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Local Distortion Techniques and Unitarity of the S-matrix for the 2-body Problem

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The two-body S-matrix for an interaction with exponential decay at infinity is defined in a time-independent way and its unitarity is proved directly by local distortion techniques. Complete sets of incoming and outgoing states, or delicate resolvent estimates are not needed for the proof.

1. INTRODUCTION

In this paper we define in a time-independent manner the scattering "matrix" $S(E)$, $E > 0$, for the reduced 2-body Hamiltonian $H = H_0 + Q$, where H_0 is the free Hamiltonian and Q is an interaction of the form $e^{-\mu r}Ue^{-\mu r}$, $\mu > 0$, where $U\colon H_1(\mathbb{R}^3) \to H_{-1}(\mathbb{R}^3)$ is a compact self-adjoint operator. Using local distortion–analytic techniques, we establish: (1) the unitarity of $S(E)$, E not a positive eigenvalue for H, as an operator on $L^2(\Omega)$, where Ω is the unit sphere of directions in momentum space (Theorem 6.4); (2) the meromorphic continuation of $S(E)$ as a function of $E^{1/2}$ to the strip $\{z \in \mathbf{C} \mid |\operatorname{Im} z| < \mu, z \neq 0\}$ (See remark preceding Proposition 5.1); (3) the compactness of $S(z^2) - 1$ for z a point of analyticity for $S(z^2)$ (Section 5); and (4) the absolute continuity of the continuous spectrum of H (Corollary 4.7). The unitarity result (1) is the main result of the paper, whereas (2), (3), and (4) are auxiliary results arising naturally in the course of the derivation of the Main Theorem.

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Of course the above results are well known at least for Yukawa-type and bounded, Hölder-continuous, exponentially decaying multiplicative potentials (see, e.g., [8, 14, 21, 22] and the references given there). Although our results are more general, allowing interactions with $r^{-2+\epsilon}$-type local singularities as well as first-order terms, the emphasis here is on methods rather than results. Our main purpose is to develop the new and in principle very simple technique of local distortions for proving unitarity of the S-matrix via a direct proof of the "generalized optical theorem" ([8, p. 191, Eq. (7.67)] and our theorem). Complete sets of incoming and outgoing (generalized) eigenfunctions or delicate resolvent estimates are not needed for the proof. Most importantly, the method is developed with the many body problem in mind, and there seems to be a reasonable chance of extending this technique at least to the three body problem. In particular, we establish the meromorphic continuation of $T(z) \equiv \underset{\sim}{Q} + \underset{\sim}{Q}R(z^2)\underset{\sim}{Q}$ to the strip described above. We also see that the extended $T(z)$ are locally "distortable." Thus, the Faddeev T_α-operators [6, 3.11] can be distorted, and we expect that similar local distortion arguments plus distorted Faddeev equations will lead to a local distortion–analytic proof of the 3-body generalized optical theorem [9, Eqs. (3.10)–(3.12)] and thus, a proof of the unitary of the 3-body S-operator.

Our approach is strictly time-independent, and we make no attempt in this paper to connect our S-matrix with the S-matrix obtained in the time-dependent approach via wave-operators. However, our definition of the S-matrix is in terms of the "on the energy shell" $\mathbb{T}$-matrix and our *definition* of the latter is essentially [8, Eq. (7.41)] (but with the center of mass factored out) which is *derived* in [8] from the time dependent approach. We should point out that we work in spherical coordinates (k, ω) in momentum space, whereas $(E = k^2, \omega)$ are used (at least implicitly) as coordinates in [8]. Thus, several formulas in this paper will have slightly different factors than the corresponding formulas in [8].

Although local distortion–analytic techniques have been in the literature for some time, they do not seem to have been used to prove the unitarity of $S(E)$. Nuttall [10] used local distortion techniques to meromorphically continue the T-matrix $\mathbb{T}(E)$ as a function of E. (We thank Lawrence Thomas for pointing out this reference to us.) Thomas [19] used local distortion techniques to prove the absolute continuity of the continuous spectrum for H. His conditions on $\underset{\sim}{Q}$ are different and allow the treatment of longer range potentials. Dilation analytic techniques [1, 4, 16, 20] are essentially *global* distortion analytic techniques. They cover longer range interactions, but it is not clear that they can be used to obtain simple proofs of unitarity. (Lovelace essentially gives a formal unitarity proof when $\underset{\sim}{Q}$ is a Yukawa potential using dilation analytic arguments, see [7, pp. 443–444]. However it is not clear that his argument can be easily made mathematically rigorous.)

There are reasons to expect that at least as far as the spectral results are concerned, and presumably also for the scattering problem, the method can be modified to include longer range interactions. Naturally, the analytic continuations in z^2 would no longer be two-sheeted.

The paper is organized as follows: Section 2 contains several definitions and introduces various locally distorted Hilbert space. Section 3 derives the local distortion analytic properties of the interaction Q. Section 4 discusses the Hamiltonian H and its distorted resolvents. Section 5 introduces the various T-operators and the S-matrix associated with H. Section 6 contains the distortion analytic proof of the unitarity of $S(E)$.

2. Definitions and Notations

Let Ω be the unit sphere in $\mathbb{R}^3$. Points in Ω are denoted by ω and the usual usual measure on Ω by $d\omega$. $L^2(\Omega)$ is the complex Hilbert space of square-integrable functions on Ω with the inner product

$$(f, g)_\Omega = \int_\Omega \bar{f}(\omega)\, g(\omega)\, d\omega.$$

The corresponding norm is denoted by $\|\cdot\|_\Omega$.

Let $\mu > 0$ be a constant, which is *fixed* throughout this paper.

For $0 \leqslant a < b \leqslant \infty$ we let

$$R_{a,b} = \{z \in \mathbf{C} \mid a < \operatorname{Re} z < b, \mid \operatorname{Im} z \mid < \mu\}.$$

We set $R = R_{0,\infty}$ and $R_\infty = \{z \in \mathbf{C} \mid \mid \operatorname{Im} z \mid < \mu\}$. Finally, $R_\infty'' = R_\infty - \{0\}$. We denote by $\mathcal{Q}_{ab}$ the space of $L^2(\Omega)$-valued measurable functions on $(0, \infty) \cup R_{ab}$, which are analytic in R_{ab}. For $z_0 \in R_{ab}$ we denote by $\mathcal{Q}_{ab}^{z_0}$ the space of $L^2(\Omega)$-valued measurable functions on $(0,\infty) \cup R_{ab}$, which are analytic in R_{ab} except for at most a pole at z_0. Thus, $\mathcal{Q}_{ab} \subset \mathcal{Q}_{ab}^{z_0}$ for $z_0 \in R_{ab}$.

We let $\mathcal{Q} = \mathcal{Q}_{0\infty}$ and $\mathcal{Q}^{z_0} = \mathcal{Q}_{0\infty}^{z_0}$. Also, $\mathcal{Q}_\infty$ is the space of $L^2(\Omega)$-valued analytic functions on R_∞.

A *positive distortion* $\Gamma_{a'b'\epsilon}$ is defined for $0 < a' < b' < \infty$, $\epsilon \geqslant 0$ as an oriented path of the form $\Gamma_{a'b',\epsilon} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$, where $\Gamma_1 = (0, a']$, $\Gamma_2 = \{k \in \mathbf{C} \mid k = a' + it, 0 < t \leqslant \epsilon\}$, $\Gamma_3 = \{k \in \mathbf{C} \mid k = i\epsilon + t, a' < t \leqslant b'\}$, $\Gamma_4 = \{k \in \mathbf{C} \mid k = b' + i(\epsilon - t), 0 \leqslant t < \epsilon\}$, $\Gamma_5 = [b', \infty)$.

A negative distortion $\Gamma_{a'b',\epsilon}$ is defined in a similar way for $0 < a' < b' < \infty$, $\epsilon \leqslant 0$. A (positive or negative) distortion is R_{ab}-admissible if $a < a' < b' < b$ and $\mid \epsilon \mid < \mu$, see Fig. 1(a, b).

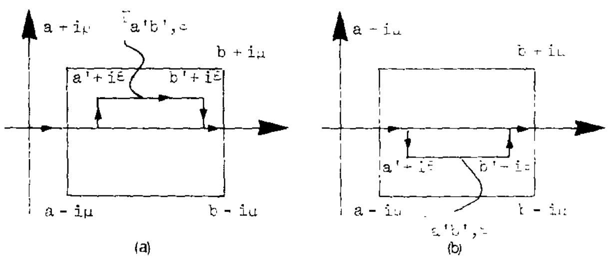

FIG. 1. (a) R_{ab}-admissible positive distortion, (b) R_{ab}-admissible negative distortion.

We shall also need the following types of positive and negative distortion, called $\Gamma_{abc,\epsilon_1,\epsilon_2}$.

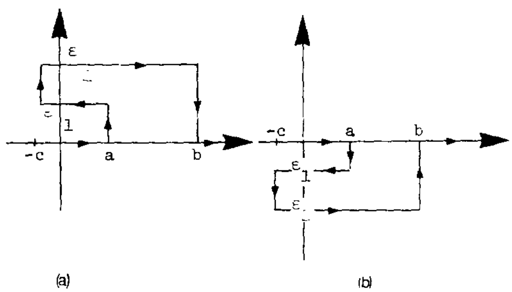

FIGURE 3

In what follows, all proofs are carried out using distortions of the $\Gamma_{ab,\epsilon}$-type, but it is clear that they are all valid for distortions of the $\Gamma_{abc,\epsilon_1\epsilon_2}$-type.

Suppose that $\Gamma_{a'b',\epsilon}$ and $\Gamma_{a'',b''\epsilon}$ are R_{ab}-admissible distortions and $z_0 \in R_{ab}$. Let $a < a_0 < \min(a', a'') < \max(b', b'') < b_0 < b$. Then we say that $\Gamma_{a'b',\epsilon}$ and $\Gamma_{a''b'',\epsilon}$ are R_{ab}-homotopic relative to z_0 if $z_0 \notin \Gamma_{a'b',\epsilon'} \cup \Gamma_{a''b'',\epsilon''}$ and $\Gamma_{a'b',\epsilon'} \setminus \{(0, a_0) \cup (b_0, \infty)\}$ and $\Gamma_{a''b'',\epsilon''} \setminus \{(0, a_0) \cup (b_0, \infty)\}$ are homotopic in $R_{ab} \setminus \{z_0\}$, see Fig. 2.

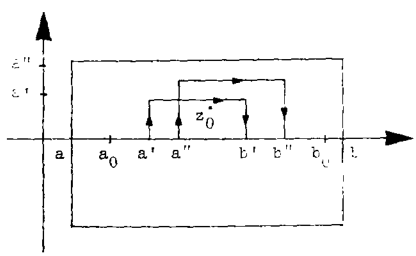

FIGURE 2

Note that $\Gamma_{a'b',\epsilon'}$ and $\Gamma_{a''b'',\epsilon''}$ are always homotopic in R_{ab}. Finally, if $\Gamma = \Gamma_{ab,\epsilon}$ is a distortion, then $\bar{\Gamma} = \Gamma_{ab,-\epsilon}$.

Let Γ be an R_{ab}-admissible distortion and let $\varphi \in \mathcal{Cl}_{ab}$. Then φ_Γ denotes the restriction of φ to Γ. For any Γ, φ_Γ determines φ by analytic continuation, and we shall identify φ with any φ_Γ, for example when we write $\varphi \in \mathcal{Cl}_{ab} \cap \mathscr{H}_\Gamma$ (see the following definition). If Γ' is another R_{ab}-admissible distortion, the analytic continuation map is defined for $\varphi \in \mathcal{Cl}_{ab}$ by $(\varphi_\Gamma)_{\Gamma'} \equiv \varphi_{\Gamma'}$. Similarly, if $z_0 \in R_{ab}$, and Γ and Γ' are R_{ab}-admissible distortions homotopic with respect to z_0, we have the restriction maps φ_Γ and $\varphi_{\Gamma'}$ and the analytic continuation map$(\varphi_\Gamma)_{\Gamma'}$ defined for $\varphi \in \mathcal{Cl}_{ab}^{z_0}$.

For a given distortion Γ we associate several Hilbert spaces with Γ. Let $\alpha = 0, 1$ or -1. Then $\mathscr{H}_\alpha^\Gamma$ is the Hilbert space of measurable $L^2(\Omega)$-valued functions φ on Γ such that

$$\| \varphi \|_{\Gamma,\alpha}^2 = \int_\Gamma \| \varphi(k) \|_\Omega^2 (1 + | k |^2)^\alpha | k |^2 | dk | \leqslant \infty.$$

The inner product is defined by

$$\langle \varphi, \psi \rangle_{\Gamma,\alpha} = \int_\Gamma (\varphi(k), \psi(k))_\Omega | k |^2 (1 + | k |^2)^\alpha | dk | .$$

We let

$$\mathscr{H}_\alpha^{(0,\infty)} = \mathscr{H}_\alpha, \qquad \langle \cdot, \cdot \rangle_{(0,\infty),\alpha} = \langle \cdot, \cdot \rangle_\alpha, \qquad \| \cdot \|_{(0,\infty),\alpha} = \| \cdot \|_\alpha, \qquad \text{for } \alpha = \pm 1,$$

$$\mathscr{H}_0^\Gamma = \mathscr{H}^\Gamma, \qquad \langle \cdot, \cdot \rangle_{\Gamma,0} = \langle \cdot, \cdot \rangle_\Gamma, \qquad \| \cdot \|_{\Gamma,0} = \| \cdot \|_\Gamma, \qquad \text{for } \Gamma \neq (0, \infty),$$

$$\mathscr{H}_0^{(0,\infty)} = \mathscr{H}, \qquad \langle \cdot, \cdot \rangle_{(0,\infty),0} = \langle \cdot, \cdot \rangle, \qquad \| \cdot \|_{(0,\infty),0} = \| \cdot \| .$$

In the usual way, $\mathscr{H}_{-\alpha}^\Gamma$ can be viewed as the dual of $\mathscr{H}_\alpha^\Gamma$ by

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_\Gamma = \int_\Gamma (\varphi(k), \psi(k))_\Omega | k |^2 | dk | , \qquad \varphi \in \mathscr{H}_\alpha^\Gamma, \quad \psi \in \mathscr{H}_{-\alpha}^\Gamma \tag{2.1}$$

Note that $\mathscr{H}_{\pm 1}$ are just the Fourier transforms of the usual Sobolev spaces $H_{\pm 1}(\mathbb{R}^3)$. Sometimes it is convenient to identify $\mathscr{H}_\alpha^\Gamma$ with $L^2(\Gamma \times \Omega; | k |^2(1 + | k |^2) | dk | \times d\omega)$.

If Γ and Γ' are R_{ab}-admissible distortions, and $\varphi_\Gamma \in \mathscr{H}_\alpha^\Gamma \cap \mathcal{Cl}_{ab}$, then clearly $(\varphi_\Gamma)_{\Gamma'} \in \mathscr{H}_\alpha^{\Gamma'} \cap \mathcal{Cl}_{ab}$. Similarly, if Γ and Γ' are R_{ab}-admissible distortions homotopic with respect to z_0, and $\varphi_\Gamma \in \mathscr{H}_\alpha^\Gamma \cap \mathcal{Cl}_{ab}^{z_0}$, then $(\varphi_\Gamma)_{\Gamma'} \in \mathscr{H}_\alpha^{\Gamma'} \cap \mathcal{Cl}_{ab}^{z_0}$.

We now define a sesqubilinear from on $\mathscr{H}^{\Gamma}_{-\alpha} \times \mathscr{H}_{\alpha}^{\Gamma}$ as follows:

$$(\varphi, \psi)_{\Gamma} = \int_{\Gamma} (\varphi(\bar{k}), \psi(k))_{\Omega}\, k^2\, dk, \qquad \varphi \in \mathscr{H}_{\alpha}^{\Gamma}, \quad \psi \in \mathscr{H}_{\alpha}^{\Gamma}.$$

We write $(\varphi, \psi)_{(0,\infty)} = (\varphi, \psi)$.

PROPOSITION 2.1. *The form $(\cdot, \cdot)_{\Gamma}$ defines a duality between $\mathscr{H}^{\Gamma}_{-\alpha}$ and $\mathscr{H}_{\alpha}^{\Gamma}$.*

Proof. $(\cdot, \cdot)_{\Gamma}$ is continuous because

$$|(\varphi, \psi)_{\Gamma}| \leqslant \| \varphi \|_{\Gamma,-\alpha} \| \psi \|_{\Gamma,\alpha}.$$

We next construct a unitary mappin $U: \mathscr{H}^{\Gamma}_{-\alpha} \to \mathscr{H}^{\Gamma}_{-\alpha}$ such that

$$(\varphi, \psi)_{\Gamma} = \langle\langle U\varphi, \psi \rangle\rangle_{\Gamma}. \tag{2.2}$$

Let $\epsilon > 0$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ be the decomposition of Γ used in the definition of distortion, and set

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5, \qquad \text{where} \qquad \varphi_i = \varphi \chi_{\Gamma_i}, \quad i = 1,\ldots, 5.$$

We define U by

$$U\varphi_1 = \varphi_1,$$

$$U\varphi_2(a + it) = \frac{-i(a - it)^2}{|a + it|^2} \varphi_2(a - it), \qquad 0 < t \leqslant \epsilon;$$

$$U\varphi_3(i\epsilon + t) = \frac{(t - i\epsilon)^2}{|t + i\epsilon|^2} \varphi_3(t - i\epsilon), \qquad a < t \leqslant b;$$

$$U\varphi_4(b + i(\epsilon - t)) = \frac{i(b - i(\epsilon - t))^2}{|b + i(\epsilon - t)|^2} \varphi_4(b - i(\epsilon - t)), \qquad 0 < t \leqslant \epsilon;$$

$$U\varphi_5 = \varphi_5.$$

A direct calculation shows that U has the required properties. But since $\mathscr{H}^{\Gamma}_{-\alpha}$ is the dual of $\mathscr{H}_{\alpha}^{\Gamma}$ by (2.1), it follows from (2.2) that $(\cdot, \cdot)_{\Gamma}$ is a duality between $\mathscr{H}^{\Gamma}_{-\alpha}$ and $\mathscr{H}_{\alpha}^{\Gamma}$.

PROPOSITION 2.2. *Let $0 < a < b < \infty$ and let Γ be an R_{ab}-admissible distortion. Then $\mathscr{H}^{\Gamma} \cap \mathfrak{A}_{ab}$ is dense in $\mathscr{H}^{\Gamma}$.*

Proof. Let $\Gamma_{ab} = \Gamma \cap R_{ab}$, and let $\mathscr{H}_{\Gamma_{ab}}$ be the Hilbert space of $L^2(\Omega)$-valued functions on Γ_{ab} with inner product $\langle f, g \rangle_{\Gamma_{ab}} = \int_{\Gamma_{ab}} \bar{f}(k) g(k) |k|^2 d|k|$. Then $\mathscr{H}_{\Gamma_{ab}}$ can be identified with the subspace of $\mathscr{H}_{\Gamma}$ consisting of functions with support contained in Γ_{ab}, and it is clearly sufficient to show that $\mathcal{O}_{ab} \cap \mathcal{H}_{\Gamma_{ab}}$ is dense in $\mathcal{H}_{\Gamma_{ab}}$.

The space $\mathcal{H}_{\Gamma_{ab}}$ also can be thought of as the tensor product $L^2(\Gamma_{ab}; |k|^2) \otimes L^2(\Omega)$, where $L^2(\Gamma_{ab}; |k|^2)$ is the usual L^2-space of scalar valued functions on Γ_{ab}. Thus, the space of functions

$$\left\{ \sum_{i=1}^{n} f_i(k)\, \varphi_i(\omega) \mid f_i \in L^2(\Gamma_{ab}; |k|^2),\, \varphi_i \in L^2(\Omega) \right\}$$

is dense in $\mathcal{H}_{\Gamma_{ab}}$.

Moreover, any function $f \in L^2(\Gamma_{ab}; |k|^2)$ can be ϵ-approximated in the L^2-norm by a continuous function g. The function g in turn can be approximated uniformly be a polynomial p in k, by the Stone–Weierstrass theorem. Thus, the following space D is dense in $\mathcal{H}_{\Gamma_{ab}}$,

$$D = \left\{ \sum_{i=1}^{m} k^i \varphi_i \mid \varphi_i \in L^2(\Omega),\, k \in \Gamma_{ab} \right\}.$$

Proposition 2.3. *Let $0 \leqslant a < b \leqslant \infty$ and $z_0 \in R_{ab}$. Suppose Γ and Γ' are R_{ab}-homotopic relative to z_0. Then*

$$(\varphi_\Gamma, \psi_\Gamma)_\Gamma = (\varphi_{\Gamma'}, \psi_{\Gamma'})_{\Gamma'} \tag{2.3}$$

for $\varphi \in \mathcal{H}_{-\alpha} \cap \mathcal{O}_{ab}^{\bar{z}_0}$, $\psi \in \mathcal{H}_\alpha \cap \mathcal{O}_{ab}^{z_0}$.

Proof. The function $(\varphi(\bar{k}), \psi(k))_\Omega$ is analytic in $R_{ab} \backslash \{z_0\}$, hence, by Cauchy's theorem

$$\int_\Gamma (\varphi(\bar{k}), \psi(k))_\Omega\, k^2\, dk = \int_{\Gamma'} (\varphi(\bar{k}), \psi(k))_\Omega\, k^2\, dk,$$

which is just 2.3.

For any pair $\mathcal{H}_1$, $\mathcal{H}_2$ of Hilbert spaces, we denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ the subspace of all compact operators from $\mathcal{H}_1$ into $\mathcal{H}_2$. We write $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{C}(\mathcal{H}) = \mathcal{C}(\mathcal{H}, \mathcal{H})$.

We let $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$.

3. The Interaction

Let μ be the fixed constant introduced in Section 2. In configuration space, let $\tilde{V}$ be the maximal operator of multiplication by $(8\pi\mu)^{-1} e^{-\mu r}$,

$r = (x^2 + y^2 + z^2)^{1/2}$. The corresponding operator V in momentum space is convolution by the function $(k^2 + \mu^2)^{-2}$. Thus, for suitable φ,

$$V\varphi(k, \omega) = \int_0^\infty \int_\Omega \frac{\varphi(k', \omega')\, k'^2}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2}\, d\omega'\, dk'$$

We notice that $k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2 = (kw - k'\omega')^2 + \mu^2 \neq 0$ for

$$k \in R_\alpha, \quad k' > 0, \quad \omega, \omega' \in \Omega,$$

and for

$$k > 0, \quad k' \in R_\alpha, \quad \omega, \omega' \in \Omega.$$

PROPOSITION 3.1. *V is a bounded operator from $\mathscr{H}_{+1}$ into $\mathscr{H}_{+1}$.*

Proof. In configuration space the operator of multiplication by $e^{-\mu r}$ is bounded from H_{+1} into H_{+1} since

$$\nabla(e^{-\mu r}\psi) = e^{-\mu r}(\nabla\psi - \mu\psi r^{-1}\bar{r}),$$

and hence,

$$\| e^{-\mu r}\psi \|_{+1} \leqslant 2(1 + \mu) \| \psi \|_{+1} .$$

Since the Fourier transform is a unitary map of H_{+1} onto $\mathscr{H}_{+1}$, it follows that V is bounded from $\mathscr{H}_{+1}$ into $\mathscr{H}_{+1}$.

Let Γ be an R-admissible distortion and define $V^\Gamma: \mathscr{H}^\Gamma_{+1} \to \mathscr{H}_{+1}$ by

$$V^\Gamma\varphi(k, \omega) = \int_\Gamma \int_\Omega \frac{\varphi(k'\omega')\, k'^2}{[(k\omega - k'\omega')^2 + \mu^2]^2}\, d\omega'\, dk'.$$

PROPOSITION 3.2. *V^Γ is a bounded operator from $\mathscr{H}^\Gamma_{+1}$ into $\mathscr{H}_{+1}$.*

Proof. Suppose $\Gamma = \Gamma^+_{ab,\epsilon}$ and write $\Gamma = \Gamma_r \cup \Gamma_c$ as in Proposition 2.2. Then $V^\Gamma = V^\Gamma\chi_{\Gamma_r} + V^\Gamma\chi_{\Gamma_c}$. It is obvious from Proposition 3.1 that $V^\Gamma\chi_{\Gamma_r}$ is a bounded operator from $\mathscr{H}^\Gamma_{+1}$ to $\mathscr{H}_{+1}$. Thus, it remains to consider $V^\Gamma\chi_{\Gamma_c}$. It is sufficient to show that

$$\int_0^\infty \int_\Omega \int_{\Gamma_c} \int_\Omega \frac{(1 + k^2)\, k^2(1 + | k' |^2)\, | k'^2 |}{| k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2 |^4}\, | dk' |\, d\omega'\, dk\, d\omega < \infty.$$

Clearly

$$\int_0^{4|b+i\epsilon|} \int_\Omega \int_{\Gamma_c} \int_\Omega \frac{(1 + k^2)\, k^2(1 + | k' |^2)\, | k'^2 |}{| k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2 |^4}\, | dk' |\, d\omega'\, dk\, d\omega < \infty.$$

For $k \geqslant 4 | b + i\epsilon |$, $k' \in \Gamma_c$

$$\left| 1 + \left(\frac{k'}{k}\right)^2 - 2\frac{k'}{k}\,\omega \cdot \omega' + \frac{\mu^2}{k^2} \right| \geqslant \frac{1}{4},$$

and hence,

$$\int_{4|b+i\epsilon|}^{\infty}\int_{\Omega}\int_{\Gamma_c}\int_{\Omega}\frac{(1+k^2)\,k^2(1+|k'|^2)\,|k'|^2}{|k^2+k'^2-2kk'\omega\cdot\omega'+\mu^2|^4}\,|dk'|\,d\omega'\,dk\,d\omega$$
$$\leqslant(16\pi)^2\int_{4|b+i\epsilon|}^{\infty}\frac{(1+k^2)\,k^2}{k^8}\,dk\int_{\Gamma_c}(1+|k'|^2)\,|k'|^2\,d\,|k'|<\infty,$$

and the proposition is proved.

For $\varphi\in\mathscr{H}_{-1}$, $k\in\Gamma$, let

$$^{\Gamma}V\varphi(k,\omega)=\int_0^{\infty}\int_{\Omega}\frac{\varphi(k',\omega')\,k'^2}{[k^2+k'^2-2kk'\omega\cdot\omega'+\mu^2]^2}\,d\omega'\,dk'. \tag{3.1}$$

Proposition 3.3. *$^{\Gamma}V$ is a bounded operator from $\mathscr{H}_{-1}$ into $\mathscr{H}^{\Gamma}_{-1}$ and for $\varphi\in\mathscr{H}^{\Gamma}_{+1}$, $\psi\in\mathscr{H}_{-1}$,*

$$(V^{\Gamma}\varphi,\psi)=(\varphi^{\Gamma},V\psi)_{\Gamma}. \tag{3.2}$$

Proof. For $\psi\in\mathscr{H}_{+1}$, by Proposition 3.2, $V\psi\in\mathscr{H}_{-1}$, and

$$\|V\psi\|_{-1}=\sup_{\substack{\varphi\in\mathscr{H}_{+1}\\ \|\varphi\|_{+1}=1}}|(V\psi,\varphi)|=\sup_{\substack{\varphi\in\mathscr{H}_{+1}\\ \|\varphi\|_{+1}=1}}|(\psi,V\varphi)|\leqslant\|V\|_{\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{+1})}\,\|\psi\|_{-1}.$$

Hence, V is bounded from the sense subspace $\mathscr{H}_{+1}$ of $\mathscr{H}_{-1}$ into $\mathscr{H}_{-1}$, and therefore, bounded from $\mathscr{H}_{-1}$ into $\mathscr{H}_{-1}$. Then it follows as in the proof of Proposition 3.2, that $^{\Gamma}V$ is bounded from $\mathscr{H}_{-1}$ onto $\mathscr{H}^{\Gamma}_{-1}$.

To prove 3.2, we apply Fubini-Tonelli's theorem to obtain

$$(\varphi,{}^{\Gamma}V\psi)_{\Gamma}$$
$$=\int_{\Gamma}\int_{\Omega}\overline{\varphi(\bar{k},\omega)}\left\{\int_0^{\infty}\int_{\Omega}\frac{\psi(k',\omega')\,k'^2}{[k^2+k'^2-2kk'\omega\cdot\omega'+\mu^2]^2}\,d\omega'\,dk'\right\}k^2\,d\omega\,dk$$
$$=\int_0^{\infty}\int_{\Omega}\overline{\left\{\int_{\Gamma}\int_{\Omega}\frac{\varphi(\bar{k},\omega)\,\bar{k}^2}{[\bar{k}^2+\bar{k}'^2-2\bar{k}\bar{k}'\omega\cdot\omega'+\mu^2]^2}\,d\bar{k}\,d\omega\right\}}\,\psi(k',\omega')\,k'^2\,d\omega'\,dk'.$$

Changing variable from k to $\bar{k}$, the last integral equals

$$\int_0^{\infty}\int_{\Omega}\overline{\left\{\int_{\Gamma}\int_{\Omega}\frac{\varphi(k,\omega)\,k^2}{[k^2+\bar{k}'^2-2k\bar{k}'\omega\cdot\omega'+\mu^2]^2}\,dk\,d\omega\right\}}\,\psi(k',\omega')\,k'^2\,d\omega'\,dk'$$
$$=(V^{\Gamma}\varphi,\psi).$$

The following Lemma contains an easy but fundamental estimate.

LEMMA 3.4. *Let $z_0 \in \mathbf{C}$, and let $\epsilon > 0$, $\delta > 0$ be such that*

$$D_{z_0,\epsilon} = \{z \in \mathbf{C} \mid |z - z_0| < \epsilon\} \subset \{z \in \mathbf{C} \mid |\operatorname{Im} z| < \mu - \delta\}.$$

Let $\varphi \in \mathscr{H}_{-1}$, $f \in L^2(\Omega)$. Then there exists $C_{z_0,\epsilon,\delta}$ such that

$$\int_0^\infty \int_\Omega \int_\Omega \frac{|\varphi(k', \omega')| \, k^2 \, |f(\omega)|}{|z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2|^2} \, d\omega \, d\omega' \, dk' \leqslant C_{z_0,\epsilon,\delta} \|\varphi\|_{-1} \|f\|_\Omega ,$$

for $z \in D_{z_0,\epsilon}$.

Proof. Let $K_{z_0,\epsilon} = K$ be such that $|z/k| < 1/4$ for $z \in D_{z_0,\epsilon}$. Let

$$M_{K,z_0,\epsilon} = M = \sup_{\substack{z \in D_{z_0,\epsilon} \\ 0 \leqslant k \leqslant K \\ \omega,\omega' \in \Omega}} \frac{1}{|z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2|^2} .$$

We have for $k \geqslant 0$, $z \in D_{z_0,\epsilon}$

$$|z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2| \geqslant (2\mu - \delta)\delta,$$

so

$$M_{k,z_0,\epsilon} \leqslant \frac{1}{(2\mu - \delta)\,\delta} .$$

Moreover, for $z \in D_{z_0,\epsilon}$, $k \geqslant K$, $\omega, \omega' \in \Omega$

$$\frac{1}{|(z/k')^2 + 1 - 2(z/k')\,\omega \cdot \omega' + (\mu^2/k'^2)|} \leqslant 16.$$

Hence,

$$\begin{aligned}
&\int_0^\infty \int_\Omega \int_\Omega \frac{|\varphi(k', \omega')| \, k'^2 \, |f(\omega)|}{|k^2 + z^2 - 2kz\omega \cdot \omega' + \mu^2|^2} \, d\omega \, d\omega' \, dk' \\
&\quad \leqslant M \int_0^\infty \int_\Omega |\varphi(k', \omega'| \, \chi_{(0,K)} k'^2 \, d\omega' \, dk' \int_\Omega |f(\omega)| \, d\omega \\
&\qquad + 16 \int_0^\infty \int_\Omega |\varphi(k', \omega')| \, \chi_{(K,\infty)} \frac{1}{k^4} k'^2 \, d\omega' \, dk' \cdot \int_\Omega |f(\omega)| \, d\omega \\
&\quad \leqslant M \|\varphi\|_{-1} \|\chi_{0,K}\|_{+1} (4\pi)^{1/2} \|f\|_\Omega + 16 \|\varphi\|_{-1} \left\| \frac{1}{k^4} \chi_{(0,\infty)} \right\|_{+1} (4\pi)^{1/2} \|f\|_\Omega ,
\end{aligned}$$

where $\chi_{(a,b)}$ is the characteristic function of (a, b) thought of as an $L^2(\Omega)$-valued function. Thus, the Lemma is proved with

$$C_{z_0,\epsilon,\delta} = (4\pi)^{1/2} \left(M \|\chi_{(0,K)}\|_{+1} + 16 \|(1/k^4)\, \chi_{(0,\infty)}\|_{+1} \right).$$

PROPOSITION 3.5. *For* $\varphi \in \mathscr{H}_{-1}$, $f \in L^2(\Omega)$, $|\operatorname{Im} z| < \mu$, *let*

$$F_{\varphi,f}(z) = \int_0^\infty \int_\Omega \int_\Omega \frac{\varphi(k', \omega')\, k'^2 \overline{f(\omega)}}{[z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2]^2}\, d\omega\, d\omega'\, dk'. \tag{3.3}$$

The function $F_{\varphi,f}(z)$ *is analytic for* $|\operatorname{Im} z| < \mu$.

Proof. Notice first that by Lemma 3.4, the integrand is absolutely integrable over $(0, \infty) \times \Omega \times \Omega$ with respect to $dk \times d\omega \times d\omega'$, so $F_{\varphi,f}(z)$ is well defined for $|\operatorname{Im} z| < \mu$. Now we prove analyticity of $F_{\varphi,f}(z)$ in any region $D_{z_0,\epsilon}$ defined in Lemma 3.4. Let C be a simple rectifiable closed curve in $D_{z_0,\epsilon}$. Then by (3.2) and Fubini-Tonelli's theorem we have

$$\oint_C F_{\varphi,f}(z)\, dz = \int_0^\infty \int_\Omega \int_\Omega \left\{ \int \frac{\varphi(k', \omega')\, k'^2 \overline{f(\omega)}}{[z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2]^2}\, d\omega\, d\omega'\, dk' \right\} dz = 0.$$

Hence, by Morera's theorem $F_{\varphi,f}(z)$ is analytic in $D_{z_0,\epsilon}$, and thus, for $|\operatorname{Im} z| < \mu$.

PROPOSITION 3.6. *Let* Γ *and* Γ' *be R-admissible distortions. Then* (a) ${}^{\Gamma}V(\mathscr{H}_{-1}) \subseteq \mathcal{A}_\infty$, (b) *for* $\varphi \in \mathscr{H}_{-1}$, $({}^{\Gamma}V_\varphi)_{\Gamma'} = {}^{\Gamma'}V\varphi$, (c) $(V\varphi)(-z, -\omega) = (V\varphi)(z, \omega)$ *for* $z \in R_\infty$, $\omega \in \Omega$.

Proof. Let $\varphi \in \mathscr{H}_{-1}$, and set

$$G_\varphi(z, \omega) = \int_0^\infty \int_\Omega \frac{\varphi(k', \omega')\, k'^2}{[z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2]^2}\, d\omega'\, dk'.$$

By Lemma 3.4, for $f \in L^2(\Omega)$

$$\left| \int_\Omega G_\varphi(z, \omega) \bar{f}(\omega)\, d\omega \right| \leqslant C_{z_0,\epsilon,\delta} \|\varphi\|_{-1} \|f\|_\Omega,$$

and hence, for fixed $z \in R_\infty$, $G_\varphi(z, \omega) \in L^2(\Omega)$.

By Proposition 3.5, the function $\int G_\varphi(z, \omega) \bar{f}(\omega)\, d\omega = F_{\varphi,f}(z)$ is analytic for $|\operatorname{Im} z| < \mu$ for every $f \in L^2(\Omega)$, so $G_\varphi(z, \cdot)$ is an analytic $L^2(\Omega)$-valued function in R_∞. Combining this with Proposition 3.3, we obtain

(a) ${}^{\Gamma}V\varphi = G_\varphi|_\Gamma \in \mathcal{A}_\infty$.

It follows from (2.3) that

(b) $({}^{\Gamma}V\varphi)_{\Gamma'} = ((G_\varphi)_\Gamma)_{\Gamma'} = (G_\varphi)_{\Gamma'} = {}^{\Gamma'}V\varphi$.

Property (c) is obvious.

PROPOSITION 3.7. *Let* $\varphi \in \mathscr{H}_{+1} \cap \mathcal{A}_{ab}^{z_0}$ *and let* Γ, Γ' *be* R_{ab}*-homotopic distortions relative to* z_0. *Then*

$$V^\Gamma \varphi_\Gamma = V^{\Gamma'} \varphi_{\Gamma'}.$$

Proof. It is clear that for fixed $k > 0$, $\omega \in \Omega$, the function

$$J_{k,\omega}(k', \omega') = \frac{1}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2}$$

is in $\mathcal{A}$ as an $L^2(\Omega)$-valued function of $k' \in R$, and therefore,

$$\int_\Omega \frac{\varphi(k', \omega')\, d\omega'}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2} = (J_{\bar{k},\omega}(\bar{k}', \cdot), \varphi(k', \cdot))_\Omega ,$$

is analytic on $R_{ab}\backslash\{z_0\}$. It then follows by Cauchy's theorem that

$$V^\Gamma \varphi_\Gamma(k \cdot \omega) = \int_\Gamma \int_\Omega \frac{\varphi(k', \omega')\, d\omega'}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2}$$

$$= \int_{\Gamma'} \int_\Omega \frac{\varphi(k', \omega')\, d\omega'}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2} = V^{\Gamma'} \varphi_{\Gamma'}(k, \omega).$$

We now define two additional families of operators associated with V, which are parameterized by $z \in R_\alpha$,

$$V_0(z) f(k, \omega) = \int_\Omega \frac{f(\omega')\, d\omega'}{[k^2 + z^2 - 2kz\omega \cdot \omega' + \mu^2]^2}, \qquad f \in L^2(\Omega),$$

and

$$_0V(z)\, \varphi(\omega) = \int_0^\infty \int_\Omega \frac{\varphi(k', \omega')\, k'^2\, d\omega'\, dk'}{[z^2 + k'^2 - 2zk'\omega \cdot \omega' + \mu^2]^2}, \qquad \varphi \in \mathscr{H}_{-1} . \tag{3.4}$$

PROPOSITION 3.8. *$V_0(z) \in \mathscr{L}(L^2(\Omega), \mathscr{H}_{+1})$, $_0V(z) \in \mathscr{L}(\mathscr{H}_{-1}, L^2(\Omega))$, for $z \in R_\alpha$, and*

$$(_0V(\bar{z})\, \varphi, f) = (\varphi, V_0(z) f), \tag{3.5}$$

for $\varphi \in \mathscr{H}_{-1}$, $f \in L^2(\Omega)$.

Proof. The first result follows from 3.2, and (3.5) follows from (3.2) and Fubini-Tonelli's theorem.

PROPOSITION 3.9. *$V_0(z)$ is a holomorphic function from R_α to $\mathscr{L}(L^2(\Omega), \mathscr{H}_{+1})$, and $_0V(z)$ is a holomorphic function from R_α to $\mathscr{L}(\mathscr{H}_{-1}, L^2(\Omega))$.*

Proof. By Proposition 3.5, for $\varphi \in \mathscr{H}_{-1}$, $f \in L^2(\Omega)$ the function

$$F_{\varphi,f}(z) = (f, {_0V}(z)\varphi),$$

is analytic in R_α . This implies that the $\mathscr{L}(\mathscr{H}_{-1}, L^2(\Omega))$-valued function $_0V(z)$ is holomorphic in R_α .

In the same way it is proved that $V_0(z)$ is holomorphic in R_α .

Notice that by the definition of V and ${}_0V(z)$ we have

$$V\varphi(k, \omega) = {}_0V(k)\, \varphi(\omega). \tag{3.6}$$

We now make our

Basic assumption on the interaction $\underset{\sim}{Q}$: There exists a self-adjoint compact operator U from $\mathscr{H}_{+1}$ into $\mathscr{H}_{-1}$ such that

$$\underset{\sim}{Q} = VUV.$$

By self-adjointness of U we mean that

$$(U\varphi, \psi) = (\varphi, U\psi), \qquad \text{for} \quad \varphi, \psi \in \mathscr{H}_{+1}.$$

Examples. The following operators $\tilde{U}$ on configuration space are compact from H_{+1} to H_{-1}. Conditions for compactness can be found in [3, 12, 13, 15].

1. A multiplication operator corresponding to a real-valued locally integrable function $u(\bar{r})$, which has at most $r^{-2+\epsilon}$-type local singularities and goes to 0 as $|\bar{r}| \to \infty$. In terms of $\tilde{Q}$, this implies multiplication by $e^{-2\mu r}r^{-\alpha}$, $0 < \alpha < 2$, (in particular the Yukawa potential $e^{-2\mu r}r^{-1}$), $e^{-(2\mu+\epsilon)r}$, $e^{-\epsilon r^{\beta}}$, $\beta > 1$, and any locally integrable potential with compact support and at most $r^{-2+\epsilon}$-type singularities are admissible.

2. A first-order differential operator $(1/i)(\bar{b}(\bar{r}) \cdot \nabla + \nabla \cdot \bar{b}(\bar{r}))$, where $\bar{b}(\bar{r}) = (b_1(\bar{r}), b_2(\bar{r}), b_3(\bar{r}))$, and the $b_i(\bar{r})$ are real-valued locally integrable functions that have at most $r^{-1+\epsilon}$-type local singularities and go to 0 as $|\bar{r}| \to \infty$, while $\nabla\bar{b}$ is locally integrable, has at most $r^{-2+\epsilon}$-type local singularities, and goes to 0 as $|\bar{r}| \to \infty$.

This condition is satisfied, for example, by the functions $r^{-1+\epsilon}$, and any function with compact support that is smooth, except for at most isolated $r^{-1+\epsilon}$-type singularities.

Thus, the corresponding operator U in momentum space will be a compact self-adjoint operator from $\mathscr{H}_{+1}$ to $\mathscr{H}_{-1}$.

4. The Hamiltonian and te Γ-distorted Resolvent

We define the Hamiltonian H by its quadratic form as follows. The sesquilinear form $\mathbf{H}$ is defined on $\mathscr{H}_1 \times \mathscr{H}_1$ by

$$\mathbf{H}[\varphi, \psi] = ((k^2 + \underset{\sim}{Q})\varphi, \psi), \qquad \varphi, \psi \in \mathscr{H}_{+1}.$$

This is a well-defined bounded sesquilinear form, since $k^2 + \underset{\sim}{Q}$ is bounded from $\mathscr{H}_{+1}$ into $\mathscr{H}_{-1}$. We can write $(\underset{\sim}{Q}\varphi, \psi)$ in the form

$$(\underset{\sim}{Q}\varphi, \psi) = ((k^2 + 1)^{-1}\underset{\sim}{Q}\varphi, \psi)_{+1},$$

where the operator $(k^2 + 1)^{-1}Q$ is compact from $\mathscr{H}_{+1}$ into $\mathscr{H}_{+1}$. Then it follows from a result of Stummel [18, Satz 9, p. 36] that for every $\epsilon > 0$

$$|(Q\varphi, \varphi)| < \epsilon \| \varphi \|_{+1}^2 + K(\epsilon) \| \varphi \|^2.$$

This in turn implies that for some $K > 0$, $K_1 > 0$,

$$\tfrac{1}{2} \| \varphi \|_{+1}^2 < \mathbf{H}[\varphi, \varphi] + K \| \varphi \|^2 < K_1 \| \varphi \|_{+1}^2 .$$

Then by the theory of Friedrichs (see, e.g., [15]), there corresponds to $\mathbf{H}$ a unique self-adjoint operator H in $\mathscr{H}$ with domain

$$\mathscr{D}_H = \{\varphi \in \mathscr{H}_{+1} \mid \mathbf{H}[\varphi, \psi] \text{ is defined for all } \psi \in \mathscr{H}\}$$

and defined by

$$\mathbf{H}[\varphi, \psi] = (H\varphi, \psi).$$

Moreover, $\mathscr{D}_H$ is a core for $\mathbf{H}$, i.e., the closure of $\mathscr{D}_H$ in $\mathscr{H}_1$ is $\mathscr{H}_1$.

The operator H is the Hamiltonian of our system. We denote its resolvent set, spectrum, essential spectrum, discrete spectrum, point spectrum, absolutely continuous spectrum, and singular continuous spectrum by $\rho(H)$, $\sigma(H)$, $\sigma_e(H)$, $\sigma_d(H)$, $\sigma_p(H)$, $\sigma_{ac}(H)$, and $\sigma_{sc}(H)$, respectively.

By the assumption on Q, H can be considered as a bounded operator from $\mathscr{H}_{+1}$ to $\mathscr{H}_{-1}$. We denote by $\tilde{\rho}(H)$, $\tilde{\sigma}(H)$, $\tilde{\sigma}_e(H)$, and $\tilde{\sigma}_d(H)$ the resolvent set, spectrum, essential spectrum, and discrete spectrum of $H \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$. Thus, $\tilde{\rho}(H) = \{z \in \mathbf{C} \mid (H - z)^{-1} \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1}\}$, $\tilde{\sigma}(H) = \mathbf{C} \backslash \tilde{\rho}(H)$, $\tilde{\sigma}_d(H)$ is the set of poles of $(H - z)^{-1}: \mathscr{H}_{-1} \to \mathscr{H}_{+1}$, and $\tilde{\sigma}_e(H) = \tilde{\sigma}(H) \backslash \tilde{\sigma}_d(H)$.

We shall now establish the identity of various parts of the spectrum of the self-adjoint operator H in $\mathscr{H}$ and that of $H \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$.

PROPOSITION 4.1. *$\sigma_e(H) = \tilde{\sigma}_e(H) = [0, \infty)$, and $\sigma_d(H) = \tilde{\sigma}_d(H)$.*

Proof. Suppose $z \in \tilde{\rho}(H)$, then for $\varphi \in \mathscr{H}$

$$\|(H - z)^{-1}\varphi\| \leqslant \|(H - z)^{-1}\varphi\|_{+1} \leqslant K \|\varphi\|_{-1} \leqslant K \|\varphi\|,$$

so $z \in \rho(H)$ and $\|(H - z)^{-1}\|_{\mathscr{L}(\mathscr{H})} \leqslant \|(H - z)^{-1}\|_{\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})}$. The self-adjoint operator H is bounded below, so there is a $K > 0$ such that $\lambda \in \rho(H)$ for $\lambda < -K$. Now let λ be any such real number, then $H - \lambda$ is positive definite, and $(H - \lambda)^{1/2}$ is self-adjoint with domain $\mathscr{H}_{+1}$ and range $\mathscr{H}$. By the open mapping theorem $(H - \lambda)^{-1/2}$ is bounded from $\mathscr{H}$ onto $\mathscr{H}_{+1}$. The adjoint of $(H - \lambda)^{-1/2} \in \mathscr{L}(\mathscr{H}, \mathscr{H}_{+1})$ is in $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H})$ and coincides on $\mathscr{H}$ with $(H - \lambda)^{-1/2}$. Hence, $(H - \lambda)^{-1}$ is bounded from $\mathscr{H}$ with the $\mathscr{H}_{-1}$-norm to $\mathscr{H}_{+1}$, and since $\mathscr{H}$ is dense in $\mathscr{H}_{-1}$, $(H - \lambda)^{-1} \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$.

Now consider the second resolvent equations

$$R(z) = R_0(z)(1 + QR_0(z))^{-1}, \tag{4.1}$$

$$R_0(z) = R(z)(1 - QR(z))^{-1}, \tag{4.2}$$

viewed as equations in $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$.

The operator $QR_0(z)$ is a $\mathscr{C}(\mathscr{H}_{-1})$-valued analytic function on $\mathbf{C}\backslash\mathbb{R}^+$, hence, $(1 + QR_0(z))^{-1}$ is a meromorphic $\mathscr{L}(\mathscr{H}_{-1})$-valued function on $\mathbf{C}\backslash\mathbb{R}^+$ if $(1 + QR_0(z))^{-1} \in \mathscr{L}(\mathscr{H}_{-1})$ exists for some z. But for $z = \lambda < -K$, $R(\lambda) \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$, and since $H_0 - \lambda \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$, it follows that $(1 + QR_0(\lambda))^{-1} \in \mathscr{L}(\mathscr{H}_{-1})$. It follows that $R(z)$ is a meromorphic $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$-valued function on $\mathbf{C}\backslash\mathbb{R}^+$, i.e., $\tilde{\sigma}_e(H) \subset \mathbb{R}^+$. Similarly, $H - \lambda \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$ for $\lambda < -K$, and $R_0(\lambda) \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$, hence, $(1 - QR(\lambda))^{-1} \in \mathscr{L}(\mathscr{H}_{-1})$. The operator $QR(z)$ is a $\mathscr{C}(\mathscr{H}_{-1})$-valued analytic function in the domain of holomorphy of $R(z) \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$, so $(1 - QR(z))^{-1}$ is a meromorphic $\mathscr{L}(\mathscr{H}_{-1})$-valued function on $\tilde{\rho}(H)$, and hence, $R_0(z)$ is an $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$-valued meromorphic function on $\tilde{\rho}(H)$.

It follows that $\tilde{\sigma}_e(H) = [0, \infty)$. Since $\tilde{\rho}(H) \subseteq \rho(H)$, it follows that $\sigma_e(H) \subseteq [0, \infty)$, and $\sigma_d(H) \subseteq \tilde{\sigma}_d(H)$.

Now let λ_0 be a pole of the meromorphic $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$-valued function $R(z)$. Then the equation $(H - \lambda_0)\varphi = 0$ has a nontrivial solution in $\mathscr{H}_{+1}$ and it is clear that φ belongs to the domain of the self-adjoint operator H. Hence, $\tilde{\sigma}_d(H) \subseteq \sigma_d(H)$.

It remains to be proved that every $\lambda > 0$ belongs to $\sigma(H)$. We prove this indirectly, assuming there exists $\lambda > 0$, $\lambda \in \rho(H)$, and hence, $\mu \in \rho(H)$ for $|\mu - \lambda| < \epsilon$. We will show that this implies $\lambda \in \tilde{\rho}(H)$, in contradiction with the above result. Let

$$H_1 = \int_{\lambda}^{\infty} \mu \, dE_\mu , \qquad H_2 = \int_{-\infty}^{\lambda} \mu \, dE_\mu ,$$

where E_λ is the spectral family of H. Then $H_1 - \lambda$ and $\lambda - H_2$ are positive definite and invertible, and

$$(H - \lambda)^{-1} = (H_1 - \lambda)^{-1} \oplus (-(\lambda - H_2)^{-1}).$$

It suffices to prove that $(H_1 - \lambda)^{-1}$ and $(\lambda - H_2)^{-1}$ are bounded from $(1 - E_\lambda)\mathscr{H}$ and $E_\lambda\mathscr{H}$, respectively, with the $\mathscr{H}_{-1}$-norm into $\mathscr{H}_{+1}$. This in turn follows if we prove that $|H - \lambda|^{-1} = |(H - \lambda)^{-1}| = (H_1 - \lambda)^{-1} \oplus (\lambda - H_2)^{-1}$ is bounded from $\mathscr{H}$ with the $\mathscr{H}_{-1}$-norm in $\mathscr{H}_{+1}$. The operator $|H - \lambda|^{1/2}$ has domain

$$\mathscr{D}(|H - \lambda|^{1/2}) = \left\{\varphi \in \mathscr{H} \;\middle|\; \int_{\lambda_0}^{\infty} |\mu - \lambda| \, d(E_\mu\varphi, \varphi) < \infty\right\},$$

where $\lambda_0 < -K$.

Clearly this equals

$$\mathscr{H}_{+1} = \mathscr{D}((H - \lambda_0)^{1/2}) = \left\{\varphi \in \mathscr{H} \,\middle|\, \int_{\lambda_0}^{\infty} (\mu - \lambda_0)\, d(E_\mu \varphi, \varphi) < \infty\right\}.$$

Thus, $| H - \lambda |^{-1/2}$ is bounded from $\mathscr{H}$ onto $\mathscr{H}_{+1}$. Then the adjoint is bounded from $\mathscr{H}_{-1}$ into $\mathscr{H}$. Since it coincides with $| H - \lambda |^{-1/2}$ on the dense subspace $\mathscr{H}$ of $\mathscr{H}_{-1}$, it follows that $| H - \lambda |^{-1/2}$ has a bounded extension as an operator from $\mathscr{H}_{-1}$ to $\mathscr{H}_{+1}$. Thus, $\lambda \in \tilde{\rho}(H)$, a contradiction.

In what follows we consider the resolvent $R(z^2)$ as a function of z in the upper half-plane. Then by Proposition 4.1, $R(z^2)$ is a meromorphic function of z for $\operatorname{Im} z > 0$ with poles on the positive imaginary axis, considered as an $\mathscr{L}(\mathscr{H})$-valued function or an $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$-valued function. The poles $\{ik_0\}$ of $R(z^2)$ are the eigenvalues $\{-k_0^2\}$ of H.

We now proceed to the definition of the Γ-distorted resolvent.

Let $\Gamma_{ab} \equiv \Gamma$ (i.e., $\Gamma = \Gamma_{ab,\epsilon}$ or $\Gamma_{abc,\epsilon_1\epsilon_2}$) be an R-admissible distortion and let $\mathbf{C}_\Gamma$ be the connected component of $\mathbf{C} - \{\Gamma \cup (-\Gamma) \cup \{0\}\}$ that contains $a + b/2$ in its interior. For $z \in \mathbf{C}_\Gamma$, let $R_0^\Gamma(z^2)$ be multiplication by $1/k^2 - z^2$ acting on $\mathscr{H}_{-1}^\Gamma$. Since $1 + k^2/k^2 - z^2$ is bounded on Γ, it is clear that $R_0^\Gamma(z^2) \in \mathscr{L}(\mathscr{H}_{-1}^\Gamma, \mathscr{H}_{+1}^\Gamma)$. $R_0^\Gamma(z^2)$ is called the *Γ-distorted free resolvent.* It is also clear that $R_0^\Gamma(z^2)$ is a holomorphic function from $\mathbf{C}_\Gamma$ into $\mathscr{L}(\mathscr{H}_{-1}^\Gamma, \mathscr{H}_{+1}^\Gamma)$. Then by Propositions 3.2 and 3.3, the operators $R_0^\Gamma(z^2)^\Gamma VUV^\Gamma$ form a holomorphic $\mathscr{C}(\mathscr{H}_{+1}^\Gamma)$-valued function on $\mathbf{C}_\Gamma$. The operators $R_0^\Gamma{}^\Gamma VUV^\Gamma$ are called the *Γ-distorted Lippman–Schwinger operators.* By Corollary 4.4 below, $(1 + R_0^\Gamma(z^2)^\Gamma VUV^\Gamma)^{-1}$ exists for some $z_0 \in \mathbf{C}_\Gamma$, and hence, $(1 + R_0^\Gamma(z^2)^\Gamma VUV^\Gamma)^{-1}$ is a meromorphic $\mathscr{L}(\mathscr{H}_{+1}^\Gamma)$-valued function on $\mathbf{C}_\Gamma$ [11]. The point z_0 is a pole of this function is and only if the equation:

$$R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma \psi_\Gamma = -\psi_\Gamma,$$

has a solution $\psi_\Gamma \neq 0$ in $\mathscr{H}_{+1}^\Gamma$.

The operator $R^\Gamma(z^2) \equiv (1 + R_0^\Gamma(z^2)^\Gamma VUV^\Gamma)^{-1} R^\Gamma{}_0(z^2)$ is called the *Γ-distorted resolvent.* Note that $R^\Gamma(z^2)$ is a $\mathscr{L}(\mathscr{H}_{-1}^\Gamma, \mathscr{H}_{+1}^\Gamma)$-valued meromorphic function on $\mathbf{C}_\Gamma$ with the same poles as $(1 + R_0^\Gamma(z^2)^\Gamma VUV^\Gamma)^{-1}$.

LEMMA 4.2. *Let $z_0 \in \mathbf{C}_\Gamma$, and let Γ_{ab} and $\Gamma'_{a'b'}$ be distortions that are R-homotopic relative to z_0. Let $c < \min(a, a') < \max(b, b') < d$ and let $\varphi \in \mathscr{H}_{+1}^\Gamma \cap \mathfrak{A}_{cd}^{z_0}$. Suppose that ψ_Γ is a solution of*

$$(1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\psi_\Gamma = \varphi_\Gamma. \tag{4.3}$$

Then $\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma \cap \mathfrak{A}_{cd}^{z_0}$ and

$$(1 + R_0^{\Gamma'}(z_0^2)^{\Gamma'} VUV^{\Gamma'})\psi_{\Gamma'} = \varphi_{\Gamma'}. \tag{4.4}$$

In particular, if ψ_Γ *is a solution of*

$$(1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\psi_\Gamma = 0, \tag{4.5}$$

then $\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma \cap \mathfrak{A}_{cd}^{z_0}$ *and*

$$(1 + R_0^{\Gamma'}(z_0^2)^{\Gamma'} VUV^{\Gamma'})\psi_{\Gamma'} = 0. \tag{4.6}$$

Proof. It follows from Proposition 3.6(a), that ${}^\Gamma VUV^\Gamma\psi_\Gamma \in \mathscr{H}_{-1}^\Gamma \cap \mathfrak{A}$. Since $R_0^\Gamma(z_0^2)$ creates at most a pole at z_0 and maps $\mathscr{H}_{-1}^\Gamma$ to $\mathscr{H}_{+1}^\Gamma$, it follows that $R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma \cap \mathfrak{A}^{z_0}$. Thus, $\psi_\Gamma = \varphi_\Gamma - R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma \cap \mathfrak{A}_{cd}^{z_0}$.

We can then apply the analytic continuation map to (4.1) to obtain, using Proposition 3.6(b),

$$\psi_{\Gamma'} + R_0^{\Gamma'}(z_0^2)^{\Gamma'} VUV^\Gamma\psi_\Gamma = \varphi_{\Gamma'}.$$

By Proposition 3.7, $V^\Gamma\psi_\Gamma = V^{\Gamma'}\psi_{\Gamma'}$, and thus (4.4).

LEMMA 4.3. *Let* $z_0 \in \mathbf{C}_\Gamma$ *and suppose* $\Gamma_{ab,\epsilon}$ *and* $\Gamma'_{a'b',\epsilon}$ *are distortions that are R-homotopic relative to* z_0. *Then* (a) $R^\Gamma(z_0^2)$ *exists if and only if* $R^{\Gamma'}(z_0^2)$ *does, and* (b) *for* $\varphi, \psi \in \mathscr{H}_{-1} \cap \mathfrak{A}_{cd}$, *with c, d as in Lemma* 4.2,

$$(\varphi_\Gamma, R^\Gamma(z_0^2)\psi_\Gamma)_\Gamma = (\varphi_{\Gamma'}, R^{\Gamma'}(z_0^2)\psi_{\Gamma'})_{\Gamma'}. \tag{4.7}$$

Proof. (a) Suppose $R^\Gamma(z_0^2)$ does not exist. Then

$$(1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\psi_\Gamma = 0,$$

has a nontrivial solution ψ_Γ, and by Lemma 4.2, $\psi_\Gamma \in \mathfrak{A}_{cd}^{z_0}$ and

$$(1 + R_0^{\Gamma'}(z_0^2)^{\Gamma'} VUV^{\Gamma'})\psi_{\Gamma'} = 0.$$

Here $\psi_{\Gamma'}$ is nontrivial, since otherwise by analytic continuation ψ_Γ would be 0, and hence, $R^{\Gamma'}(z_0^2)$ does not exist.

(b) Assume now, that $R^\Gamma(z_0^2)$, and hence, by (a), $R^{\Gamma'}(z_0^2)$ exist. Then

$$(1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\vartheta_\Gamma = R_0^\Gamma(z_0^2)\psi_\Gamma,$$

has a solution ϑ_Γ in $\mathscr{H}_{+1}^\Gamma$, and by Lemma 4.2, $\vartheta_\Gamma \in \mathfrak{A}_{cd}^{z_0}$ and

$$(1 + R_0^{\Gamma'}(z_0^2)^{\Gamma'} VUV^{\Gamma'})\vartheta_{\Gamma'} = R_0^{\Gamma'}(z_0^2)\psi_{\Gamma'}.$$

Note that $\vartheta_\Gamma = R^\Gamma(z_0^2)\psi_\Gamma$ and $\vartheta_{\Gamma'} = R^{\Gamma'}(z_0^2)\psi_{\Gamma'}$. Then applying Proposition 2.2 we have

$$(\varphi_\Gamma, R^\Gamma(z_0^2)\psi_\Gamma)_\Gamma = (\varphi_\Gamma, \vartheta_\Gamma)_\Gamma = (\varphi_{\Gamma'}, \vartheta_{\Gamma'}) = (\varphi_{\Gamma'}, R^{\Gamma'}(z_0^2)\psi_{\Gamma'})_{\Gamma'}.$$

COROLLARY 4.4. *The poles of the meromorphic* $\mathscr{L}(\mathscr{H}_{-1}^{\Gamma}, \mathscr{H}_{+1}^{\Gamma})$*-valued function* $R^{\Gamma}(z^2)$ *can only occur on* (a, b) *or in the interior of the rectangle* $\{\Gamma_c \cup [a, b]\}$.

Proof. If $\Gamma' = (0, \infty)$, $R^{\Gamma'}(z^2)$ has no nonreal poles. If z_0 belongs to the exterior of the rectangle $\{\Gamma_c \cup [a, b]\}$, then Γ and Γ' are R-homotopic with respect to z_0, and by Lemma 4.3, z_0 cannot be a pole of $R^{\Gamma}(z^2)$, since this would imply that z_0 was a pole of $R(z^2)$.

DEFINITION. A point z_0 such that $\operatorname{Im} z_0 \leqslant 0$ is called a *resolvent resonance* if either of the following holds,

(a) $\operatorname{Re} z_0 > 0$, $R^{\Gamma}(z^2)$ has a pole at z_0 for some negative distortion Γ (Fig. 4a).

(b) $\operatorname{Re} z_0 < 0$, $R^{\Gamma}(z^2)$ has a pole at $-z_0$ for some positive distortion Γ (Fig. 4b).

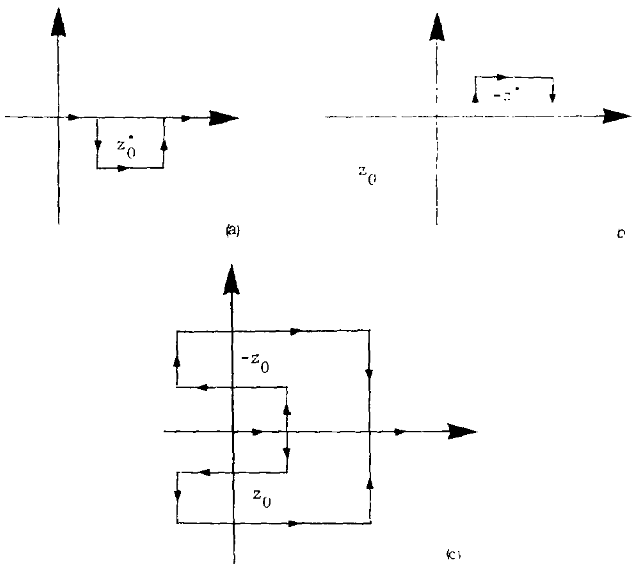

FIGURE 4

Remark. The function z^2 maps R_0^{∞} in a one-to-one way onto a parabolic subset of the two-sheeted Riemann surface of $z^{1/2}$. The upper half-plane corresponds to the first (physical) sheet and the lower half-plane to the second sheet, hence, the definition of resonances.

A *virtual pole* is a point z_0 on the negative imaginary axis such that $z_0(-z_0)$ is a pole of $R^{\Gamma}(z^2)$ for some negative (positive) distortion $\Gamma = \Gamma_{abc,\epsilon_1\epsilon_2}$ (Fig. 4).

In agreement with the above definitions, a discrete eigenvalue of H is a point z_0 on the positive imaginary axis such that $R(z^2)$ has a pole at z_0, i.e., such that z_0^2 is an eigenvalue of H in the usual sense.

We denote by G the set of points $z \in R_\infty^0$, such that z is not a discrete eigenvalue, a resonance or a virtual pole of H.

It follows from Lemma 4.3, that a resonance z_0 is a pole of $R^\Gamma(z^2)$ for all distortions Γ that are not homotopic to $(0, \infty)$ relative to z_0 (if $\operatorname{Re} z_0 > 0$) or $-z_0$ (if $\operatorname{Re} z_0 < 0$).

Remark. If z_0 is a resonance, then any nontrivial solution $\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma$ of

$$(1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\psi_\Gamma = 0$$

is called a resonance state corresponding to z_0. Note that $\psi_\Gamma \in \mathfrak{A}^{z_0}$ (resp. $\mathfrak{A}^{-z_0}$) and z_0 $(-z_0)$ is a simple pole of ψ_Γ.

LEMMA 4.5. *Let* $0 \leqslant c < d \leqslant \infty$ *and let* $\varphi, \psi \in \mathscr{H}_{-1} \cap \mathfrak{A}_{cd}$. *Then the function* $(\varphi, R(z^2)\psi)$ *has a meromorphic continuation to the region* $\{z \in \mathbf{C} \mid \operatorname{Im} z > 0\} \cup R_{cd} \cup \{-R_{cd}\}$, *with poles occurring at most at discrete eigenvalues, resonances and virtual poles of H.*

Proof. By the compactness of $R_0(z^2)VUV$ on $\mathscr{H}_{+1}$, $\operatorname{Im} z > 0$, the function $R(z^2) = (1 + R_0(z^2)VUV)^{-1}R_0(z^2)$ is a meromorphic $\mathscr{L}(\mathscr{H}_{+1})$-valued function for $\operatorname{Im} z > 0$ with poles at the discrete eigenvalues of H. Hence, $(\varphi, R(z^2)\psi)$ is meromorphic in the same region.

We define the meromorphic continuation $\mathscr{F}_{\varphi,\psi}(z)$ of $(\varphi, R(z^2)\psi)$ as follows. Let z be a point in R_{cd} with $\operatorname{Im} z \leqslant 0$, which is not a resonance, and let Γ_z be a negative R_{cd}-admissible distortion which is not R_{cd}-homotopic to $(0, \infty)$ with respect to z. Set

$$\mathscr{F}_{\varphi,\psi}(z) = (\varphi_{\Gamma_z}, R^{\Gamma_z}(z^2)\psi_{\Gamma_z})_{\Gamma_z}.$$

By Lemma 4.3 this does not depend on Γ_z and

$$\mathscr{F}_{\varphi,\psi}(z) = (\varphi, R(z^2)\psi), \qquad \text{for} \quad \operatorname{Im} z > 0.$$

A fixed negative R_{cd}-admissible distortion $\Gamma = \Gamma_{c'd',\epsilon}$ is a Γ_z for every z in the interior of the rectangle bounded by Γ and $[c', d']$ and on (c', d'). The function $R^\Gamma(z^2)$ is a meromorphic $\mathscr{L}(\mathscr{H}_{-1}^\Gamma, \mathscr{H}_{+1}^\Gamma)$ valued function in $\{z \mid \operatorname{Im} z > 0\} \cup R_{c'd',\epsilon}$ with poles at the resonances in $\{z \mid \operatorname{Im} z \leqslant 0\} \cap R_{c'd',\epsilon}$. Hence, $\mathscr{F}_{\varphi,\psi}(z)$ is meromorphic in the same region. Since $\{z \mid \operatorname{Im} z \leqslant 0\} \cap R_{cd}$ is the union of all such regions, we have obtained the meromorphic continuation of $(\varphi, R(z^2)\psi)$ into $\{z \mid \operatorname{Im} z \leqslant 0\} \cap R_{cd}$. Similarly, we define the meromorphic continuation into $\{z \mid \operatorname{Im} z \leqslant 0\} \cap \{-R_{cd}\}$, using positive distortions not homotopic to $(0, \infty)$ with respect to $-z_0$.

LEMMA 4.6. *A point $z_0 \in R^+$ is a resonance if and only if it is a positive eigenvalue of H, and in that case z_0 is a simple pole of $R^{\Gamma}_{ab}(z^2)$ for $0 < a < z_0 < b < \infty$.*

Proof. Let $z_0 > 0$, and let $R_{ab,\epsilon}$ be such that $0 < a < z_0 < b < \infty$ and $R_{ab,\epsilon} \setminus z_0$ does not contain any resonances. Let Γ be an $R_{ab,\epsilon}$-admissible negative distortion, and let $\varphi, \psi \in \mathscr{H} \cap \mathcal{O}_{ab}$, so $\varphi_\Gamma \in \mathscr{H}_\Gamma \cap \mathcal{O}_{ab}$, $\psi_\Gamma \in \mathscr{H}_\Gamma \cap \mathcal{O}_{ab}$. Denote by P_{z_0} the projection on the eigenspace of H corresponding to z_0, so that $P_{z_0} \neq 0$ if and only if z_0 is an eigenvalue of H. Then for $0 < \delta < \pi/2$,

$$P_{z_0} = \lim_{\substack{z \to z_0 \\ \delta \leqslant \arg(z - z_0) \leqslant \pi - \delta}} (z - z_0) R(z),$$

and hence,

$$(\varphi, P_{z_0}\psi) = \lim_{\substack{z \to z_0 \\ \delta \leqslant \arg(z - z_0) \leqslant \pi - \delta}} (z - z_0)(\varphi, R(z)\psi).$$

By Lemma 4.3, for $\operatorname{Im} z > 0$

$$(\varphi, R(z^2)\psi) = (\varphi_\Gamma, R^\Gamma(z^2)\psi_\Gamma).$$

Thus,

$$(\varphi, P_{z_0}\psi) = \lim_{\substack{z \to z_0 \\ \delta \leqslant \arg(z - z_0) \leqslant \pi - \delta}} (z - z_0)(\varphi_\Gamma, R^\Gamma(z^2)\psi_\Gamma).$$

By Proposition 2.2, $\mathscr{H} \cap \mathcal{O}_{ab}$ is dense in $\mathscr{H}$ and $\mathscr{H}_\Gamma \cap \mathcal{O}_{ab}$ is dense in $\mathscr{H}^\Gamma$.
If $P_{z_0} \neq 0$, there exists $\varphi_0, \psi_0 \in \mathscr{H} \cap \mathcal{O}_{ab}$, (such that $(\varphi_0, P_{z_0}, \psi_0) \neq 0$, so $(\varphi_\Gamma, R^\Gamma(z^2)\psi_\Gamma)$ has a simple pole at z_0, and z_0 is a resonance.
Conversely, if z_0 is a resonance, there exist $\varphi_\Gamma, \psi_\Gamma \in \mathscr{H}_\Gamma \cap \mathcal{O}_{ab}$ such that $(\varphi_\Gamma, R^\Gamma(z^2)\psi_\Gamma)$ has a pole at z_0, so $(\varphi, P_{z_0}\psi) \neq 0$, and z_0 is an eigenvalue. We have also show that z_0 is a simple pole of $R^\Gamma(z^2)$.
If $z_0 < 0$, the proof is similar, using positive distortions.

COROLLARY 4.7. *The singular continuous spectrum of H is empty, the absolutely continuous spectrum is* $[0, \infty)$ *and the point spectrum accumulates at most at* 0.

Proof. By Proposition 4.1, $\sigma_e(H) = [0, \infty)$.
It follows from Lemma 4.5, as in [1, 4], that the spectral family of H has no singular continuous spectrum on any interval $[\epsilon, K] \subset (0, \infty)$. Moreover, by Lemma 4.6, the positive eigenvalues have no accumulation point in any interval $[\epsilon, K] \subset (0, \infty)$. This implies, that $\sigma_{ac}(H) = [0, \infty)$, and that the point spectrum accumulates at most at 0.

Remark. Under rather weak conditions on Q positive eigenvalues are ruled out. If for example Q is a radial, multiplicative, dilation–analytic potential, this is known (cf. [17]). Thus, for the potentials $r^{-\alpha}e^{-\mu r^{\beta}}$, $\alpha > 0$, $\beta \geqslant 1$, it holds. We conjecture, that this holds generally under our assumptions.

The next Lemma will be useful in defining the analytic continuations of the various T-operators discussed in the next section.

LEMMA 4.8. *The $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$-valued meromorphic function $W(z) = VR(z^2)V$ in $\{z \mid \operatorname{Im} z > 0\}$ has a meromorphic continuation $W(z)$ to $R_\infty{}^0$ that is analytic in G.*

Proof. For $z \in G$, z not on the negative imaginary axis, let $\Gamma = \Gamma_z$ be as defined in the proof of Lemma 4.5. For z on the negative imaginary axis let Γ_z be a distortion of the type $\Gamma_{abc,\epsilon_1\epsilon_2}$. Then we define $\tilde{W}(z)$ by

$$\tilde{W}(z) = V^{\Gamma_z}R^{\Gamma_z}(z^2)^{\Gamma_z}V.$$

Then $\tilde{W}(z)$ is a bounded operator from $\mathscr{H}_{-1}$ to $\mathscr{H}_{+1}$. For z not on negative imaginary axis, this definition is independent of Γ_z, since the matrix elements $(\varphi, V^{\Gamma_z}R^{\Gamma_z}(z^2)^{\Gamma_z}V\psi) = ({}^{\Gamma_z}V\varphi, R^{\Gamma_z}(z^2)^{\Gamma_z}V\psi)$, $\varphi, \psi \in \mathscr{H}_{-1}$, are independent of Γ_z be Proposition 3.6 and the argument of Lemma 4.5. For z on the negative imaginary axis, the same argument shows the independence of Γ_z within the class of positive distortions as well for Γ_z within the class of negative distortions. The following proof will then show that distortions $\Gamma_{abc,\epsilon_1\epsilon_2}$ lead to the same operator for $\epsilon_1, \epsilon_2 > 0$ and $\epsilon_1, \epsilon_2 < 0$. In order to prove the Lemma it is now sufficient to show that for $\varphi, \psi \in \mathscr{H}_{-1}$, the function

$$\mathscr{F}_{\varphi,\psi}(z) = (\varphi, VR(z^2)V\psi) = (V\varphi, R(z^2)V\psi),$$

has a meromorphic continuation to $R_\infty{}^0$, analytic in G. By Proposition 3.6, the functions $V\varphi$ and $V\psi$ are analytic in R_∞, so by Lemma 4.5, $\mathscr{F}_{\varphi,\psi}(z)$ has a meromorphic continuation to $R \cup (-R)$ with the above restriction on the poles. Using contours of the type $\Gamma_{abc,\epsilon_1\epsilon_2}$ we obtain by the argument of the proof of Lemma 4.5 meromorphic continuations $\mathscr{F}^+_{\varphi,\psi}(z)$ and $\mathscr{F}^-_{\varphi,\psi}(z)$ of $\mathscr{F}_{\varphi,\psi}(z)$ across R^+ and R^- to the regions $R_\infty{}^0\backslash R^-$ and $R_\infty{}^0\backslash R^+$, respectively. It remains to be shown that for z on the negative imaginary axis, z not a virtual pole,

$$\mathscr{F}^+_{\varphi,\psi}(z) = \mathscr{F}^-_{\varphi,\psi}(z).$$

Let $z_0 = -it_0$, $t_0 > 0$ be such a point and let $\Gamma = \Gamma_{abc,\epsilon_1\epsilon_2}$ be a negative distortion not homotopic to R^+ with respect to z_0. Then $\mathscr{F}_{\varphi,\psi}(z_0)$ and $\mathscr{F}_{\varphi,\psi}(z_0)$ given by

$$\mathscr{F}^+_{\varphi,\psi}(z_0) = ((V\varphi)_\Gamma, R^\Gamma(z_0{}^2)(V\psi)_\Gamma)_\Gamma,$$

$$\mathscr{F}^-_{\varphi,\psi}(z_0) = ((V\varphi)_\Gamma, R^\Gamma(z_0{}^2)(V\psi)_\Gamma)_\Gamma.$$

Writing

$$\chi_\Gamma^+ = (1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2))^{-1}(V\psi)_\Gamma ,$$

and

$$\chi_{\bar\Gamma}^- = (1 + {}^{\bar\Gamma} VUV^{\bar\Gamma} R_0^{\bar\Gamma}(z_0^2))^{-1}(V\psi)_{\bar\Gamma} ,$$

we have

$$\chi_\Gamma^+ = (V\psi)_\Gamma - {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2)\chi_\Gamma^+, \tag{4.8}$$

and

$$\chi_{\bar\Gamma}^- = (V\psi)_{\bar\Gamma} - {}^{\bar\Gamma} VUV^{\bar\Gamma} R^{\bar\Gamma}_0(z_0^2)\chi_{\bar\Gamma}^-. \tag{4.9}$$

By Proposition 3.6 (a) and (b), the functions χ_Γ^+ and $\chi_{\bar\Gamma}^-$ are in R_∞ , and setting $\chi_{\bar\Gamma}^+ = (\chi_\Gamma^+)_{\bar\Gamma}$, we have by Proposition 3.6(c),

$$\chi_{\bar\Gamma}^+(z_0 , \omega) = \chi_{\bar\Gamma}^+(-z_0 , -\omega). \tag{4.10}$$

Then

$$\begin{aligned}
&V^\Gamma R_0^\Gamma(z_0^2)\, \chi_\Gamma^+(k, \omega) \\
&= \int_\Gamma \frac{k'^2}{[k^2 + k'^2 - 2kk'\omega \cdot \omega']^2} \frac{\chi_\Gamma^+(k', \omega')}{k' + z_0} \frac{1}{k' - z_0}\, d\omega'\, dk' \\
&= \int_{R^+} \frac{k'^2}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2} \frac{(\chi_\Gamma^+)_{R^+}(k', \omega')}{k' - z_0} \frac{1}{k' + z_0}\, d\omega'\, dk' \\
&\quad + \pi i z_0 \int_\Omega \frac{\chi_\Gamma^+(z_0 , \omega')}{[k^2 + z_0^2 - 2kz_0\omega \cdot \omega' + \mu^2]^2}\, d\omega' \\
&= \int_{\bar\Gamma} \frac{k'^2}{[k^2 + k'^2 - 2kk'\omega \cdot \omega' + \mu^2]^2} \frac{\chi_{\bar\Gamma}^+(k', \omega')}{k'^2 - z_0^2}\, d\omega'\, dk' \\
&\quad - \pi i z_0 \int_\Omega \frac{\chi_{\bar\Gamma}^+(-z_0 , \omega')}{[k^2 + z_0^2 + 2kz_0\omega \cdot \omega' + \mu^2]^2}\, d\omega' \\
&\quad + \pi i z_0 \int_\Omega \frac{\chi_\Gamma^+(z_0 , \omega')}{[k^2 + z_0^2 - 2kz_0\omega \cdot \omega' + \mu^2]^2}\, d\omega'.
\end{aligned}$$

By (4.10), the last two terms cancel, so

$$V^\Gamma R_0^\Gamma(z_0^2)\, \chi_\Gamma^+(k, \omega) = V^{\bar\Gamma} R_0^{\bar\Gamma}(z_0^2)\, \chi_{\bar\Gamma}^+(k, \omega).$$

Then, by analytic continuation of Eq. (4.8) to $\bar\Gamma$ we have

$$\chi_{\bar\Gamma}^+ + {}^{\bar\Gamma} VUV^{\bar\Gamma} R_0^{\bar\Gamma}(z_0^2)\, \chi_{\bar\Gamma}^+ = (V\psi)_{\bar\Gamma} .$$

Thus, $\chi_{\bar\Gamma}^+$ is solution of the same Eq. (4.9) as $\chi_{\bar\Gamma}^-$, and hence, $\chi_{\bar\Gamma}^+ = \chi_{\bar\Gamma}^-$, i.e., the function χ_Γ^+ and $\chi_{\bar\Gamma}^-$ are analytic continuations of each other.

Let χ be the analytic function on R_∞, such that $\chi_\Gamma = \chi_{\Gamma}{}^+$ and $\chi_{\bar\Gamma} = \chi_{\bar\Gamma}{}^-$. Then we have

$$\mathscr{F}^+_{\varphi,\psi}(z_0) = ((V\varphi)_\Gamma, R_0{}^\Gamma(z_0{}^2)\, \chi_\Gamma)_\Gamma,$$

$$\mathscr{F}^-_{\varphi,\psi}(z_0) = ((V\varphi)_{\bar\Gamma}, R_0{}^{\bar\Gamma}(z_0{}^2)\, \chi_{\bar\Gamma})_{\bar\Gamma}.$$

Now, by a similar residue calculation,

$$\mathscr{F}^+_{\varphi,\psi}(z_0) = (V\varphi, R_0(z_0{}^2)\, \chi) + \pi i z_0(\overline{V\varphi(-z_0)}, \chi(z_0))_\Omega \tag{4.11}$$

and

$$\mathscr{F}^-_{\varphi,\psi}(z_0) = (V\varphi, R_0(z_0{}^2)\, \chi) + \pi i z_0(\overline{V\varphi(z_0)}, \chi(-z_0))_\Omega. \tag{4.12}$$

By Proposition 3.6(c), $\overline{V\varphi(-z_0)}$ and $\chi(z_0)$ are reflections in the origin of $\overline{(V\varphi z_0)}$ and $\chi(-z_0)$, respectively; hence, the two last terms in (4.11) and (4.12) are equal, so

$$\mathscr{F}^+_{\varphi,\psi}(z_0) = \mathscr{F}^-_{\varphi,\psi}(z_0).$$

We now derive the adjoint properties of the distorted resolvents.

PROPOSITION 4.9. *If $-\bar z_0$ is a pole of $R^{\bar\Gamma}(z^2)$, then z_0 is a pole of $R^\Gamma(z^2)$. If Γ is an R-admissible distortion and z_0 is neither a pole of $R^\Gamma(z^2)$ nor an element of $\Gamma \cup \{0\} \cup \{-\Gamma\}$, then for $\varphi \in \mathscr{H}^{\bar\Gamma}_{-1}$, $\psi \in \mathscr{H}^\Gamma_{-1}$, we have*

$$(\varphi_{\bar\Gamma}, R^\Gamma(z_0{}^2)\psi_\Gamma)_\Gamma = (R^{\bar\Gamma}(\bar z_0{}^2)\varphi_{\bar\Gamma}, \psi_\Gamma)_\Gamma. \tag{4.13}$$

Proof. Assume that $-\bar z_0$ is a pole of $R^{\bar\Gamma}(z^2)$; we shall prove that z_0 is a pole of $R^\Gamma(z^2)$. Since $-\bar z_0$ is a pole of $R^{\bar\Gamma}$, there exists $\varphi_{\bar\Gamma} \in \mathscr{H}^{\bar\Gamma}_{+1}$, $\varphi_{\bar\Gamma} \neq 0$, such that

$$(1 + R_0{}^{\bar\Gamma}(z_0{}^2)^{\Gamma}VUV^{\bar\Gamma})\varphi_{\bar\Gamma}, \psi_\Gamma)_\Gamma = 0,$$

for all $\psi_\Gamma \in \mathscr{H}^\Gamma_{-1}$. But it is clear that

$$(R_0{}^{\bar\Gamma}(\bar z_0{}^2)\vartheta_{\bar\Gamma}, \psi_\Gamma)_\Gamma = (\vartheta_{\bar\Gamma}, R_0{}^\Gamma(z_0{}^2)\psi_\Gamma)_\Gamma,$$

for all $\vartheta^{\bar\Gamma} \in \mathscr{H}^{\bar\Gamma}_{-1}$, and thus, by Proposition 3.3 and self-adjointness of U we have

$$(\varphi_{\bar\Gamma}, (1 + {}^\Gamma VUV^\Gamma R_0{}^\Gamma(z_0{}^2))\psi_\Gamma)_\Gamma = 0,$$

for all $\psi_\Gamma \in \mathscr{H}^\Gamma_{-1}$. Thus, $1 + {}^\Gamma VUV^\Gamma R_0{}^\Gamma(z_0{}^2)$ is not onto $\mathscr{H}^\Gamma_{-1}$ because $(\cdot, \cdot)_\Gamma$ is a duality. But

$$1 + {}^\Gamma VUV^\Gamma R_0{}^\Gamma(z_0{}^2) = (k^2 - z_0{}^2)(1 + R_0{}^\Gamma(z_0{}^2)^\Gamma VUV^\Gamma\, R_0{}^\Gamma(z_0{}^2), \tag{4.14}$$

and $R_0^\Gamma(z_0^2)$ is an isomorphism from $\mathscr{H}_{-1}^\Gamma$ to $\mathscr{H}_{+1}^\Gamma$, whereas $k^2 - z_0^2$ is an isomorphism from $\mathscr{H}_{+1}^\Gamma$ to $\mathscr{H}_{-1}^\Gamma$. Thus, $1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma$ is not onto $\mathscr{H}_{+1}^\Gamma$, which means that z_0 is a pole of $R^\Gamma(z^2)$.

Thus, if z_0 is not a pole of $R^\Gamma(z^2)$, then $-\bar{z}_0$ is not a pole of $R^\Gamma(z^2)$, and moreover, by (4.14), $1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2)$ is an isomorphism of $\mathscr{H}_{-1}^\Gamma$ to $\mathscr{H}_{-1}^\Gamma$ and

$$R^\Gamma(z_0^2) = R_0^\Gamma(z_0^2)(1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2))^{-1}.$$

Note also that for $\varphi_\Gamma \in \mathscr{H}_{-1}^\Gamma$, $\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma$,

$$((1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2))\varphi_\Gamma, \psi_\Gamma)_\Gamma = (\varphi_\Gamma, (1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)\psi_\Gamma)_\Gamma,$$

and thus, for $\varphi_\Gamma \in \mathscr{H}_{-1}^\Gamma$, $\psi_\Gamma \in \mathscr{H}_{+1}^\Gamma$

$$((1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2))^{-1}\varphi_\Gamma, \psi_\Gamma)_\Gamma = (\varphi_\Gamma, (1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)^{-1})\psi_\Gamma)_\Gamma.$$

Finally, for $\varphi_\Gamma \in \mathscr{H}_{-1}^\Gamma$, $\psi_\Gamma \in \mathscr{H}_{-1}^\Gamma$,

$$\begin{aligned}(\varphi_\Gamma, R^\Gamma(z_0^2)\psi_\Gamma)_\Gamma &= (\varphi_\Gamma, (1 + R_0^\Gamma(z_0^2)^\Gamma VUV^\Gamma)^{-1} R_0^\Gamma(z_0^2)\psi_\Gamma)_\Gamma \\ &= (R_0^\Gamma(\bar{z}_0^2)(1 + {}^\Gamma VUV^\Gamma R_0^\Gamma(z_0^2))^{-1}\varphi_\Gamma, \psi_\Gamma)_\Gamma = (R^\Gamma(\bar{z}_0^2)\varphi_\Gamma, \psi_\Gamma)_\Gamma.\end{aligned}$$

5. Various T-operators and the S-matrix

Let Q be as above. There are several T-operators that are used in scattering theory. They are essentially all the "same operator," but defined on different spaces. We will also introduce distorted T-operators that bear the same relation to the corresponding usual T-operators as the distorted operators ${}^\Gamma V$, ${}^\Gamma V_0(z)$, etc. to V, $V_0(z)$ etc. The basic T-operator is

$$\begin{aligned}T(z) &= Q - QR(z^2)Q = VUV - VUVR(z^2)\, VUV \\ &= VUV - VUW(z^2)\, UV, \qquad \operatorname{Im} z > 0.\end{aligned}$$

From Section 4 we see that $T(z)$ is a meromorphic $\mathscr{C}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$-valued function in the upper half-plane with poles at discrete eigenvalues of H.

Let Γ and Γ' be R-admissible distortions. Then the $\Gamma - \Gamma'$-distorted T-operator is defined by

$${}^\Gamma T^{\Gamma'}(z) = {}^\Gamma VUV^{\Gamma'} - VUW(z^2)\, UV^{\Gamma'}.$$

By the results of Sections 3 and 4 we see that ${}^\Gamma T^{\Gamma'}(z)$ is a $\mathscr{C}(\mathscr{H}_{+1}^{\Gamma'}, \mathscr{H}_{-1}^\Gamma)$-valued meromorphic function in the upper half-plane with poles at discrete eigenvalues of H.

By Lemma 4.8, $T(z)$ and ${}^{\Gamma}T^{\Gamma'}(z)$ can be extended to meromorphic $\mathscr{C}(\mathscr{H}_{+1}^{\Gamma'}, \mathscr{H}_{-1}^{\Gamma})$-valued functions on $R_\infty{}^0$, analytic in G. The continuations $\tilde{T}(z)$ and ${}^{\Gamma}\tilde{T}^{\Gamma'}(z)$ are given by

$$\tilde{T}(z) = VUV - VU\tilde{W}(z^2)\, UV,$$

and

$${}^{\Gamma}\tilde{T}^{\Gamma'}(z) = {}^{\Gamma}VUV^{\Gamma'} - {}^{\Gamma}VU\tilde{W}(z^2)\, UV^{\Gamma'}.$$

The operators $\tilde{T}(z)$ and ${}^{\Gamma}\tilde{T}^{\Gamma'}(z)$ are called the full and the full $\Gamma - \Gamma'$ distorted T-operators, respectively. They will be important in the three-body problem. More important for two-body scattering theory are the following distorted partial T-operators. Let $z \in R_\infty{}^0$ and let Γ be an R-admissible distortion. Then we define the Γ-distorted partial T-operators ${}^{\Gamma}T_0(z)$ and ${}_0T^{\Gamma}(z)$ by

$${}^{\Gamma}T_0(z) = {}^{\Gamma}VUV_0(z) - {}^{\Gamma}VU\tilde{W}(z)\, UV_0(z), \tag{5.1a}$$

$${}_0T^{\Gamma}(z) = {}_0V(z)\, UV^{\Gamma} - {}_0V(z)\, U\tilde{W}(z)\, UV^{\Gamma}. \tag{5.1b}$$

From Sections 3 and 4 we see that ${}^{\Gamma}T_0(z)$ is a $\mathscr{C}(L^2(\Omega), \mathscr{H}_{-1}^{\Gamma})$-valued function meromorphic on $R_\infty{}^0$, analytic on G. ${}_0T^{\Gamma}(z)$ is a $\mathscr{C}(\mathscr{H}_{+1}^{\Gamma}, L^2(\Omega))$-valued meromorphic function on $R_\infty{}^0$, analytic on G.

We now define a family of T-operators on $L^2(\Omega)$ that are close to the standard T-matrix. For $z \in G$ we define

$${}_0T_0(z) = {}_0V(z)\, UV_0(z) - {}_0V(z)\, U\tilde{W}(z)\, UV_0(z). \tag{5.2}$$

Again from Sections 3 and 4 we see that ${}_0T_0(z)$ is a $\mathscr{C}(L^2(\Omega))$-valued meromorphic function on $R_\infty{}^0$ that is analytic on G.

Let $z \in G$. The *T-matrix for H* is defined by

$$\mathbb{T}(z) = z\, {}_0T_0(z).$$

The function $\mathbb{T}(z)$ has the same analyticity properties as ${}_0T_0(z)$.

DEFINITION. The S-matrix of H is the operator in $\mathscr{L}(L^2(\Omega))$, defined for $E > 0$ (E not a positive eigenvalue) by

$$S(E) = 1 - \pi i \mathbb{T}(E^{1/2}).$$

Remark. If $z > 0$, $\mathbb{T}(z)$ is sometimes called the "on the energy shell T-matrix." It also follows automatically from the properties of $\mathbb{T}(z)$, that $S(E)$ has a meromorphic continuation as an $\mathscr{L}(L^2(\Omega))$-valued function of $E^{1/2}$ to the region $R_\infty{}^0$, which is analytic in G.

PROPOSITION 5.1. *$S(E)$ is unitary if and only if*

$$\pi i(\mathbb{T}(E^{1/2}))^* \mathbb{T}(E^{1/2}) = \pi i \mathbb{T}(E^{1/2})(\mathbb{T}(E^{1/2}))^* = (\mathbb{T}(E^{1/2}))^* - \mathbb{T}(E^{1/2}).$$

Proof. This is a simple calculation.

PROPOSITION 5.2. *$S(E)$ is unitary if and only if*

$$\begin{aligned} \pi E^{1/2}({}_0T_0(E^{1/2}))^* T_0(E^{1/2}) &= \pi i E^{1/2}\, {}_0T_0(E^{1/2})\, ({}_0T_0(E^{1/2}))^* \\ &= ({}_0T_0(E^{1/2}))^* - {}_0T_0(E^{1/2}). \end{aligned} \tag{5.3}$$

Proof. Obvious.

In our formulation, the main objective of this paper is to prove (5.3). We now derive several properties of ${}_0T(z)$ and $T_0(z)$ that follow easily from the properties of ${}_0V(z)$ and $V_0(z)$ and $\tilde{W}(z)$. We make use of the following.

NOTATION. If $A \in \mathscr{L}(L^2(\Omega), \mathscr{H}_{-1}^{\Gamma})$, then $A^* \in \mathscr{L}(\mathscr{H}_{+1}^{\Gamma}, L^2(\Omega))$ is defined by

$$(\varphi_\Gamma, Af)_\Gamma = (A^*\varphi_\Gamma, f)_\Omega, \qquad f \in L^2(\Omega), \quad \varphi_\Gamma \in \mathscr{H}_{+1}^{\Gamma}.$$

Similarly, if $B \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$, then $B^* \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$ is defined by

$$(\psi, B\varphi) = (B^*\psi, \varphi), \qquad \text{for} \quad \psi, \varphi \in \mathscr{H}_{-1}.$$

PROPOSITION 5.3. *Let $z \in G$ and let Γ be an R-admissible distortion. Then*

$$({}^{\Gamma}T_0(z))^* = {}_0V(\bar{z})\, UV^{\Gamma} - {}_0V(\bar{z})\, U(\tilde{W}(z))^*\, UV^{\Gamma}, \tag{5.4a}$$

$$({}_0T^{\Gamma}(z))^* = {}^{\Gamma}VUV_0(\bar{z}) - {}^{\Gamma}VU(\tilde{W}(z))^*\, UV_0(\bar{z}), \tag{5.4b}$$

$$({}_0T_0(z))^* = {}_0V(\bar{z})\, UV_0(\bar{z}) - {}_0V(\bar{z})\, U(\tilde{W}(z))^*\, UV_0(\bar{z}). \tag{5.5}$$

Note that if $\operatorname{Im} z > 0$, $(\tilde{W}(z))^* = (W(z))^* = VR(\bar{z}^2)V$. Also if $E > 0$ is not an eigenvalue, then

$$({}_0T_0(E^{1/2}))^* = {}_0V(E^{1/2})\, UV_0(E^{1/2}) - {}_0V(E^{1/2})\, U(VR(E - i0)V)\, UV_0(E^{1/2}). \tag{5.6}$$

Proof. Using the above notation, Propositions 3.3 and 3.8 say that

$$({}_0V(z))^* = V_0(\bar{z}), \qquad (V_0(z))^* = {}_0V(\bar{z}),$$

and

$$(V^{\Gamma})^* = {}^{\Gamma}V, \qquad ({}^{\Gamma}V)^* = V^{\Gamma}.$$

Using this, the self-adjointness of U and the usual order-reversing properties of adjoints, the proof is immediate.

PROPOSITION 5.4. *Let* $z \in G$, *let* Γ *be an R-admissible distortion and* $f \in L^2(\Omega)$. *Then* $T_0(z)f \in \mathscr{H}_{-1} \cap \mathfrak{A}$, $({}_0T(z))^*f \in \mathscr{H}_{-1} \cap \mathfrak{A}$ *and*

$$(T_0(z)f)_\Gamma = {}^\Gamma T_0(z)f, \tag{5.7}$$

$$(({}_0T(z))^*f)_\Gamma = ({}_0T^\Gamma(z))^*f. \tag{5.8}$$

Moreover, for $E^{1/2} > 0$, *not an eigenvalue,*

$$T_0(E^{1/2})f(E^{1/2}) = {}_0T_0(E^{1/2})f, \tag{5.9}$$

$$({}_0T(E^{1/2}))^* f(E^{1/2}) = ({}_0T_0(E^{1/2}))^*f. \tag{5.10}$$

Proof. By (5.1a) and (5.4b) with $\Gamma = (0, \infty)$, $T_0(z)f$ and $({}_0T(z))^*f$ are given by

$$T_0(z)f = V(UV_0(z) - U\tilde{W}(z)\, UV_0(z)), \tag{5.11}$$

$$({}_0T(z))^*f = V(UV_0(\bar{z}) - U(\tilde{W}(z))^*\, UV_0(\bar{z})). \tag{5.12}$$

Then the first assertion and (5.7) and (5.8) follow from Proposition 3.6, and (5.9) and (5.10) follow from (3.6), (5.2), and (5.6). We now prove an off the energy shell version of the unitarity condition (5.3).

PROPOSITION 5.5. *Let* z *be such that* $\operatorname{Re} z > 0$, $0 < \operatorname{Im} z < \mu$. *Then*

$$\begin{aligned}(T_0(z))^*[R_0(z^2) - R_0(\bar{z}^2)]\, T_0(z)\\ = {}_0V(\bar{z})\, UVR(z^2)\, VUV_0(z) - {}_0V(\bar{z})\, UVR(\bar{z}^2)\, VUV_0(z),\end{aligned} \tag{5.13}$$

$$\begin{aligned}{}_0T(z)[R_0(z^2) - R_0(\bar{z}^2)]({}_0T(z))^*\\ = {}_0V(z)\, UVT(z^2)\, VUV_0(\bar{z}) - {}_0V(z)\, UVR(\bar{z}^2)\, VUV_0(\bar{z}).\end{aligned} \tag{5.14}$$

Proof. Recall the first and second resolvent equations

$$R(z^2) - R(\bar{z}^2) = (z^2 - \bar{z}^2)\, R(z^2)\, R(\bar{z}^2) = (z^2 - \bar{z}^2)\, R(\bar{z}^2)\, R(z^2),$$

and

$$R(z^2) = R_0(z^2)(1 - VUVR(z^2)) = (1 - R(z^2)\, VUV)\, R_0(z^2).$$

These equations are valid in $\mathscr{L}(\mathscr{H})$, and since $\mathscr{H}$ is a dense subspace in $\mathscr{H}_{-1}$, and the left- and right-hand side are in $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$, they can be extended by continuity to $\mathscr{H}_{-1}$. We shall also use the first resolvent equation for $R_0(z^2)$ in $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H}_{+1})$.

By the definition of $T_0(z)$ and Propositions 5.3 and 5.4 with $\Gamma = (0, \infty)$, the left-hand side of (5.13) equals

$${}_0V(\bar{z})\, UV[1 - R(\bar{z}^2)\, VUV](R_0(z^2) - R_0(\bar{z}^2))[1 - VUVR(z^2)]\, VUV_0(z). \tag{5.15}$$

Then applying successively the first, the second, and the first resolvent equation, we obtain the following expression for (5.15),

$$\begin{aligned}(z^2 - \bar{z}^2)_0V(\bar{z})\, UV[1 - R(\bar{z}^2)\, VUV](R_0(\bar{z}^2)\, R_0(z^2))[1 - VUVR(z^2)]\, VUV_0(z)\\ = (z^2 - \bar{z}^2)_0V(\bar{z})\, UVR(\bar{z}^2)\, R(z^2)\, VUV_0(z) \qquad (5.16)\\ = {}_0V(\bar{z})\, UVR(z^2)\, VUV_0(z) - {}_0V(\bar{z})\, UVR(\bar{z}^2)\, VUV_0(z),\end{aligned}$$

which proves (5.13). A similar calculation establishes (5.14).

6. Unitarity of the S-matrix

Throughout this section, E will be a positive number that is not an eigenvalue in the usual sense. It is the purpose of this section to prove unitarity of $S(E)$ by establishing (5.3). This is obtained from (5.13) and (5.14) through a limiting process.

PROPOSITION 6.1. *Suppose $E > 0$ is not an eigenvalue of H in the usual sense and $f \in L^2(\Omega)$. Then*

$$\begin{aligned}\lim_{\epsilon\to 0}[{}_0V(E - i\epsilon)^{1/2}\, UVR(E + i\epsilon)\, VUV_0(E + i\epsilon)^{1/2}\\ - {}_0V(E - i\epsilon)^{1/2}\, UVR(E - i\epsilon)\, VUV_0(E + i\epsilon)^{1/2}]f \qquad (6.1)\\ = ({}_0T_0(E^{1/2}))^* - {}_0T_0(E^{1/2})]\, f\end{aligned}$$

$$\begin{aligned}\lim_{\epsilon\to 0}[{}_0V(E + i\epsilon)^{1/2}\, UVR(E + i\epsilon)\, VUV_0(E - i\epsilon)^{1/2}\\ - {}_0V(E + i\epsilon)^{1/2}\, UVR(E - i\epsilon)\, VUV_0(E - i\epsilon)^{1/2}]f \qquad (6.2)\\ = [({}_0T_0(E^{1/2}))^* - {}_0T_0(E^{1/2})]\, f,\end{aligned}$$

where the limits are in the $L^2(\Omega)$-norm.

Proof. Since the operator-valued functions ${}_0V(z)$, $V_0(z)$ and $\tilde{W}(z)$ are analytic and hence continuous at $z = E^{1/2}$ in their respective operator-norms, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\, {}_0V(E - i\epsilon)^{1/2}\, UVR(E + i\epsilon)\, VUV_0(E + i\epsilon)^{1/2} f\\ = \lim_{\epsilon\to 0}\, {}_0V(E - i\epsilon)^{1/2}\, U\tilde{W}(E + i\epsilon)\, UV_0(E + i\epsilon)^{1/2} f\\ = {}_0V(E^{1/2})\, U\tilde{W}(E)\, UV_0(E^{1/2}) f = [-{}_0T_0(E^{1/2}) + {}_0V_0(E^{1/2})]\, f.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{\epsilon\to 0}\, {}_0V(E - i\epsilon)^{1/2}\, UVR(E - i\epsilon)\, VUV_0(E + i\epsilon)^{1/2} f\\ = [-({}_0T_0(E^{1/2}))^* + {}_0V_0(E^{1/2})]\, f,\end{aligned}$$

and (6.1) follows. In exactly the same manner we prove (6.2).

PROPOSITION 6.2. *Let $f, g \in L^2(\Omega)$, then*

$$\begin{aligned}\lim_{\epsilon\to 0}&(f, (T_0(E+i\epsilon)^{1/2})^* [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, T_0(E+i\epsilon)^{1/2} g)_\Omega \\ &= \pi i E^{1/2}({}_0T_0(E^{1/2}) f, {}_0T_0(E^{1/2}) g)_\Omega\end{aligned} \tag{6.3}$$

$$\begin{aligned}\lim_{\epsilon\to 0}&(f, {}_0T(E+i\epsilon)^{1/2} [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, ({}_0T(E+i\epsilon)^{1/2})^* g)_\Omega \\ &= \pi i E^{1/2}({}_0T_0(E^{1/2})^* f, {}_0T_0(E^{1/2})^* g)_\Omega .\end{aligned} \tag{6.4}$$

Proof. Let $\delta, a, b > 0$ be such that $a^2 < E < b^2$, and such that $R_{ab,\delta} \cup \{-R_{ab,\delta}\}$ does not contain any resonances or positive eigenvalues. Let $\epsilon_0 > 0$ be such that $(E \pm i\epsilon)^{1/2} \in R_{ab,\delta}$ for $0 \leqslant \epsilon < \epsilon_0$. Let Γ_+ be a positive R-admissible distortion such that $\Gamma_+ \cap R_{ab,\delta} = \varnothing$, and thus, $\bar{\Gamma}_+ \cap R_{ab,\delta} = \varnothing$. Note that, letting $\Gamma_- = \bar{\Gamma}_+$;

(a) because ${}^{\Gamma_\pm}T_0(z)$ is analytic in $R_\infty{}^0$,

$${}^{\Gamma_\pm}T_0(z) f, \qquad {}^{\Gamma_\pm}T_0(z) g, \qquad ({}_0T^{\Gamma_\pm}(z))^* f \qquad \text{and} \qquad ({}_0T^{\Gamma_\pm}(z))^* g,$$

are continuous as functions of z from $R_{ab,\delta}$ to $\mathscr{H}^{\Gamma_\pm}_{-\bar{1}}$,

(b) Γ^+ is R-homotopic to $(0, \infty)$ relative to $(E - i\epsilon)^{1/2}$ and Γ^- is R-homotopic to $(0, \infty)$ relative to $(E + i\epsilon)^{1/2}$,

(c) because $R^{\Gamma_\pm}(z^2)$ is analytic as an $\mathscr{L}(\mathscr{H}^{\Gamma_\pm}_{-\bar{1}}, \mathscr{H}^{\Gamma_\pm}_{+\bar{1}})$-valued function ina neighborhood of

$$z = E^{1/2}, \| R^{\Gamma_\pm}(z^2)\|_{\mathscr{L}(\mathscr{H}^{\Gamma_\pm}_{-\bar{1}}, \mathscr{H}^{\Gamma_\pm}_{+\bar{1}})},$$

is bounded for z in a sufficiently small neighborhood of $E^{1/2}$.

By Proposition 2.3 and the first part of Proposition 5.4 we have, using (b),

$$\begin{aligned}
&(f, T_0(E+i\epsilon)^{1/2})^* [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, T_0(E+i\epsilon)^{1/2} g)_\Omega \\
&\quad = (T_0(E+i\epsilon)^{1/2} f, [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, T_0(E+i\epsilon)^{1/2} g) \\
&\quad = (T_0(E^{1/2}) f, [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, T_0(E^{1/2}) g) \\
&\qquad + ([{}^{\Gamma_+}T_0(E+i\epsilon)^{1/2} - {}^{\Gamma_+}T_0(E^{1/2})] f, R_0{}^{\Gamma_-}(E+i\epsilon)\, {}^{\Gamma_-}T_0(E^{1/2}) g)_{\Gamma_-} \\
&\qquad + ({}^{\Gamma_+}T_0(E^{1/2}) f, R_0{}^{\Gamma_-}(E+i\epsilon)\, [{}^{\Gamma_-}T_0(E+i\epsilon^{1/2}) - {}^{\Gamma_-}T_0(E^{1/2})]\, g)_{\Gamma_-} \\
&\qquad + ([{}^{\Gamma_+}T_0(E+i\epsilon)^{1/2} - {}^{\Gamma_+}T_0(E^{1/2})] f, R_0{}^{\Gamma_-}(E+i\epsilon)^{1/2}\, [{}^{\Gamma_-}T_0(E+i\epsilon)^{1/2} \\
&\qquad - {}^{\Gamma_-}T_0(E^{1/2})]\, g)_{\Gamma_-} \\
&\qquad - ([{}^{\Gamma_-}T_0(E+i\epsilon)^{1/2} - {}^{\Gamma_-}T_0(E)^{1/2})] f, R_0{}^{\Gamma_+}(E-i\epsilon)\, {}^{\Gamma_+}T_0(E^{1/2}) g)_{\Gamma_+} \\
&\qquad - ({}^{\Gamma_-}T_0(E^{1/2}) f, R_0{}^{\Gamma_+}(E-i\epsilon)\, [{}^{\Gamma_+}T_0(E+i\epsilon)^{1/2} - {}^{\Gamma_+}T_0(E^{1/2})]\, g)_{\Gamma_+} \\
&\qquad - ([{}^{\Gamma_-}T_0(E+i\epsilon)^{1/2} - {}^{\Gamma_-}T_0(E^{1/2})] f, R_0{}^{\Gamma_+}(E-i\epsilon)\, [{}^{\Gamma_+}T_0(E+i\epsilon)^{1/2} \\
&\qquad - {}^{\Gamma_+}T_0(E)^{1/2}]\, g)_{\Gamma_+} .
\end{aligned}$$

But by (a) and (c), all but the first term of the last expression go to 0 as $\epsilon \to 0$. Thus, to prove (6.3) we need to establish

$$\lim_{\epsilon \to 0}(T_0(E^{1/2})f, [R_0(E+i\epsilon) - R_0(E-i\epsilon)]\, T_0(E^{1/2})g) = \pi i E^{1/2}({}_0T_0(E^{1/2})f, {}_0T_0(E^{1/2})g)_\Omega . \tag{6.5}$$

To accomplish this we need the following

LEMMA 6.3. *Suppose $\psi, \varphi \in \mathscr{H}_{-1}$ and that ψ and φ viewed as functions from $(0, \infty)$ into $L^2(\Omega)$ are continuous at $E^{1/2} > 0$. Then*

$$\lim_{\epsilon \to 0}(\psi, (1/\pi i)\,[R_0(E+i\epsilon) - R_0(E-i\epsilon)]\,\varphi) = E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega .$$

In particular this holds if $\psi, \varphi \in \mathscr{H}_{-1} \cap \mathscr{A}$.

Proof. First observe that our assumption implies

$$\lim_{k \to E^{1/2}} k(\psi(k), \varphi(k))_\Omega = E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega . \tag{6.6}$$

Next, for $E, \epsilon > 0$, let

$$N(\epsilon, E) = \frac{2\epsilon}{\pi}\int_0^\infty \frac{k\,dk}{(k^2 - E)^2 + \epsilon^2} = \frac{1}{\pi}\left(\frac{\pi}{2} + \operatorname{Arc\,tan}\frac{E}{\epsilon}\right).$$

Note that

$$N(\epsilon, E) \xrightarrow[\epsilon \to 0]{} 1 \tag{6.7}$$

and

$$\frac{1}{\pi i}[R_0(E+i\epsilon) - R_0(E-i\epsilon)] = \frac{2\epsilon}{\pi}\,\frac{1}{(k^2 - E)^2 + \epsilon^2}.$$

Then

$$\begin{aligned}
&\left(\psi, \frac{1}{\pi i}[R_0(E+i\epsilon) - R_0(E-i\epsilon)]\,\varphi\right) - E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega \\
&= \frac{2\epsilon}{\pi}\int_0^\infty k(\psi(k), \varphi(k))_\Omega \frac{k\,dk}{(k^2-E)^2 + \epsilon^2} - E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega \\
&= \frac{2\epsilon}{\pi}\int_0^\infty [k(\psi(k), \varphi(k))_\Omega - E^{1/2}(\psi(E^{1/2}, \varphi(E^{1/2}))_\Omega]\frac{k\,dk}{(k^2-E)^2 + \epsilon^2} \\
&\quad + (N(\epsilon, E) - 1)\, E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega .
\end{aligned}$$

By (6.7), the second term of the right-hand side goes to 0 as $\epsilon \to 0$.

It remains to be proved, that the first term goes to 0 as $\epsilon \to 0$. Let $\delta > 0$, then by (6.6), there exists $\delta_1 > 0$ such that

$$| k(\psi(k), \varphi(k))_\Omega - E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega | < \delta \qquad \text{for} \quad | k - E^{1/2} | < \delta_1 .$$

We obtain the estimate

$$\begin{aligned}
&\left| \frac{2\epsilon}{\pi} \int_0^\infty [k(\psi(k), \varphi(k))_\Omega - E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega] \frac{k\,dk}{(k^2 - E)^2 + \epsilon^2} \right| \\
&\quad \leqslant \delta \frac{2\epsilon}{\pi} \int_{|k-E^{1/2}|<\delta_1} \frac{k\,dk}{(k^2 - E)^2 + \epsilon^2} \\
&\qquad + \frac{2\epsilon}{\pi} \int_{|k-E^{1/2}|\geqslant\delta_1} \left| (\psi(k), \varphi(k))_\Omega \frac{k^2}{1 + k^2} \frac{1 + k^2\,dk}{(k^2 - E)^2} \right. \\
&\qquad + \frac{2\epsilon}{\pi} E^{1/2} |(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega , \int_{|k-E^{1/2}|\geqslant\delta_1} \frac{k^2\,dk}{(k^2 - E)^2} \\
&\quad \leqslant \delta N(\epsilon, E) + \frac{2\epsilon}{\pi} \left[\| \psi \|_{-1} \| \varphi \|_{-1} \sup_{|k-E^{1/2}|\geqslant\delta_1} \frac{1 + k^2}{(k^2 - E)^2} \right] \\
&\qquad + \frac{2\epsilon}{\pi} E^{1/2} |(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega | \int_{|k-E^{1/2}|\geqslant\delta_1} \frac{k^2\,dk}{(k^2 - E)^2} .
\end{aligned}$$

Thus

$$\limsup_{\epsilon\to 0} \left| \frac{2\epsilon}{\pi} \int_0^\infty [k(\psi(k), \varphi(k))_\Omega - E^{1/2}(\psi(E^{1/2}), \varphi(E^{1/2}))_\Omega] \frac{k\,dk}{(k^2 - E)^2 + \epsilon^2} \right| \leqslant \delta.$$

But δ can be chosen arbitrarily small, so the Lemma follows.

We can now complete the proof of Proposition 6.2. By the first part of Proposition 5.4, $T_0(E^{1/2})f$ and $T_0(E^{1/2})g \in \mathscr{H}_{-1}$, so Lemma 6.3 yields

$$\begin{aligned}
&\lim_{\epsilon\to 0}(T_0(E^{1/2}) f, [R_0(E + i\epsilon) - R_0(E - i\epsilon)]\, T_0(E^{1/2})\, g) \\
&\quad = \pi i E^{1/2}(T_0(E^{1/2}) f(E^{1/2}), T_0(E^{1/2})\, g(E^{1/2}))_\Omega .
\end{aligned} \tag{6.8}$$

But by (5.9) and (5.10) of Proposition 5.4, the right-hand side equals

$$\pi i\, E^{1/2}({}_0T_0(E^{1/2})f, {}_0T_0(E^{1/2})g)_\Omega .$$

Thus, we have established (6.5) and thereby (6.3). The proof of (6.4) follows in exactly the same way.

THEOREM 6.4 (Unitarity of the S-matrix). *Suppose $E > 0$ is not an eigenvalue. Then $S(E)$ is unitary.*

Proof. By Proposition 5.2, the unitary of $S(E)$ is equivalent to Eq. (5.3), which is clearly equivalent to the following statement. For $f, g \in L^2(\Omega)$,

$$\begin{aligned}
\pi i E^{1/2}({}_0T_0(E^{1/2}) f, {}_0T_0(E^{1/2})\, g)_\Omega &= \pi i E^{1/2}({}_0T_0(E^{1/2})^* f, {}_0T_0(E^{1/2})^*\, g)_\Omega \\
&= ({}_0T_0(E^{1/2}) f, g)_\Omega - (f, {}_0T_0(E^{1/2})\, g)_\Omega .
\end{aligned} \tag{6.9}$$

But (6.9) follows directly from Propositions 5.5, 6.1, and 6.2.

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