. We have the following weighted spectral multiplier result.

Proposition 6.1. *Let L be the homogeneous sub-Laplacian defined by the formula* (6.1) *acting on a homogeneous group* $\mathbf{G}$. *Then Theorem* 3.1 *holds for spectral multipliers $F(L)$ with $q = 2$, $D = 0$ and the doubling dimension given by $n = n_0 = n_\infty$.*

Proof. It is well known that the heat kernel corresponding to the operator L satisfies Gaussian bound (*GE*). It is also not difficult to check that for some constant $C > 0$

$$\|F(\sqrt{L})\|_{2\to\infty}^2 = C \int_0^\infty |F(t)|^2 t^{n-1}\, dt.$$

See for example Eq. (7.1) of [14] or [5, Proposition 10]. It follows from the above equality that the operator L satisfies estimate (3.1) with $q = 2$. Hence Theorem 3.1 holds for spectral multipliers $F(L)$ with $q = 2$, $D = 0$ and $n = n_0 = n_\infty$. □

This result can be extended to "quasi-homogeneous" operators acting on homogeneous groups, see [31] and [14].

In the setting of general Lie groups of polynomial growth, spectral multipliers were investigated by Alexopoulos. The following weighted spectral multiplier result extends Alexopoulos's unweighted result in [1].

Proposition 6.2. *Let L be a group invariant operator acting on a Lie group* **G** *of polynomial growth defined by* (6.1). *Then Theorem* 3.2 *holds for spectral multipliers $F(L)$ with the doubling dimension $n = \max\{n_0, n_\infty\}$ and $D = 0$.*

Proof. It is well known that the heat kernel corresponding to the operator L satisfies Gaussian bound (*GE*) so the operator L satisfies estimate (3.1) for $q = \infty$, see the proof of Theorem 3.2 above and Lemma 2.2 of [14]. Hence Theorem 3.2 holds for spectral multipliers $F(L)$. □

6.2. Compact manifolds

For a general non-negative self-adjoint elliptic operator on a compact manifold, Gaussian bound (*GE*) holds by general elliptic regularity theory. Further, one has the Avakumovič–Agmon–Hörmander theorem.

Theorem 6.3. *Let L be a non-negative elliptic pseudo-differential operator of order m on a compact manifold X of dimension n. Then*

$$\big\| \chi_{[R,R+1]}\big(L^{1/m}\big)\big\|^2_{L^1(X)\to L^2(X)} \leqslant CR^{n-1}, \quad \forall R \in \mathbb{R}^+. \tag{6.3}$$

Theorem 6.3 was proved by Hörmander [19]. This theorem has the following useful consequence.

Corollary 6.4. *Condition* (3.3) *with $q = 2$ holds for non-negative elliptic pseudo-differential operators on compact manifolds.*

Proof. By spectral theorem

$$\begin{aligned}\sup_{y\in X}\big\| K_{F(\sqrt[m]{L})}(\cdot, y)\big\|^2_{L^2(X)} &\leqslant \left(\sum_{\ell=1}^{N}\big\| \chi_{[\ell-1,\ell]}F\big(\sqrt[m]{L}\big)\big\|^2_{L^1(X)\to L^2(X)}\right)^{1/2}\\ &\leqslant CN^{n/2}\|\delta_N F\|_{N,2}\end{aligned}$$

as required. □

The importance of estimate (6.3) for multiplier theorems was noted by Sogge [33], who used it to establish the convergence of the Riesz means up to the critical exponent $(n-1)/2$ (see also [6] and [30]).

Proposition 6.5. *Suppose that L is a non-negative self-adjoint elliptic differential operator of order $m \geqslant 2$ acting on a compact Riemannian manifold X of dimension n. Then the operator L*

satisfies estimate (3.3) *for* $q = 2$*, and hence Theorem* 3.3 *holds for spectral multipliers* $F(L)$ *under the same conditions with* $q = 2$*,* $D = 0$ *and with the doubling dimension* n*. That is the exponent* n *in* (2.2) *is equal to the topological dimension of the manifold* X*.*

Proof. This result is a direct consequence of Theorem 3.3 and Corollary 6.4. □

Proposition 6.5 applied to an elliptic operator on a compact Lie group gives a stronger result than Proposition 6.2. One can say that for elliptic operators on a compact Lie group Proposition 6.1 holds. However, we do not know if the Avakumovič–Agmon–Hörmander condition holds for sub-elliptic operators on a compact Lie group (see also [7]). Hence, Proposition 6.2 gives the strongest known result for sub-elliptic operators on a compact Lie group.

6.3. Laplace operators on irregular domains with Dirichlet boundary conditions

Let Ω be a connected open subset of $\mathbb{R}^n$. Note that if the boundary of Ω is not smooth enough, then Ω is not necessarily a homogeneous space because the doubling condition might not hold.

In this section we are interested in dealing with weighted norm estimates in those contexts. As it is pointed out in [13], one can extend the singular operators defined in Ω to the space $\mathbb{R}^n$. Since there is no assumption on the regularity of the kernels in space variables, the extension of the kernel still satisfies similar conditions. Given T, a bounded linear operator on $L^p(\Omega)$, $1 < p < \infty$, the extension of T to $\mathbb{R}^n$ is defined as $\widetilde{T} f(x) = T(f\chi_\Omega)(x)\chi_\Omega(x)$ for $f \in L^p(\mathbb{R}^n)$. Then, T is bounded on $L^p(\Omega)$ if and only if $\widetilde{T}$ is bounded on $L^p(\mathbb{R}^n)$. If K is the kernel of T, then the associated kernel of $\widetilde{T}$ is given by $\widetilde{K}(x, y) = K(x, y)$ for $(x, y) \in \Omega \times \Omega$ and $\widetilde{K}(x, y) = 0$ otherwise. As it is observed in [13], the assumptions on the kernels do not involve their regularity so they imply similar properties on the kernels of the extended operators.

We are going to use the notation $A_p(\mathbb{R}^n)$ in order to make clear that the Muckenhoupt weights are considered in the whole space $\mathbb{R}^n$. The following result gives examples of singular integral multipliers on spaces without the doubling condition.

Proposition 6.6. *Suppose that* Δ_Ω *is the Laplace operator with Dirichlet boundary condition* $\Omega \subset \mathbb{R}^n$*. Let* $s > n/2$ *and* $r_0 = \max(1, n/s)$*. Then for any bounded Borel function* F *such that* $\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty$*, the operator* $F(\Delta_\Omega)$ *is bounded on* $L^p(\Omega, w)$ *for all* p *and* w *satisfying* $r_0 < p < \infty$ *and* $w \in A_{p/r_0}(\mathbb{R}^n)$*. In addition,*

$$\big\|F(\Delta_\Omega)\big\|_{L^p(\Omega,w)\to L^p(\Omega,w)} \leqslant C_s\Big(\sup_{t>0}\|\eta\delta_t F\|_{W_s^\infty} + \big|F(0)\big|\Big).$$

Proof. Note that

$$0 \leqslant K_{\exp(-t\Delta_\Omega)}(x, y) \leqslant \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right)$$

(see e.g., Example 2.18 [9]). That is the heat kernels corresponding to Δ_Ω satisfy Gaussian bound (*GE*), and the operator Δ_Ω satisfies estimate (3.1) for $q = \infty$. Then, Proposition 6.6 follows from estimate (3.2) and Theorem 3.2 applied to the extended operator $\widetilde{F(\Delta_\Omega)}$. Hence the same weighted norm estimates hold for the original operator $F(\Delta_\Omega)$. □

6.4. Schrödinger operators

In this section we discuss applications of our main results to spectral multipliers of Schrödinger operators.

Let Δ be the standard Laplace operator acting on $\mathbb{R}^n$. We consider the Schrödinger operator $L = -\Delta + V$ where $V : \mathbb{R}^n \to \mathbb{R}$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \geqslant 0$. The operator L is defined by the quadratic form. If $p_t(x, y)$ denotes the heat kernel corresponding to L then as a consequence of the Trotter product formula

$$0 \leqslant p_t(x, y) \leqslant \tilde{p}_t(x, y), \tag{6.4}$$

where $\tilde{p}_t(x, y)$ denotes the standard Gauss heat kernel corresponding to Δ, see also [27, Section 2.3].

The estimate (6.4) holds also for heat kernel $p_t(x, y)$ of Schrödinger operator with electromagnetic potentials, see [32, Theorem 2.3] and [14, (7.9)]. For the Schrödinger operator in this setting, estimate (3.1) holds for $q = \infty$ as in the next result.

Proposition 6.7. *Assume that $L = -\Delta + V$ where Δ is the standard Laplace operator acting on $\mathbb{R}^n$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a non-negative function. Then the operator L satisfies estimate* (3.1) *for $q = \infty$, and hence Theorem* 3.2 *holds for spectral multipliers $F(L)$ under the same conditions with $q = \infty$, $D = 0$ and the doubling constant n.*

We note that under suitable additional assumptions this result can be extended by a similar proof to situation of magnetic Schrödinger operators acting on a complete Riemannian manifold with non-negative potentials.

Proof. This result is a consequence of (6.4) and Theorem 3.2. □

6.5. Estimates on operator norms of holomorphic functional calculi

For $\theta > 0$, we put $\sum_\theta = \{z \in \mathbf{C} - \{0\}\colon |\arg z| < \theta\}$. Let F be a bounded holomorphic function on $\sum_\theta$. By $\|F\|_{\theta,\infty}$ we denote the supremum of F on $\sum_\theta$. We are interesting in finding sharp bounds, in terms of θ, of the norm of $F(L)$ as the operator acting on $L^p(X, w)$. The following proposition, which is a weighted version of [14, Proposition 8.1], is a consequence of Theorem 3.2.

Proposition 6.8. *Let L be an operator satisfying assumptions of Theorem* 3.2. *Let $s > \frac{n}{2}$ and let $r_0 = \max\{1, \frac{2(n+D)}{2s+D}\}$. Then the operator $F(L)$ is bounded on $L^p(X, w)$ for all p and w satisfying $r_0 < p < \infty$ and $w \in A_{\frac{p}{r_0}}$. In addition,*

$$\big\|F(L)\big\|_{L^p(X,w)\to L^p(X,w)} \leqslant \frac{C_\epsilon}{\theta^{\frac{n}{2}+\epsilon}} \|F\|_{\theta,\infty}$$

for every $\epsilon > 0$, $r_0 < p < \infty$ and $w \in A_{\frac{p}{r_0}}$.

Proof. It is easy to check, using the Cauchy formula that there exists a constant C independent of F and θ such that

$$\sup_{\lambda>0}\left|\lambda^k F^{(k)}(\lambda)\right| \leqslant \frac{C}{\theta^k}\|F\|_{\theta,\infty}, \quad \forall k \in \mathbb{Z}_+.$$

For any $\epsilon > 0$, $\sup_{t>0}\|\eta\delta_t F\|_{W^\infty_{k-\epsilon}} \leqslant C\sup_{\lambda>0}|\lambda^k F^{(k)}(\lambda)|$ so by interpolation

$$\sup_{t>0}\|\eta\delta_t F\|_{W^\infty_s} \leqslant \frac{C_\epsilon}{\theta^{s+\epsilon}}\|F\|_{\theta,\infty}.$$

Applying the above inequality and Theorem 3.2 we obtain Proposition 6.8 (see also, Theorem 4.10 [8]). □

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