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# Minimal Steiner Trees for $2^k \times 2^k$ Square Lattices*

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We prove a conjecture of Chung, Graham, and Gardner (*Math. Mag.* **62** (1989), 83–96), giving the form of the minimal Steiner trees for the set of points comprising the vertices of a $2^k \times 2^k$ square lattice. Each full component of these minimal trees is the minimal Steiner tree for the four vertices of a square. 

## 1. Introduction

Consider a finite set of points in the Euclidean plane. The *Steiner problem* asks us to find a *minimal* network connecting these points, that is,

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a network such that the sum of the lengths of all the edges is as small as possible (over all choices of suitable networks). This differs from other path-minimizing problems, such as the Travelling Salesman Problem, in that the network can contain extra vertices not belonging to the original set of points.

In general the Steiner problem is a difficult one, principally because minimizing the length of a network locally does not guarantee it will be absolutely minimal among all possible choices. There is no local reduction process whereby we can transform a network into a minimal network. If, however, we restrict the problem to sets of points which lie in special geometric configurations it is often possible to find simple algorithms to construct a minimal network. In 1989, Chung, Gardner and Graham [2] examined what they described as the checkerboard problem, namely, what do the minimal networks look like when the points are arranged in a regular lattice of unit squares like the corners of the cells of a checkerboard. They gave a number of constructions for various cases which they conjectured, but were unable to prove, were minimal. These constructions use as a basic building block the Steiner tree for the corners of a unit square, which we will denote by $X$, shown in Fig. 1.

In the case of a square array of $2^k \times 2^k$ points, there is a simple recursive construction for building a network using only $X$s. This is illustrated for the $8 \times 8$ square lattice in Fig. 2. Note that the network is built from four networks for $4 \times 4$ square lattices connected in the center by a single $X$. In a similar way a network composed only of $X$s for a $2^k \times 2^k$ square lattice can be constructed from four networks for $2^{k-1} \times 2^{k-1}$ square lattices.

In this paper we will show that such networks, or *Steiner trees*, are indeed minimal. The basic idea is to show that the local properties of any other network inevitably result in a longer Steiner tree. The details of this appear in Section 3, after we provide some useful background and preliminary results in Section 2.

The general checkerboard problem for square lattices of other dimensions is substantially more difficult, as it requires a detailed structural understanding of precisely which Steiner trees can potentially be used as building blocks for the minimal network. The solution to this problem and some of its generalizations will appear in a forthcoming paper [1].

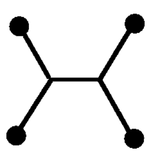


FIG. 1. The Steiner tree $X$.

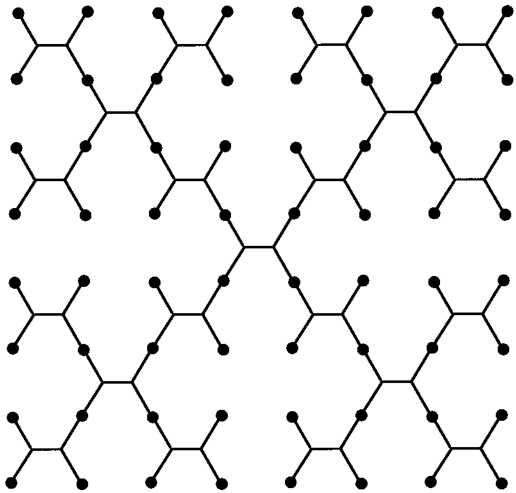


FIG. 2. A Steiner tree for the $8 \times 8$ checkerboard.

## 2. PRELIMINARIES

In order to begin an analysis of Steiner trees on square lattices we should first review some fundamental facts about Steiner trees, and some of the basic tools available. Let $M$ be a set of $m$ fixed points in the Euclidean plane. We will refer to these points as *terminals*. A minimal network, $S$, consisting of vertices and edges connecting the points in $M$ must have the following properties [6]:

— all edges are straight lines;

— the network is a tree;

— the angle between any two edges meeting at a vertex is at least $120°$;

— all vertices of $S$ not in $M$, known as *Steiner points*, have degree 3, and hence the edges meeting at a Steiner point make angles of precisely $120°$ with each other;

— the number of Steiner points is at most $m-2$.

Any network on $M$ with these properties is known as a *Steiner tree*. However, these properties are not sufficient to ensure that the length of the Steiner tree $S$, which we denote by $L(S)$, is minimal. They do guarantee that for any particular topology of $S$ the tree is the shortest possible, but the number of different topologies grows exponentially as $m$ increases. In fact it has been shown [5] that the Steiner problem is NP hard, which suggests that there can be no polynomial-time algorithm for generating minimal Steiner trees.

For $2^k \times 2^k$ square lattices, however, we will show there is an explicit construction for a minimal Steiner tree, namely the tree built solely from $X$s. We say that the $X$s constitute the full components of this Steiner tree. A Steiner tree is *full* if each of its terminals have degree 1. The *full components* of a Steiner tree can be thought of as being the smallest irreducible "blocks" from which the Steiner tree is composed (by union at the terminals).

The aim of our analysis is to show that the minimal Steiner trees here can contain no type of full component other than an $X$. Much of this analysis requires only basic notions from geometry and trigonometry. The following three well-known facts about minimal Steiner trees will also prove useful.

1. MELZAK'S THEOREM [7]. *Let $a$, $b$ and $c$ be vertices of a triangle, all of whose angles are less than* $120°$. *Let the point* $(ac)$ *be the third vertex of the equilateral triangle based on $ac$ whose interior does not intersect the interior of $\triangle abc$. Let $S$ be the minimal Steiner tree on $a$, $b$ and $c$, with Steiner point $s$. Melzak showed that the point* $(ac)$ *lies on the extended line $bs$* (*see* Fig. 3). *Furthermore,* $L(S) = \mathbf{d}[b, (ac)]$, *the distance between $b$ and* $(ac)$. *This observation often proves valuable in helping calculate or estimate the length of a Steiner tree.*

2. POLLAK'S THEOREM [8]. *Let $abcd$ be a convex quadrilateral. Let $o$ be the point where the diagonals $ac$ and $bd$ intersect, and assume* $\angle aod < 90°$. *Pollak's theorem says that if the two Steiner trees on* $\{a, b, c, d\}$ *shown in* Fig. 4 *both exist then the one on the right is a minimal Steiner tree, whereas the other is strictly non-minimal. Du, Hwang, Song and Ting* [4] *have given explicit conditions for determining when both topologies exist.*

3. THE VARIATIONAL TECHNIQUE [9]. *It is possible to use techniques from the Calculus of Variations to understand how $L(S)$ changes when the terminals of $S$ undergo small perturbations. A simple consequence of this*

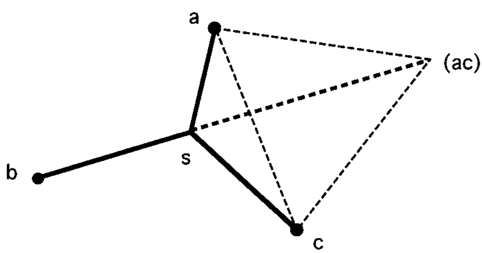



FIG. 3. Melzak's theorem.

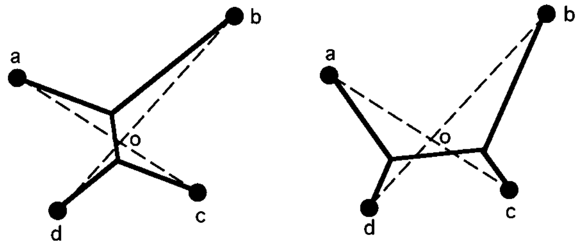



FIG. 4. The Steiner tree on the right is shorter than the one on the left since $\angle aod < 90°$.

*variational approach is the following result, which will be of use in this paper. Let e be an edge of S ending at a terminal t, and let v be a vector in the Euclidean plane such that the angle $\theta$ between e and v is acute. If we perturb S by moving t in the direction of v while keeping all other terminals fixed (and moving the Steiner points in such a way that S continues to be a Steiner tree) then $L(S)$ decreases in length, and the rate of decrease with respect to v is proportional to* $\cos(\theta)$. *Hence the larger $\theta$ is, the slower the rate at which $L(S)$ decreases as t moves along v.*

Having established these definitions and techniques, we can now make some simple observations about minimal Steiner trees on square lattices.

PROPOSITION 2.1. *A minimal Steiner tree for a square lattice contains no edge of length greater than* 1.

*Proof.* This is clear; it is an immediate property of the square lattice that any edge of length greater than 1 in the Steiner tree could be replaced by one of the edges of the square lattice to form a shorter tree. ∎

Let $T^*$ be a minimal Steiner tree spanning a square lattice. Let $T$ denote a full Steiner subtree of $T^*$ spanning a set of $m$ terminals in this square lattice. In other words, $T$ is a full component of $T^*$. Let $S$ denote the set of connected components of the union of the vertices and edges of $T$ resulting when the boundaries of squares of the square lattice are deleted. Define the graph $G(T)$ to be the graph whose vertex set is $S$, two components in $S$ being adjacent in $G(T)$ if they both contain parts of the same edge of $T$ or if they both contain edges adjacent to a Steiner point on the boundary of a square. It is immediate that $G(T)$ is a tree since $T$ is a tree.

PROPOSITION 2.2. *Every element of S contains at least one Steiner vertex of T.*

*Proof.* Since each edge of $T$ has length at most 1, the only way the proposition could fail to hold is if a situation such as that shown in Fig. 5 could occur. We will show by contradiction that this is not possible. Since

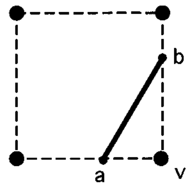



FIG. 5. This picture cannot occur in $T$.

$v$ is a vertex of the square lattice, by symmetry we can assume there is a path in $T^*$ from $a$ to $v$ not passing through $b$ (otherwise swap the roles of $a$ and $b$). But this implies $T^*$ is not minimal as we can replace the line segment $ab$ by the shorter line segment $bv$ to create a shorter tree. ∎

We define a *leaf* of $G(T)$ to be a vertex of $G(T)$ of degree 1. Much of our argument in this paper is based on examining the behaviour of the structure of $T^*$ that occurs in the leaves of $G(T)$. Such a leaf is just a connected part of $T$ contained in a unit square of the lattice which intersects the interior of precisely one side of the square at a single point.

PROPOSITION 2.3. *The only possible topologies (up to reflection and rotation) for the parts of $T$ corresponding to leaves in $G(T)$ are the ones shown in Fig.* 6.

*Proof.* The idea of the proof is to eliminate all other possibilities by non-optimality. Let $a$, $b$, $c$, and $d$ be the vertices of the square containing a leaf of $G(T)$ and let $T$ intersect the interior of the line segment $cd$.

If only one of $a$ and $b$ (say $b$) is a vertex of $T$ then, since each edge of $T$ has length at most 1, it follows that the only possible topologies for the leaf are those in Figures 6(ii) and (iii). If neither $c$ nor $d$ are vertices of $T$ we obtain the topology given in Figure 6(iv).

Now consider the situation where the leaf contains the vertices $a$ and $b$ and precisely one of $c$ and $d$ (say $c$). There are two possible topologies,

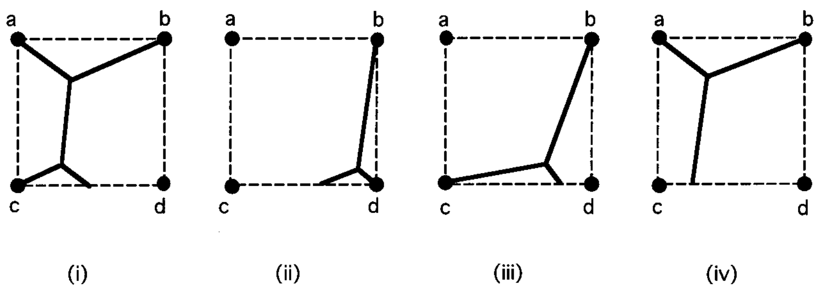



FIG. 6. The only possible topologies for leaves of $G(T)$.

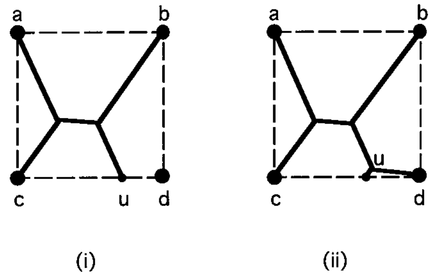



FIG. 7. Two non-minimal leaves.

those given in Figs. 6(i) and 7(i). In each case let $u$ be the point of intersection of $T$ with the line segment $cd$ and let $o$ be the point of intersection of the lines $au$ and $bc$. Clearly $\angle aob < 90°$ so it follows from Pollak's Theorem that the topology in Fig. 7(i) is non-minimal.

Finally, if the leaf contains all four vertices of the square, then the only possible topology (up to reflection) is that shown in Figure 7(ii). Let $u$ be the Steiner point adjacent to $d$ and let $o$ be the point of intersection of the lines $au$ and $bc$. If $\angle udc < 45°$ it again follows that $\angle aob < 90°$. But it is clear that in fact $\angle udc < 30°$, hence there exists an alternative full topology for the quadrilateral $abcu$ which by Pollak's Theorem is minimal. Hence, the topology in Fig. 7(ii) is non-minimal. ▮

Throughout this paper, we will refer to a leaf of $G(T)$ with the topology in Fig. 6(i) as an *X-type leaf*.

## 3. Nothing Succeeds Like Excess

Let $T^*$ be a minimal Steiner tree on an $n \times n$ square square lattice. In this section we prove the following conjecture from [2].

*Conjecture* [Chung *et al.*] The minimal Steiner tree for a $2^k \times 2^k$ square lattice, is the Steiner tree all of whose full components are $X$s.

Chung *et al.* show such a tree exists on a square lattice precisely when $n = 2^k$; we will show such trees are minimal. The key to proving the conjecture is the observation that, per terminal, $X$ appears in some sense to be the most efficient possible full component of $T^*$. In particular, we will prove that if $T$ is a full component of $T^*$ spanning a set of $m$ terminals in the $n \times n$ square lattice, then $L(T)/(m-1)$ is minimized for $T = X$.

We first establish the notion of the excess of any tree $T$. Define

$$\rho = \frac{L(X)}{3} = \frac{1+\sqrt{3}}{3} = .91068 \cdots.$$

Define the *excess* of $T$ to be

$$e(T) = L(T) - (m-1)\,\rho.$$

Note that the excess is additive in the sense that if $T$ is a subtree of $T^*$ such that $T = \bigcup_{i=1}^{k} T_i$ where each $T_i$ is a full component of $T^*$ then $e(T) = \sum_{i=1}^{k} e(T_i)$. In Table I we give the excesses of some small full trees that can occur in square lattices, as well as the terminology we will use to refer to these trees.

We have defined excess so that $e(X) = 0$. By the additivity property of excess it now follows that the conjecture can be proved by establishing the following theorem.

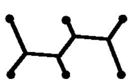
THEOREM 3.1. *Let $T$ denote a full component of $T^*$ spanning a set of $m$ terminals in the square lattice. Then either $T = X$ or $e(T) > 0$.*

*Proof.* Suppose $e(T) \leqslant 0$; we aim to show that $T = X$. This is clearly true if the number of terminals spanned by $T$ is at most 4. So assume $T$ spans $m \geqslant 5$ terminals. By induction on $m$, we can assume that any connected graph spanning at most $m-1$ terminals of the square lattice has excess at least 0. The theorem is proved by reaching a contradiction to our assumption that $e(T) \leqslant 0$, or in other words $L(T) \leqslant (m-1)\,\rho$.

Define $G(T)$ as in the previous section. We will now prove a series of lemmas which will enable us to eliminate all possibilities for $T$.

TABLE I

Excesses for Some Full Trees

| $T$ | Name | $m$ | Length | Approx. excess |
|---|---|---|---|---|
|  | edge | 2 | 1 | $0.089316\cdots$ |
|  | $Y$ | 3 | $(1+\sqrt{3})/\sqrt{2}$ | $0.11048\cdots$ |
|  | $X$ | 4 | $1+\sqrt{3}$ | 0 |
|  | $2\times 3$ ladder | 6 | $\sqrt{11+6\sqrt{3}}$ | $0.07176358\cdots$ |

LEMMA 3.2. *The only possible topology (up to reflection and rotation) for each part of $T$ occurring in a leaf of $G(T)$ is that in Figure* 6(i).

*Proof.* From Proposition 2.3 it follows that we need to prove that none of the topologies in Figure 6(ii)–(iv) can occur as a leaf of $G(T)$. In each case let $T_1$ denote the part of the branch of $T$ which lies above the line $cd$ and let $u$ be the point where $T$ intersects the interior of $cd$.

First consider the leaf shown in Figure 6(ii). Let $x$ be the distance from $d$ to $u$. Since $L(T) \leqslant (m-1)\rho$, then by the minimality of $T$ we require that $L(T_1) \leqslant x + \rho$, that is,

$$L(T_1) - x \leqslant \rho,$$

for otherwise we can remove $T_1$ from $T$ and replace it by $du$. This would reduce the excess of the tree and the number of terminals, thereby contradicting the inductive hypothesis. A simple calculation shows that

$$L(T_1) = \sqrt{x^2 + x\sqrt{3} + 1}.$$

The function $\sqrt{x^2 + x\sqrt{3} + 1} - x$ is monotone decreasing for $x > 0$. But when $x = 1$,

$$L(T_1) - x = L(Y) - 1 = 0.93185 \cdots > \rho,$$

which gives a contradiction. So this is eliminated as a possible leaf of $G(T)$.

For the leaf in Figure 6(iii), it is easily seen that the Steiner tree on the points $b$, $c$ and $u$ is longer than the Steiner tree on $a$, $c$ and $u$, so again using the previous argument this cannot be a leaf of $G(T)$.

The final possibility to be considered is that in Fig. 6(iv). Again let $x$ be the distance from $c$ to $u$. If we consider the excess of $T - T_1$ it is clear we require that $L(T_1) < 2\rho$. Now note that $L(T_1)$ increases as $x$ decreases from 1/2 to 0. When $x = 1/2$ a simple calculation shows that $L(T_1) = 1 + \sqrt{3}/2$. But this implies that for any $x$ such that $0 \leqslant x \leqslant 1/2$ we have

$$L(T_1) \geqslant 1 + \sqrt{3}/2 > 2\rho.$$

So again this topology is not possible. ∎

Using excess, we next show that the range of angles that can occur in an $X$-type leaf is restricted.

LEMMA 3.3. *If $T_1$ is an $X$-type leaf of $G(T)$, as in Figure* 6(i), *then the angle of the near-vertical line from the vertical must be less than* 7°. *Furthermore, if $c$ and $d$ are adjacent to adjacent Steiner points, as shown in*

*Figure* 10, *then the angle of the near-vertical line from the vertical must be less than* $5^\circ$.

*Proof.* To prove this result we make use of Melzak's Theorem. Let $u$ be the point where $T$ crosses the line $cd$, let $x = \mathbf{d}[c, u]$ and let $\theta$ be the angle of the near-vertical edge of $T_1$ from the vertical. By the same excess argument used above it follows that it suffices to prove that $L(T_1) - x - 2\rho > 0$ for all $\theta$ such that $7^\circ \leqslant \theta < 15^\circ$.

Let $(ab)$ be the third vertex of the equilateral triangle not lying in $abcd$ whose other vertices are $a$ and $b$. Similarly, let $(cu)$ be the third vertex of the equilateral triangle on $c$ and $u$ outside $abcd$. By Melzak's Theorem, the near-vertical edge of $T_1$ when extended passes through both $(ab)$ and $(cu)$, and $L(T_1) = \mathbf{d}[(ab), (cu)]$. Now extend the line segments $(ab)a$ and $(cu)c$ to form an isosceles triangle on the edge $ac$, as illustrated in Fig. 8. Note that $\angle(cu)(ab)a = (30 - \theta)^\circ$ and $\angle(ab)(cu)c = (30 + \theta)^\circ$. By the sine rule we obtain

$$\frac{L(T_1)}{\sin(120)} = \frac{1 + 1/\sqrt{3}}{\sin(30 + \theta)},$$

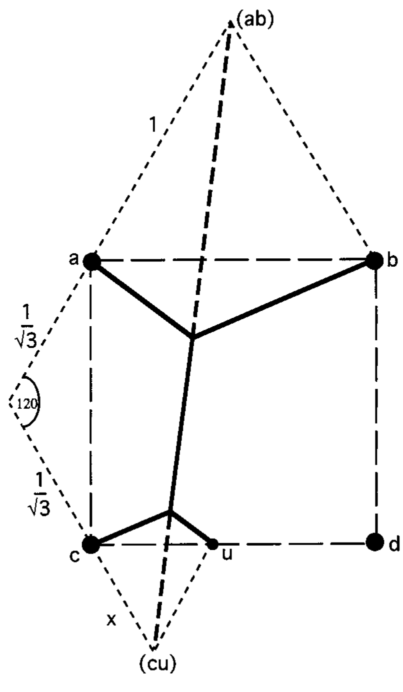



FIG. 8. Construction for the first part of Lemma 3.3.

that is,

$$L(T_1) = \frac{1+\sqrt{3}}{2\sin(30+\theta)}.$$

To find $x$ in terms of $\theta$ again apply the sine rule, giving

$$\frac{x+1/\sqrt{3}}{\sin(30-\theta)} = \frac{L(T_1)}{\sin(120)},$$

which implies

$$x = (1+1/\sqrt{3})\,\frac{\sin(30-\theta)}{\sin(30+\theta)} - 1/\sqrt{3}. \tag{3.1}$$

Computing $L(T_1) - x - 2\rho$ as a function of $\theta$ it is now easily checked, using for example Mathematica, that this function, $f_1(\theta)$, is a monotonically increasing function and equals 0 at $\theta \approx 6.8699°$ (see Fig. 9). Consequently $f_1(\theta)$ is positive for all $\theta \geqslant 7°$.

For the second angle result we can strengthen the above argument. Let $s$ be the Steiner point of $T$ adjacent to $d$ and define $y = \mathbf{d}[c, s]$ and $z = \mathbf{d}[u, s]$, as shown in Fig. 10. By induction, we have that $L(T - (T_1 + us) + cs) \geqslant (m-3)\rho$. Thus, by our assumption on $e(T)$,

$$L(T_1) + z - y - 2\rho \leqslant 0.$$

Given that we have calculated $x$ as a function of $\theta$, it is not difficult to calculate the left-hand side of this inequality as a function of $\theta$. By the sine rule,

$$z = (1-x)\,\frac{\sin(30-\theta)}{\sin(120)}$$

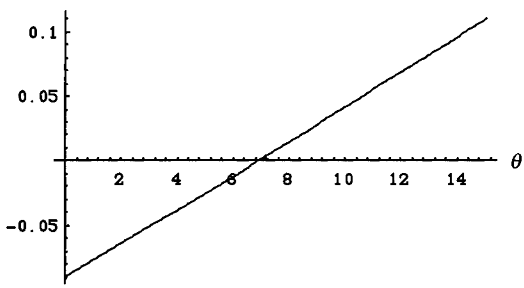



FIG. 9. $f_1(\theta) = L(T_1) - x - 2\rho$.

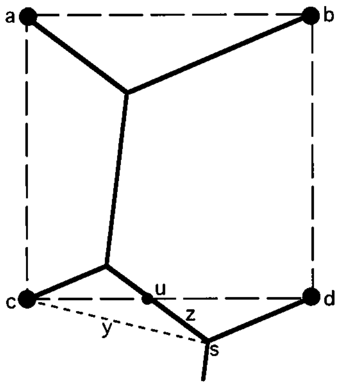



FIG. 10. Construction for the second part of Lemma 3.3.

since $\angle uds = (30 - \theta)^\circ$. Noting that $\angle cus = (150 - \theta)^\circ$, we obtain, by the cosine rule,

$$y = \sqrt{x^2 + z^2 - 2xz\cos(150 - \theta)}.$$

By substitution we can now compute $f_2(\theta) = L(T_1) + z - y - 2\rho$ as a function of $\theta$. Again using Mathematica, $f_2$ can be shown to be an increasing function for $0^\circ < \theta < 7^\circ$, and is positive for all $\theta \geqslant 5^\circ$ (see Fig. 11). ∎

LEMMA 3.4. *Two leaves of $G(T)$ which are in squares which share exactly one corner cannot both be adjacent to the same vertex of $G(T)$.*

*Proof.* This follows immediately from Lemmas 3.2 and 3.3 since the leaves are oriented at 90° to each other and hence the angles do not match up. ∎

LEMMA 3.5. *No two leaves of $G(T)$ can both be adjacent to the same vertex of $G(T)$ of degree* 3.

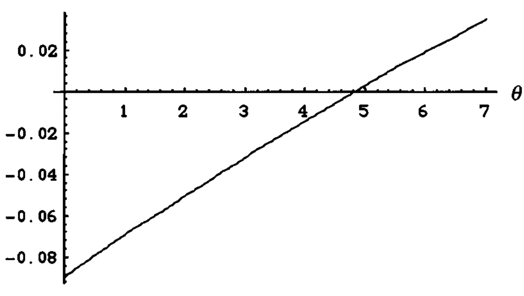



FIG. 11. $f_2(\theta) = L(T_1) + z - y - 2\rho$.

*Proof.* Assume there exist two leaves of $G(T)$ both adjacent to the same vertex of $G(T)$ of degree 3. Both leaves must be $X$-leaves, and by Lemma 3.4 they must be in squares not sharing a corner. Up to reflection and rotation, there are only three possible situations: those shown in Figs. 12(i), 13(i) and 13(ii). Other situations immediately cause the tree to intersect with itself (noting Proposition 2.2).

First consider the situation in Figure 12(i). We will show this can be eliminated by optimality. Let $s$ be the Steiner point not adjacent to any vertex of the square lattice. Consider the left-hand downwards branch at $s$. By Proposition 2.2, this edge must branch at another Steiner point $s_1$ in the square $efgh$, and the downwards branch at $s_1$ must cross $eg$ in order that $T$ not intersect itself (see Figure 12(ii)). By the same argument this branch must branch again at another Steiner point, $s_2$, before reaching the extension of $gh$. In the same way, the rightmost downwards branch at $s_2$ is forced to cross the extension of $gh$ and branch at $s_3$, the rightmost upwards branch of which must cross the extension of $eg$ and branch at $s_4$, the upwards branch of which must cross $gh$. But it is clear that $T$ now cannot avoid self-intersection, hence this topology is impossible.

Figures 13(i) and (ii) can be shown not to occur using excess. In each case, comparing the part of $T$ contained in the upper and middle squares with the $2 \times 3$ ladder shown in Figure 13(iii), it is clear that the angle of the near-vertical edge from the vertical must be greater than the same angle in the $2 \times 3$ ladder. But in the $2 \times 3$ ladder this angle is approximately $6.2°$ (see, for example, [3]), which implies that these possibilities can be eliminated by Lemma 3.3. ▌

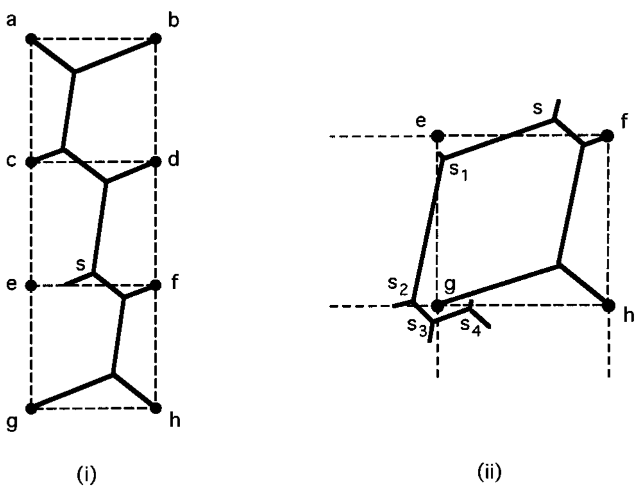



FIG. 12. Figure for Lemma 3.5.

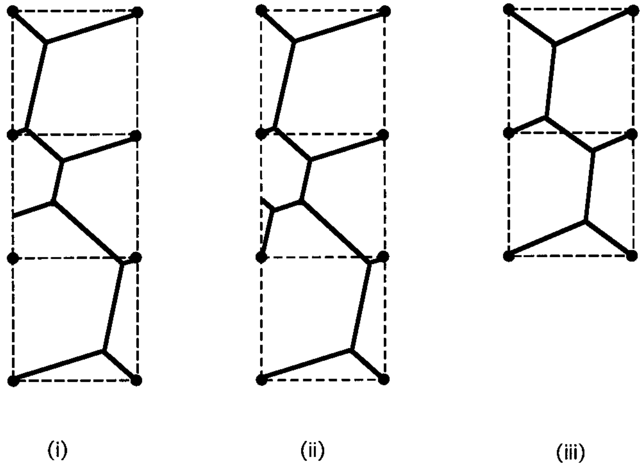



FIG. 13. Second figure for Lemma 3.5.

To conclude the proof of the main theorem we will show that $G(T)$ cannot have an $X$-type leaf adjacent to a vertex of degree 2. For this, we begin by using an optimality argument rather than excess. So let $T'$ denote any full component of $T^*$. Then there are ten possible topologies for the tree in the square corresponding to a vertex of degree 2 in $G(T')$ adjacent to an $X$-type leaf. These are shown in Fig. 14, and it is easily checked that the list is complete by considering all possible ways this branch can connect to the rest of the tree in such a way that Propositions 2.1 and 2.2 are satisfied. Six of these ten possibilities can immediately be eliminated by being shown to

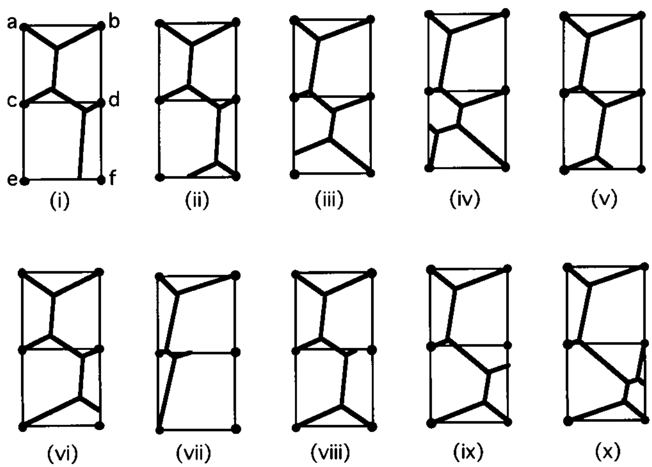



FIG. 14. The 10 possible topologies for a vertex of degree 2 adjacent to an $X$-type leaf.

be non-optimal. We will prove this result first (before showing that none of the other four can occur in $T$) as it may also be useful in approaching the more general problem of classifying minimal Steiner trees on any rectangular square lattice.

PROPOSITION 3.6. *Let $T'$ be a full component of $T^*$. The only possible topologies* (*up to reflection*) *for the part of $T'$ corresponding to an X-type leaf and adjacent vertex of degree* 2 *in $G(T')$ are those shown in Figure* 14(i), (ii), (vi) *and* (ix).

*Proof.* We will show that the other six topologies in Figure 14 are all non-optimal. In each case let $u$ be the point on the boundary of *cdef* where this branch connects to the rest of $T'$.

To deal with alternative (v), consider the instance of (ii) in which the distance, $x$, from $u$ to $e$ in (ii) equals the distance from $u$ to $f$ in (v). We now argue using the variational technique. In alternative (ii) the acute angle between *ef* and the edge through $u$ is smaller than the corresponding angle in the reflection of (v), so a small decrease in $x$ increases the length of (ii) faster than it increases the length of (v). But if $x = 0$ then (ii) and (v) coincide. Thus (v) is longer than (ii) and hence non-minimal, since we can replace it by the reflection of (ii) to obtain a shorter subtree. Similarly, (iii) is non-optimal since it is longer than (vi) by the same variational argument.

To see that (x) is non-optimal consider Fig. 15. Let *dhfg* be the square to the right of *cdef* and let $v_0$ be the Steiner point in $T'$ adjacent to $d$. By Proposition 2.2 the edge of $T'$ that extends from the square *cdef* to the square *dhfg* cannot extend all the way to the line *fg* or *gh*. Thus, this edge must branch at a Steiner point $v_1$ in the square *cdef*. Assume the downwards branch at $v_1$ meets *fg* at the point $w_1$. From the geometry of the situation, it is clear that $\angle w_1 f v_0 > 90°$ and $\angle f w_1 v_1 > 90°$, hence $\mathbf{d}[v_0, v_1] > \mathbf{d}[f, w_1]$. Thus we can replace the edge $v_0 v_1$ by the new edge $f w_1$ to create a shorter tree. But if the downwards branch at $v_1$ branches again before reaching *fg* the situation is clearly even less optimal. Thus (x)

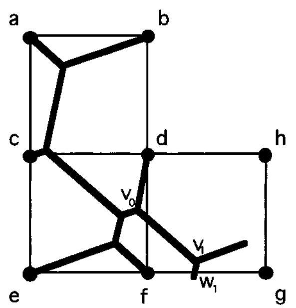



FIG. 15. Alternatives (x) and (iv) cannot be part of optimal trees.

cannot be part of a minimal tree. Alternative (iv) is also non-optimal by a very similar argument.

Finally, (vii) and (viii) are non-optimal by the same argument we used to eliminate Figure 12(ii) in Lemma 3.5. ∎

LEMMA 3.7. *$G(T)$ cannot have a leaf adjacent to a vertex of degree* 2.

*Proof.* Suppose to the contrary that $G(T)$ has a vertex $v$ of degree 2 adjacent to a leaf. Since the leaf $T_1$ must be an $X$-type leaf, there are four cases to consider by the previous proposition: specifically, those in Figure 14(i), (ii), (vi) and (ix).

For alternative (i) let $T_2$ be the part of $T$ above $ef$ shown in (i), and let $v$ be the point where $T$ intersects the interior of $ef$. By induction, $L(T_2) \leqslant 4\rho$. Now note that $L(T_2)$ decreases as $\mathbf{d}[v, f]$ decreases and hence $T_2$ is longer than an $X$ plus a unit edge. That is,

$$L(T_2) > L(X) + 1 = 3\rho + 1 > 4\rho.$$

This eliminates (i).

Alternative (ix) can also be easily eliminated using excess. First note that we can extend the result in the second part of Lemma 3.3 (where the tree $T_2$ connects to $d$) to this case (where $T_2$ crosses the interior of $df$). By the variational argument, as the intersection with $df$ moves towards $f$, in the proof of Lemma 3.3, $z$ increases faster than $y$. So the upper bound $2\rho + y - z$ for $T_1$ decreases, and the estimate of the angle, from the vertical, of the near vertical edge, as less than $5°$, holds. Next, as in Lemma 3.5, we observe that this angle must be greater than that in a vertical $2 \times 3$ ladder, and hence greater than $6.2°$, which provides the required contradiction.

For (ii) and (vi) we require more computational arguments. In each case let $T_2$ be the part of $T$ in the square $cdef$ adjacent to $T_1$, let $u$ be the point where $T$ intersects the interior of $cd$, and define $x = \mathbf{d}[c, u]$.

First consider alternative (ii). Let $v$ be the point where $T$ crosses $ef$ and let $z = \mathbf{d}[v, f]$. Our construction here is almost identical to that in Lemma 3.3. Let $(ud)$ and $(vf)$ be the third points of the equilateral triangles on $ud$ and $df$ lying outside the square $cdef$, as in Fig. 16. By Melzak's Theorem, $L(T_2) = \mathbf{d}[(ud), (vf)]$. We wish to calculate $L(T_2)$ and $z$ in terms of $\theta$, the angle of the near-vertical edge from the vertical. Equation 3.1, in the proof of Lemma 3.3, gives us an expression for $x$ in terms of $\theta$, and the method employed in that proof will allow us to find $L(T_2)$ and $z$ in terms of $x$ and $\theta$. As before, extend $(ud)d$ and $(vf)f$ to form an isoceles triangle on $df$. Applying the sine rule, we obtain

$$L(T_2) = \frac{1 + (1 - x)\sqrt{3}}{2 \sin(30 - \theta)}$$

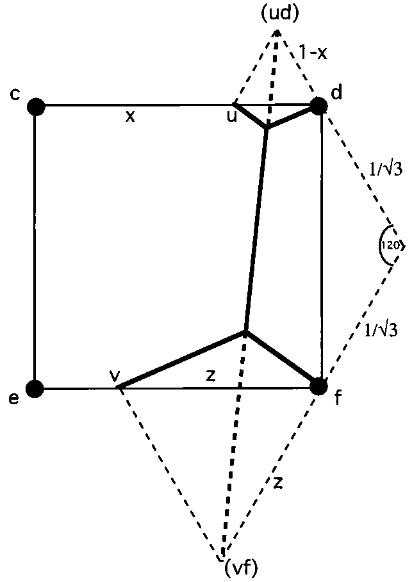



FIG. 16. Construction for calculating the excess of alternative (ii).

and

$$z = (1 - x + 1/\sqrt{3})\,\frac{\sin(30+\theta)}{\sin(30-\theta)} - 1/\sqrt{3}.$$

Let $F = L(T_1) + L(T_2) - z - 4\rho$. By substitution we can express $F$ as a function of $\theta$, and by induction we have that $F(\theta) \leqslant 0$. However a computation of $F(\theta)$, using Mathematica, shows that $F(\theta) > 0$ for all $\theta \leqslant 5°$ (Fig. 17) and hence, by the second part of Lemma 3.3, this topology is not possible.

For alternative (vi), let $(ud)$ be the third vertex of the equilateral triangle based on $ud$ and let $(e(ud))$ be the third vertex of the equilateral triangle

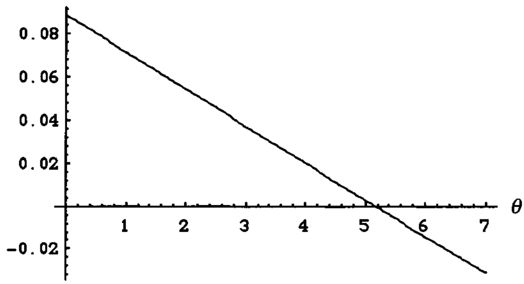



FIG. 17. Graph of $F(\theta)$.

based on $e(ud)$, as shown in Fig. 18. Now let $y = \mathbf{d}[e, (ud)]$ and let $\phi = \angle ce(ud)$. A simple geometric argument shows that

$$y = \sqrt{(1 + (1 - x)\sqrt{3}/2)^2 + ((1 + x)/2)^2}$$

and

$$\phi = \arcsin\left(\frac{1 + x}{2y}\right).$$

Next let the extension of $(e(ud))(ud)$ meet $bd$ at the point $p$, let $p_1 = \mathbf{d}[(ud, p]$ and let $p_2 = \mathbf{d}[p, d]$. Noting that $\angle(ud)\,pd = (60 + \phi)°$, we obtain from the sine rule:

$$p_1 = \frac{1 - x}{2\sin(60 + \phi)}$$

and

$$p_2 = \frac{(1 - x)\cos(\phi)}{\sin(60 + \phi)}.$$

Now define $v$ to be the point where $T$ crosses $df$. By Melzak's theorem, $L(T_2) = \mathbf{d}[(e(ud)), v]$. Since $\angle(e(ud))\,vp = (60 - \theta)°$, we deduce that

$$L(T_2) = \frac{(y + p_1)\sin(60 + \phi)}{\sin(60 - \theta)}.$$

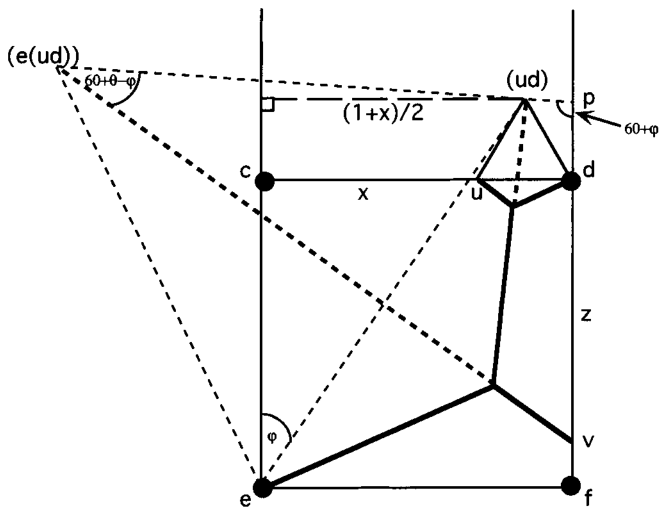



FIG. 18. Construction for calculating the excess of alternative (vi).

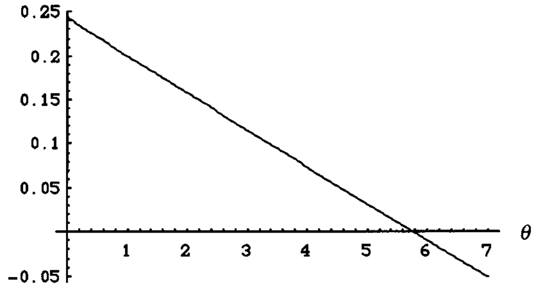



FIG. 19. Graph of $G(\theta)$.

Furthermore, if we define $z = \mathbf{d}[d, v]$, then we obtain

$$z = \frac{L(T_2)\sin(60 + \theta - \phi)}{\sin(60 + \phi)} - p_2.$$

Now let $G = L(T_1) + L(T_2) - z - 4\rho$. By extensive back-substitution it is clear that we can express $G$ as a function of $\theta$, and by induction we have $G(\theta) < 0$. Analysing the function using Mathematica again shows this is not the case for $\theta \leqslant 5°$, as illustrated in Fig. 19. This eliminates the final case and hence proves the lemma. ∎

*Proof of Theorem* 3.1 *continued.* The *diameter*, diam$(T')$, of a tree $T'$ is the length of a longest path in $T'$. By our assumption, $T$ spans more than four vertices of the square lattice, and hence diam$(G(T)) > 0$. Suppose that $G(T)$ has diameter at least 3. Then deleting all leaves of $G(T)$ must produce a tree, with a leaf $v$. In $G(T)$, $v$ must either be a vertex of degree 2 adjacent to one leaf, or of degree 3 adjacent to two leaves, or of degree 4 adjacent to three leaves. These are ruled out by Lemmas 3.7, 3.5 and 3.4 respectively. Hence $G(T)$ has diameter at most 2. If it is 2, then it is again eliminated by these lemmas. If it is 1, then by Lemma 3.2 $T$ spans the $2 \times 3$ ladder and thus has excess more than 0 (see Table 1). This eliminates all possibilities for $T$ spanning more than four terminals. The induction is complete. ∎

This proves Theorem 1. Since $n \times n$ boards with $n = 2^k$ can be covered with $X$s, this proves the Conjecture for such $n$.

## REFERENCES

1. M. BRAZIL, J. H. RUBINSTEIN, D. A. THOMAS, J. F. WENG, AND N. C. WORMALD, Full minimal Steiner trees on lattice sets, preprint.
2. F. R. K. CHUNG, M. GARDNER, AND R. L. GRAHAM, Steiner trees on a checkerboard, *Math. Mag.* **62** (1989), 83–96.

3. F. R. K. Chung and R. L. Graham, Steiner trees for ladders, *Ann. Discrete Math.* **2** (1978), 173–200.
4. D. Z. Du, F. K. Hwang, G. D. Song, and G. Y. Ting, Steiner minimal trees on sets of four points, *Discrete Comput. Geom.* **2** (1987), 401–414.
5. M. R. Garey, R. L. Graham, and D. S. Johnson, The complexity of computing Steiner minimal trees, *SIAM J. Appl. Math.* **32** (1977), 835–859.
6. E. N. Gilbert and H. O. Pollak, Steiner minimal trees, *SIAM J. Appl. Math.* **16** (1968), 1–29.
7. Z. A. Melzak, On the problem of Steiner, *Canad. Math. Bull.* **4** (1961), 143–148.
8. H. O. Pollak, Some remarks on the Steiner problem, *J. Combin. Theory Ser. A* **24** (1978), 278–295.
9. J. H. Rubinstein and D. A. Thomas, A variational approach to the Steiner network problem, *Ann. Oper. Res.* **33** (1991), 481–499.