On the Analytical Representation of Chip Area and Tool Geometry when Oblique Turning with Round Tools. Part 2: Variation of Tool Geometry along the Edge Line


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Abstract

This paper presents the models, analytical equations and results of analysis for determination of major cutting edge, orthogonal and normal clearance and rake angles, and cutting edge inclination angle. The analysis is carried for variable side and back rake angles used by the tool manufacturers and a corresponding variation of the geometry along the edge line for the case of round tools. The influence of tool nose radius, chamfer/chipbreaker angle and depth-of-cut was considered as well. Significant variation in orthogonal geometry from that stated in catalogues was found in most cases. The developed analytical solutions and algorithms allow the establishment of fundamental geometrical relationships and giving a correct prediction when developing new tools for conventional and rotary turning.

Keywords: Oblique turning; tool geometry; round tool

1. Introduction

Tool geometry, in terms of rake (γ), clearance (α), and inclination angles (λ), is one of the main factors influencing mechanics of machining process and tool performance. Most of descriptive and predictive theories for metal cutting are based on a straight-edge tool geometry, which has been given a detailed analysis [1]. Apart from that, back γb and side γf rake angles should be used for indexable inserts because they are specified on the toolholder/insert assembly [2] and not rake γ and inclination λ angles. In a case of oblique machining, a series of orthogonal and normal tool angles is calculated [1] for given back γb and side γf rake toolholder angles. Application of a nose radius leads to an existence of a non-stationary orthogonal and normal tool geometry along the cutting edge. Use of equivalent edge has become a practice in this case [3-10]. Several methods for calculation of an equivalent edge are developed which are used in chip-flow analysis [3]. The methods vary in computational complexity and consideration of different geometrical parameters. Colwell [4], in his chip-flow model, proposed to treat a straight-line segment connecting extreme points of the engaged edge (Fig. 1.a) as an equivalent edge. Okushima and Minato [5], treated the undeformed chip as infinitesimal elements along the engaged edge line, with chip flowing in normal direction to each element. Position of an equivalent edge was then calculated as the average direction of the resultant flow. This method gives identical result to Colwell model [4] for the case of round tool but results in higher accuracy for nose-radiused tools. Both methods operate under the assumption of zero rake and inclination angles. For oblique machining, the equivalent edge obtained by these methods should be subjected to back γb and side γf rake angles by applying ISO 3002 formulæ [1]. Young et al. [6] introduced the inclination angle and the variation of chip thickness along the edge line when calculating the equivalent edge.
Further development of this model by Wang [7] included effects of the rake angle. In this model a position of the equivalent edge is defined as a resultant of elemental chip-flow directions. Elemental chip-flow is then weighted against the local chip thickness and the spatial orientation in each part of the engaged cutting edge (Fig. 1b). Similar approach for the calculation of the equivalent edge has been developed by El-Wardany and Elbestawi [8].

The major disadvantage of the above models for round tools is that they treat the edge as a single line with constant geometry along it. In reality the geometry variation can exceed the stationary value manifolds, depending on the initial tool geometry and cutting data. The presented models and analysis in this paper address the extent of geometry variation along the edge line and address the parameters of the tool and the process that are the most influential on the sought variation.

2. Model description

It is recognized [1] that tool angles may vary from point to point along the cutting edge, thus definitions set for such angles may refer only to the angles in the selected point. For the case of round or nose-radiused insert the angles at a selected arbitrary point on the edge described by general definition should be traced along the entire edge length engaged in the cutting process. Two steps are required for the realization of this approach. First, a coordinate system in which the cutting edge is defined should be established. Second, a series of coordinate systems moving along the cutting edge from a selected edge beginning to its end should be established. The angles of interest are defined and sought in these moving coordinate systems.

Approach for establishment of the first system is identical to the chip-area model [11]. It involves the system \( x_2y_2z_2 \) where the edge can be described as an intersection of the tool flank and pseudo-rake, by given (Eq. 1) formulae:

\[
x^2 + (y - r)^2 = r^2 \quad \text{and} \quad z = 0
\]

where \( x_0, y_0, z_0 \) – coordinates belonging to the anvil coordinate system \( x_2y_2z_2 \), \( r \) – nose radius.

In oblique turning, the round insert is inclined on back \( \gamma_r \) and side \( \gamma_r \) rake angles where each angular position has its own coordinate system (Fig. 2a). The system \( xy_2z_2 \) where the edge line is defined, is located at the virtual tip of the tool and contains tool reference \( Pr \), back \( Pr_b \) and working \( Pr_w \) planes intersecting at the point of origin. In oblique machining, the tool flank significantly differs (Fig. 2a) from Eq. 1, because the coordinate system of the anvil is inclined on side \( \gamma_r \) rake, in system \( x_3y_3z_3 \), and back \( \gamma_r \) rake angles, in system \( xy_3z_3 \), with respect to the tool reference plane. Using the transformations between three coordinate systems presented in [11] Eq. 1 will respectively transform into:

\[
\begin{align*}
x \cdot \cos(\gamma_f) + z \cdot \sin(\gamma_f) &= x_3, \\
y \cdot \cos(\gamma_f) - z \cdot \sin(\gamma_f) &= y_3, \\
z \cdot \cos(\gamma_f) - x \cdot \sin(\gamma_f) &= z_3.
\end{align*}
\]

Additional consideration concerns the appearance of the rake surface. Practical tools have either a flat rake on neutral inserts, a chipbreaker or a chamfer on positive and negative inserts respectively. Mathematical difference between them is only in the sign and value of the chamfer/chipbreaker angle. Calculations are exemplified for the chamfer case in this study.

Fig. 2b shows that the main cutting edge angle \( \kappa_n \) identified in any arbitrary point on the edge, can be found as an angle between the tangent to \( pr_x \) projection and the axis \( x \) (see Eq. 3).

\[
\kappa_n = \arctan \left( \frac{\partial (pr_x)}{\partial x} \right) |_{x=x_4}
\]

The moving orthogonal coordinate system \( x_3y_3z_3 \) has a different orientation with respect to the reference system \( xyz \). It has a special relocation \( x=x_4, y=y_4, z=z_4 \) and is rotated around \( z_0 \) axis on the angle \( \kappa_n \), where index 4 designates coordinates of a movable arbitrary point on the edge defined in the reference system \( xyz \). Such transformation of coordinates between the systems can be described as:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\cos(\kappa_n) & -\sin(\kappa_n) & 0 & x_4 \\
\sin(\kappa_n) & \cos(\kappa_n) & 0 & y_4 \\
0 & 0 & 1 & z_4
\end{bmatrix} \begin{bmatrix}
x_3 \\
y_3 \\
z_3
\end{bmatrix}
\]

where values of \( x_4, y_4, z_4 \) and \( \kappa_n \) are found from Eq. 4 and equations of both \( pr_x \) and \( pr_y \) for the arbitrary point of interest along the edge. After the transformation, the tool flank (Eq. 2) becomes a function of several variables \( F_e = f(x_4, y_4, z_4, \kappa_n, \gamma_r, \gamma_f, \eta) \) (see Appendix).

The orthogonal clearance angle \( \alpha_c \) is measured in the cross-section perpendicular to the edge projection onto the reference plane. Since the cylindrical tool flank is not collinear with the reference axis \( z \), the section at \( x_4=0 \) will be an ellipse. This means that \( \alpha_c \) is an angle between the tangent to the ellipse and \( z_4 \) axis. Then the orthogonal clearance angle \( \alpha_c \) equals:
\[ \alpha_o = \arctan \left( \frac{\partial (F_o |_{\alpha=0})}{\partial z_o} \right) \]  

(5)

While knowing tool flank in \( x_o, y_o, z_o \) system is enough for determination of clearance angle \( \alpha_o \), finding the inclination angle \( \alpha_i \) requires knowing the edge position. Edge is defined as an intersection between tool flank and pseudo-rake. Term “pseudo” is used to indicate that the chip formation process takes place on the chamfer, yet the tool edge is formed by a plane perpendicular to the flank. Application of the transformation Eq. 4 to the rake face (Eq. 2) defines the pseudo-rake \( R_o = f(x_o, y_o, \kappa_o, r_o, \gamma_o, \phi) \). Intersection between the orthogonal flank \( F_o \) and the rake \( R_o \) defines the projection of the edge on the \( x_o, z_o \) coordinate plane:

\[ p_{r, z_o} = F_o - R_o = f(x_o, y_o, \gamma_o, x_o, z_o) \]

The inclination angle \( \alpha_i \) is then found on the determined \( p_{r, z_o} \), projection as an angle between the tangent to the projection and the orthogonal axis \( x_o \) (Fig. 3a):

\[ \lambda_i = \arctan \left( \frac{\partial (p_{r, z_o} |_{\gamma=0})}{\partial z_o} \right) \]

(6)

Since the chamfer surface is the factual rake then the orthogonal rake angle \( \gamma_o \) is the angle between the intersection line of the chamfer conical surface with the orthogonal plane \( x_o, y_o \), and the coordinate axis \( y_o \) (Fig. 3a). Plane \( x_o, y_o \) creates a parabola in the cross section because it passes on a side to the cone center due to the inclination of an insert. Then the exact value of the orthogonal rake \( \gamma_o \) is the angle between the tangent to the parabola and the axis \( x_o \).

![Fig. 3 Tool surfaces and angles in orthogonal (a) and normal (b) coordinate systems.](Image)

\[
\begin{align*}
\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} - \left( z - r_z \cdot \tan \gamma_{o, \phi} \right)^2 &= 0,
\end{align*}
\]

where \( \gamma_{o, \phi} \) is the angle of the tool chamfer.

After the transformations between tool anvil and reference systems presented in part 1 ([11] Eq. 2) and the orthogonal transformation (Eq. 4) the chamfer cone becomes a function of the model parameters and the orthogonal coordinates:

\[ C_o = f(x_o, y_o, \gamma_o, x_o, z_o) \]

(7)

Following the definition of the orthogonal rake angle \( \gamma_o \) and considering \( x_o=0 \) cross-section of the cone obtained via substitution \( x_o=0 \) into Eq. 7, the angle \( \gamma_o \) will equal:

\[ \gamma_o = \arctan \left( \frac{\partial (C_o |_{x_o=0})}{\partial y_o} \right) \]

(8)

Normal coordinate system is rotated on the value of inclination angle \( \lambda_i \) with respect to the orthogonal coordinate system. Transformation between the orthogonal and the normal coordinate systems is given below.

\[
\begin{bmatrix}
x_o \\
y_o \\
z_o
\end{bmatrix} =
\begin{bmatrix}
\cos(\lambda_i) & 0 & \sin(\lambda_i) \\
0 & 1 & 0 \\
-\sin(\lambda_i) & 0 & \cos(\lambda_i)
\end{bmatrix}
\begin{bmatrix}
x_N \\
y_N \\
z_N
\end{bmatrix}.
\]

(9)

Application of the normal transformation (Eq. 9) to the orthogonal tool flank \( (F_o) \) and its subsequent cross-sectioning \( (x_N=0) \) results in an ellipse. Then the normal clearance angle \( \alpha_N \) is the angle between the tangent to the ellipse and the coordinate axis \( z_N \) (Fig. 3b).

\[ \alpha_N = \arctan \left( \frac{\partial (F_o |_{x_N=0})}{\partial z_N} \right) \]

(10)

Similar transformations (Eq. 9) applied to the conical chamfer surface (Eq. 7) and respective cross-sectioning \( (x_N=0) \) results in a parabola. The normal rake angle \( \gamma_N \) is an angle between the tangent to the parabola and \( y_N \) coordinate axis (Fig. 3b), which is expressed as:

\[ \gamma_N = \arctan \left( \frac{\partial (C_N |_{y_N=0})}{\partial z_N} \right) \]

(11)

It should be noted that all the above equations for tool angles are expressed via the \( x \) coordinate of an arbitrary point \( A \) on the edge - \( x_A \) and model parameters. Conversion to the length of the edge was performed similarly to the methods used in part 1 [11].

3. Results and comparison

3.1. Input data and variation range

The model parameters of toolholder geometry and cutting conditions were varied in the same range as in part 1 [11]. Chamfer angle has a significant influence on the angular geometry of a tool and therefore the chamfer angle \( \gamma_{o, \phi} \) was varied from 0 to 30 degrees within this study. Two principal types of studies were performed: analysis of the variation of tool angles along the edge line; and the analysis of the influence of model parameter on the orthogonal and normal tool geometry at the depth-of-cut level. Use of tool angles at depth-of-cut level allows an assessment of mutual influences from several model inputs, while the first type of study identifies the variation of an angle of interest along the edge line as a function of only one model parameter. The developed model was initially compared to the four selected existing models [4-7]. Relative error analysis has shown that the absolute error value remains the same regardless of the model compared to. Colwell model [4], given by the equation:
\[
\kappa_r = \arctan \left( \frac{2 \cdot \alpha_r - 2 \cdot r_x + \sqrt{4 \cdot r_x^2 - f^2}}{f + 2 \cdot \sqrt{r_x^2 - (r_y - \alpha_r)^2}} \right)
\]

provides the most symmetrically distributed error compared to the rest of models. Therefore the Colwell model combined with ISO 3002 [1] angle conversion formulae was used for the comparison purposes. More than 250 individual cases were analysed in total.

3.2. Major cutting edge angle \( \kappa_r \)

It is known that the major cutting edge angle \( \kappa_r \) varies from zero at the tool tip to its maximum value at the depth-of-cut line for nose-radiused tools. Relative error analysis has shown that neither of models [4-7] gives correct \( \kappa_r \) value due to its large variation (Fig. 4.a) along the edge line. Increase of the depth-of-cut \( \alpha_r \) and back rake angle \( \gamma_p \) tends to decrease the overall angle variation. In some of the cases the error and the angle variation was reduced by more than 40%. Of all input data, feed \( f \) has no influence on the major cutting edge angle, while depth-of-cut and nose radius are the most influential.

It was found that the application of both the side \( \gamma_r \) and the back \( \gamma_p \) rake angles changes the edge projection from circular to an elliptic one.

The angles \( \gamma_r \) and \( \gamma_p \) have an opposite influence on the semiaxes of the ellipse. Such an influence results in formation of a local minimum (Fig. 4.b) for \( \kappa_r \). The position of this minimum can be controlled by the nose radius and depth-of-cut – smaller values of both tend to shift the minimum closer to a zero value of the side rake angle \( \gamma_r \).

3.3. Orthogonal clearance angle \( \alpha_c \)

Orthogonal clearance angle \( \alpha_c \) plays an important role in the performance of a tool. Small clearance, if selected below 7 to 9 degrees, can lead to rubbing and excessive heat generation on the tool flank [2] and thus the reduced tool life. Analysis of the developed model (Eq. 5) and the experimental results [12] has shown a strong variation of orthogonal clearance \( \alpha_c \) along the edge line (Fig. 5.a). The dominant influence on the orthogonal clearance is imposed by \( \gamma_p \) angle. It was found that the minor cutting edge has the \( \alpha_c \) clearance angle lower than back rake angle \( \gamma_p \) specified on the toolholder, yet this deviation is rather small. Relative error analysis (Fig. 5.b), when comparing the developed \( \alpha_c \) model (Eq. 5) to the formula of \( \cot(\alpha_c) = \cos(\kappa_r) \cdot \cot(\gamma_p) + \sin(\kappa_r) \cdot \cot(\gamma_p) \) [1], has shown an error having a positive sign – i.e. the true orthogonal clearance is larger than the value estimated with the help of the equivalent edge.

This error increases with the depth-of-cut \( \alpha_r \) from 70% to 230% and less intensively decreases with an increase of the nose radius \( r_x \) and the back rake angle \( \gamma_p \). Analysis of the variation of the orthogonal clearance angle at the depth-of-cut level \( \alpha_{m_{\text{max}}} \) has revealed its strong increase with back and lesser with side rake angles (Fig. 5.c). Application of depth-of-cut to nose radius ratio \( m \) in the model analysis (Fig. 5.d) made it possible to find a potential for stabilization of \( \alpha_{m_{\text{max}}} \). The side rake angle \( \gamma_r \) within the range of -10 to 0 degrees gives highly stable and practically applicable angular values of \( \alpha_{m_{\text{max}}} \), irrespective of other model parameters. If higher side rake angle \( \gamma_r \) is to be applied, for example in self-propelled turning, then a small depth-of-cut to nose ratio \( m \) provides practical orthogonal clearance \( \alpha_{m_{\text{max}}} \). Application of higher \( m \) values leads to a dramatically increased orthogonal clearance angle.

3.4. Orthogonal rake angle \( \gamma_r \)

The orthogonal rake angle \( \gamma_r \) plays the least important role in the case of oblique machining among other stationary or kinematic rake angles found on a tool [15]. It is believed [15], that normal, effective or speed rake angles control the chip formation mechanics. However orthogonal rake \( \gamma_r \) is consistently used by tool manufacturers and is specified in most of the tooling catalogues. Chamfer angle \( \gamma_{cm} \), back \( \gamma_p \) and side rake \( \gamma_r \) angles have the biggest impact on the orthogonal rake angle \( \gamma_r \) and its variation along the edge line. The \( \gamma_r \) angle practically does not change (Fig. 6.a) at side rake angle \( \gamma_r \) values less than -15 degrees. Further increase of side rake angle strongly and nonlinearly influences the orthogonal rake. Additionally, a variation of \( \gamma_r \) angle along the edge line is significantly lower compared to all other studied tool angles. When comparing the developed model (Eq. 8) to the formula of \( \tan(\gamma_r) = \tan(\gamma_p) \cos(\kappa_r) + \tan(\gamma_p) \sin(\kappa_r) \) [1] it was found that
the relative error lies within 12% to 25% - increasing with bigger depth-of-cut and decreasing with other model inputs.

Both back rake $\gamma_p$ and chamfer $\gamma_c$ angles (Fig. 6.b) have close to a linear influence on the orthogonal rake, yet the effect of chamfer angle is much stronger.

3.5. Cutting edge inclination angle $\lambda_e$

Cutting edge inclination angle $\lambda_e$ is strongly affected by the side rake angle $\gamma_r$ and proportionally increases with the later (Fig. 7.a). Variation of the inclination angle along the edge line is relatively insignificant and, as a rule, the angle decreases from the tool tip to the depth-of-cut level. Such a decrease is more intensive at higher $\alpha_o$ values and can lead to a change in the $\lambda_e$ value from a positive to a negative. Application of large back rake angle $\gamma_p$ intensifies the decrease of the inclination angle $\lambda_e$, which can lead to highly negative local inclination angles.

Comparison of $\lambda_e$ (Eq. 6) to the inclination angle determined as $\tan(\beta_e) = \sin(\alpha_c) \tan(\gamma_p) \cdot \cos(\alpha_c) \tan(\gamma_r)$ [1] has shown that the absolute error is relatively small and increases with side rake $\gamma_r$ back rake $\gamma_p$ and the depth-of-cut $a_p$. However the relative error has rather high values when $\gamma_r$ and $\gamma_p$ range from -6 to 0 degrees (Fig. 7.b). In such case the $\lambda_e$ angle determined according to ISO 3002 [1] always has a positive sign, while the true inclination angle changes its sign along the edge line. Analysis of the inclination angle at the depth-of-cut line $\lambda_{max}$ has additionally shown that application of large negative back rake $\gamma_p$ can lead to a strong decrease in the inclination angle and to its highly negative values (Fig. 7.c). Similar behaviour of $\lambda_{max}$ is observed when a large depth-of-cut to nose radius ratio $m$ is applied (Fig. 7.d). It can be concluded that under finishing conditions and low back rake angles smaller variations of the inclination angle $\lambda_e$ along the edge line can be achieved, even at high side rake angles.

3.6. Normal clearance angle $\alpha_N$

The normal clearance angle $\alpha_N$ strongly changes the pattern of its variation along the edge line with side rake angle $\gamma_r$. At low $\gamma_r$ it decreases along the edge line and with $\gamma_r$ being bigger than -10 degrees it increases instead. Application of side rake angle being within -5 to -10 degrees leads to a stable normal clearance $\alpha_N$ along the edge line. The error analysis has revealed that the ISO 3002 [1] formula of $\cot(\alpha_o) = \cos(\lambda_e) \cot(\alpha_c)$ in combination with Colwell model [3] generally underestimates the normal clearance angle $\alpha_N$ (Fig. 8.a). Both the depth-of-cut and the back rake angle tend to increase the relative error, while nose radius has no significant influence.

Analysis of the normal clearance angle at the depth-of-cut level $\alpha_{Nmax}$ shows that the back rake angle has a stronger impact on $\alpha_{Nmax}$ than the side rake $\gamma_r$ (see Fig. 8.b). Increased depth-of-cut changes such a pattern - side rake angle tends to have a stronger influence than the back rake. Overall, it can be concluded that the normal clearance $\alpha_N$ is less affected by the the model parameters than the orthogonal clearance $\alpha_o$ and is highly stable under a wide range of side rake $\gamma_r$ variation.

3.7. Normal rake angle $\gamma_N$

Normal rake angle $\gamma_N$ is the most frequently and readily measured angle on a tool and is believed [15] to have bigger impact on the mechanics of chip formation in oblique machining than the orthogonal rake $\gamma_r$. It can be seen (Fig. 9.a) that the normal rake $\gamma_N$ is strongly and nonlinearly influenced by the side rake angle - it dramatically increases at side rake angles being over -15 degrees. The deviation of the normal rake along the edge line is insignificant within the entire range of side rake variation. Both back $\gamma_p$ and side $\gamma_r$ angles increase the normal rake $\gamma_N$ at the depth-of-cut level. Back rake angle has stronger influence (Fig. 9.b) than the side rake angle. Additionally, the influence from the side rake practically ceases at back rake angles larger than - 20 degrees. Chamfer angle has a linear and a proportional influence on the
normal rake $\gamma_N$. Relative error analysis, when comparing Eq. 11 to ISO 3002 [1] angle of $\tan(\gamma_N) = \cos(\lambda) \tan(\gamma_o)$, has mainly shown the error of up to 60% and having a positive sign: i.e. ISO formula applied to an equivalent edge underestimates the actual value of the normal rake $\gamma_N$.

Fig. 9. Normal rake angle variation along the edge (a) and its value at the depth-of-cut level (b).

It should however be noted that at small side and back rakes the error is within 5% over the entire edge length. This means that ISO equations can be safely used in conventional machining and they lead to significant errors in cases of self-propelled or rotary machining.

4. Conclusion

A set of new analytical models for determination of orthogonal and normal geometry for round tools which accounts for tool back $\gamma_o$ and side $\gamma_N$ rake angles is developed. The influence of model input parameters on the geometry variation along the edge line was analysed as well as their effects on the angles of interest at the depth-of-cut level. The relative error analysis, which compares the developed models to models employing equivalent edge approach, has shown that the application of a single equivalent edge in oblique machining with round tools is inadequate due to large geometry variation along the edge. The relative error, for certain tool angles, can reach up to 150-300%. Side rake angle $\gamma_N$ does not significantly influence or, in some cases, even stabilizes the tool geometry along the edge line. Increase in back rake $\gamma_o$ and chamfer $\gamma_{ch}$ angles tends to increase respective tool geometry substantially.

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References


Appendix

$$p_{r_0} = \frac{x \cdot \sin(\gamma_o) \cdot \sin(\gamma_f) + r_o \cdot \cos(\gamma_f)}{\cos(\gamma_f) \cdot \cos(\gamma_o) + \sin(\gamma_f)}$$

$$p_{r_e} = \frac{x \cdot \cos(\gamma_f) \cdot \sin(\gamma_f) + r_e \cdot \cos(\gamma_f)}{\cos(\gamma_f) \cdot \cos(\gamma_e) + \sin(\gamma_f)}$$

$$R_o = \left[(z_o + z_A)\cos(\gamma_f) - (x_o \cdot \cos(\gamma_o) - y_o \cdot \sin(\gamma_o) + x_A)\sin(\gamma_f)\right] \cos(\gamma_f) + (x_o \cdot \sin(\gamma_o) + y_o \cdot \cos(\gamma_o) + y_A) \sin(\gamma_f)$$

$$F_o = \left[(x_o \cdot \cos(\gamma_o) - y_o \cdot \sin(\gamma_o) + x_A)\cos(\gamma_f) + (z_o + z_A) \sin(\gamma_f)\right]^2 + \left[- \cos(\gamma_f) + y_o \cdot \cos(\gamma_o) + y_A\right] \sin(\gamma_f) + \left[(z_o + z_A) \cos(\gamma_f) - (x_o \cdot \cos(\gamma_o) - y_o \cdot \sin(\gamma_o) + x_A) \sin(\gamma_f)\right]^2 - r_o^2 = 0$$