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Identification of the unknown diffusion coefficient in a linear parabolic equation by the semigroup approach

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Abstract

In this article, we study the semigroup approach for the mathematical analysis of the inverse coefficient problems of identifying the unknown coefficient $k(x)$ in the linear parabolic equation $u_t(x,t) = (k(x)u_x(x,t))_x$, with Dirichlet boundary conditions $u(0,t) = \psi_0$, $u(1,t) = \psi_1$. Main goal of this study is to investigate the distinguishability of the input–output mappings $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$, $\Psi[\cdot] : \mathcal{K} \to C^1[0,T]$ via semigroup theory. In this paper, we show that if the null space of the semigroup $T(t)$ consists of only zero function, then the input–output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$ have the distinguishability property. Moreover, the values $k(0)$ and $k(1)$ of the unknown diffusion coefficient $k(x)$ at $x = 0$ and $x = 1$, respectively, can be determined explicitly by making use of *measured output data* (boundary observations) $f(t) := k(0)u_x(0,t)$ or/and $h(t) := k(1)u_x(1,t)$. In addition to these, the values $k'(0)$ and $k'(1)$ of the unknown coefficient $k(x)$ at $x = 0$ and $x = 1$, respectively, are also determined via the input data. Furthermore, it is shown that *measured output data* $f(t)$ and $h(t)$ can be determined analytically, by an integral representation. Hence the input–output mappings $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$, $\Psi[\cdot] : \mathcal{K} \to C^1[0,T]$ are given explicitly in terms of the semigroup. Finally by using all these results, we construct the local representations of the unknown coefficient $k(x)$ at the end points $x = 0$ and $x = 1$.

Keywords: Semigroup approach; Coefficient identification; Parabolic equation

1. Introduction

Consider the following initial boundary value problem:

$$\begin{cases} u_t(x,t) = \big(k(x)u_x(x,t)\big)_x, & (x,t) \in \Omega_T, \\ u(x,0) = g(x), & 0 < x < 1, \\ u(0,t) = \psi_0, \quad u(1,t) = \psi_1, & 0 < t < T, \end{cases} \tag{1}$$

where $\Omega_T = \{(x,t) \in R^2\colon\ 0 < x < 1,\ 0 < t \leqslant T\}$. The left and right boundary values ψ_0, ψ_1 are assumed to be constants. The functions $c_1 > k(x) \geqslant c_0 > 0$ and $g(x)$ satisfy the following conditions:

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(C1) $k(x) \in C^1[0,1]$;
(C2) $g(x) \in C^2[0,1]$, $g(0) = \psi_0$, $g(1) = \psi_1$.

Under these conditions the initial boundary value problem (1) has a unique solution $u(x,t) \in C^{2,1}(\Omega_T) \cap C^{2,0}(\overline{\Omega}_T)$.

Consider *the inverse problem* of determining the unknown coefficient $k = k(x)$ from the following observations at the boundaries $x = 0$ and $x = 1$:

$$k(0)u_x(0,t) = f(t), \quad k(1)u_x(1,t) = h(t), \quad t \in (0,T]. \tag{2}$$

Here $u = u(x,t)$ is the solution of the parabolic problem (1). The functions $f(t)$, $h(t)$ are assumed to be *noisy free measured output data*. In this context the parabolic problem (1) will be referred as a *direct* (*forward*) *problem*, with the *inputs* $g(x)$ and $k(x)$. It is assumed that the functions $f(t)$ and $h(t)$ belong to $C^1[0,T]$ and satisfy the consistency conditions $f(0) = k(0)g'(0)$, $h(0) = k(1)g'(1)$.

We denote by $\mathcal{K} := \{k(x) \in C^1[0,1]\colon c_1 > k(x) \geqslant c_0 > 0,\ x \in [0,1]\} \subset C^1[0,1]$, the set of admissible coefficients $k = k(x)$, and introduce the input–output mappings $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$, $\Psi[\cdot] : \mathcal{K} \to C^1[0,T]$, where

$$\Phi[k] = k(x)u_x(x,t;k)|_{x=0}, \quad \Psi[k] = k(x)u_x(x,t;k)|_{x=1}, \quad k \in \mathcal{K},\ f(t), h(t) \in C^1[0,T]. \tag{3}$$

Then the inverse problem with the measured output data $f(t)$ and $h(t)$ can be formulated as the following operator equations:

$$\Phi[k] = f, \quad \Psi[k] = h, \quad k \in \mathcal{K},\ f, h \in C^1[0,T]. \tag{4}$$

The monotonicity, continuity, and hence invertibility of the input–output mappings $\Phi[\cdot]:\mathcal{K} \to C^1[0,T]$ and $\Psi[\cdot]:\mathcal{K} \to C^1[0,T]$ are investigated in [2–4].

The purpose of this paper is to study a distinguishability of the unknown coefficient via the above input–output mappings. We say that the mapping $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$ (or $\Psi[\cdot] : \mathcal{K} \to C^1[0,T]$) has the distinguishability property, if $\Phi[k_1] \neq \Phi[k_2]$ ($\Psi[k_1] \neq \Psi[k_2]$) implies $k_1(x) \neq k_2(x)$. This, in particular, means injectivity of the inverse mappings Φ^{-1} and Ψ^{-1}.

The paper is organized as follows. In Section 2, an analysis of the semigroup approach is given for the inverse problem with the measured data $f(t)$. The similar analysis is applied to the inverse problem with the single measured output data $h(t)$ given at the point $x = 1$, in Section 3. The inverse problem with two Neumann measured data $f(t)$ and $h(t)$ is discussed in Section 4. The local representations of the unknown coefficient $k(x)$ at the endpoints $x = 0$ and $x = 1$ are given in Section 5. Finally, some concluding remarks are given in Section 6.

2. An analysis of the inverse problem with measured output data $f(t)$

Consider now the inverse problem with one measured output data $f(t)$ at $x = 0$. In order to formulate the solution of the parabolic problem (1) in terms of semigroup, let us first arrange the parabolic equation as follows:

$$u_t(x,t) - k(0)u_{xx}(x,t) = \big(\big(k(x) - k(0)\big)u_x(x,t)\big)_x, \quad (x,t) \in \Omega_T.$$

Then the initial boundary value problem (1) can be rewritten in the following form:

$$\begin{aligned} &u_t(x,t) - k(0)u_{xx}(x,t) = \big(\big(k(x) - k(0)\big)u_x(x,t)\big)_x, \quad (x,t) \in \Omega_T, \\ &u(x,0) = g(x), \quad 0 < x < 1, \\ &u(0,t) = \psi_0, \quad u(1,t) = \psi_1, \quad 0 < t < T. \end{aligned} \tag{5}$$

For the time being we assume that $k(0)$ were known, later this value will be determined. In order to formulate the solution of the parabolic problem (5) in terms of semigroup, we need to define a new function

$$v(x,t) = u(x,t) - \psi_0(1-x) - \psi_1 x, \quad x \in [0,1], \tag{6}$$

which satisfies the following parabolic problem:

$$\begin{aligned}
&v_t(x,t) + A\big[v(x,t)\big] = \big(\big(k(x)-k(0)\big)\big(v_x(x,t)+\psi_1-\psi_0\big)\big)_x, \quad (x,t)\in\Omega_T,\\
&v(x,0) = g(x) - \psi_0(1-x) - \psi_1 x, \quad 0<x<1,\\
&v(0,t)=0, \quad v(1,t)=0, \quad 0<t<T.
\end{aligned} \tag{7}$$

Here $A[.] := -k(0)\,d^2[.]/dx^2$ is a second order differential operator, whose domain is $D_A = \{u \in C^2(0,1) \cap C^2[0,1]\colon u(0)=u(1)=0\}$. It is obvious that $g(x)\in D_A$, since the initial value function $g(x)$ belongs to $C^2[0,1]$.

Denote by $T(t)$ the semigroup of linear operators generated by the operator A [5,7]. Note that we can easily find the eigenvalues and eigenfunctions of the differential operator A. Moreover, the semigroup $T(t)$ can be easily constructed by using the eigenvalues and eigenfunctions of the infinitesimal generator A. Hence we first consider the following eigenvalue problem:

$$\begin{aligned}
&A\phi(x) = \lambda\phi(x),\\
&\phi(0)=0, \qquad \phi(1)=0.
\end{aligned}$$

We can easily determine that the eigenvalues are $\lambda_n = k(0)n^2\pi^2$ for all $n=0,1,\ldots,$ and the corresponding eigenfunctions are $\phi_n(x)=\sqrt{2}\sin(n\pi x)$. In this case the semigroup $T(t)$ can be represented in the following way:

$$T(t)U(x,s) = \sum_{n=0}^{\infty}\big\langle\phi_n(.), U(.,s)\big\rangle e^{-\lambda_n t}\phi_n(x), \tag{8}$$

where $\langle\phi_n(\zeta), U(\zeta,s)\rangle := \int_0^1 \phi_n(\zeta)U(\zeta,s)\,d\zeta$. Under this representation, the null space of the semigroup $T(t)$ of the linear operators can be defined as follows:

$$N(T) = \big\{U(x,s)\colon \big\langle\phi_n(x), U(x,s)\big\rangle = 0, \text{ for all } n=0,1,2,3,\ldots\big\}.$$

From the definition of the semigroup $T(t)$, we can say that the null space of it consists of only zero function, i.e., $N(T)=\{0\}$. As we will see later that this result is very important for the uniqueness of the unknown coefficient $k(x)$.

The unique solution of the initial-boundary value problem (7) in terms of semigroup $T(t)$ can be represented in the following form:

$$v(x,t) = T(t)v(x,0) + \int_0^t T(t-s)\big(\big(k(x)-k(0)\big)\big(v_x(x,t)+\psi_1-\psi_0\big)\big)_x\,ds.$$

Hence by using the identity (6) and taking the initial value $u(x,0)=g(x)$ into account, the solution $u(x,t)$ of the parabolic problem (5) in terms of semigroup can be written in the following form:

$$u(x,t) = \psi_0(1-x) + \psi_1 x + T(t)\big(g(x)-\psi_0(1-x)-\psi_1 x\big) + \int_0^t T(t-s)\big(\big(k(x)-k(0)\big)u_x(x,s)\big)_x\,ds. \tag{9}$$

In order to arrange the above solution representation, let us define the followings:

$$\begin{aligned}
&\zeta(x) = \big(g(x)-\psi_0(1-x)-\psi_1 x\big),\\
&\xi(x,t) = \big(\big(k(x)-k(0)\big)u_x(x,t)\big)_x,\\
&z(x,t) = \sum_{n=0}^{\infty}\big\langle\phi_n(.), \zeta(.)\big\rangle e^{-\lambda_n t}\phi_n'(x),
\end{aligned} \tag{10}$$

$$w(x,t,s) = \sum_{n=0}^{\infty}\big\langle\phi_n(.), \xi(.,s)\big\rangle e^{-\lambda_n t}\phi_n'(x). \tag{11}$$

Then we can rewrite the solution representation (9) in terms of $\zeta(x)$ and $\xi(x,s)$ in the following form:

$$u(x,t) = \psi_0(1-x) + \psi_1 x + T(t)\zeta(x) + \int_0^t T(t-s)\xi(x,s)\,ds.$$

Differentiating both sides of the above identity with respect to x and using semigroup properties at $x=0$ yields

$$u_x(0,t) = -\psi_0 + \psi_1 + z(0,t) + \int_0^t w(0,t-s,s)\,ds.$$

Taking into account the measured output data $k(0)u_x(0,t) = f(t)$ we get

$$f(t) = k(0)\left(-\psi_0 + \psi_1 + z(0,t) + \int_0^t w(0,t-s,s)\,ds\right). \tag{12}$$

Using the measured output data $k(0)u_x(0,t) = f(t)$, we can write $k(0) = f(t)/u_x(0,t)$ for all $t>0$ which can be rewritten in terms of semigroup in the following form:

$$k(0) = f(t)/\left(-\psi_0 + \psi_1 + z(0,t) + \int_0^t w(0,t-s,s)\,ds\right).$$

Taking limit as $t \to 0$ in the above identity, we obtain the following explicit formula for the value $k(0)$ of the unknown coefficient $k(x)$:

$$k(0) = f(0)/\big(-\psi_0 + \psi_1 + z(0,0)\big). \tag{13}$$

Note that in [1] the value $k(0)$ is defined via the same input data, by different way. However compare with formula given in [1], formula (13) is more convenient for practical purposes.

Let us differentiate now the both sides of identity (9) with respect to t

$$\begin{aligned} u_t(x,t) &= T(t)A\big(u(x,0) - \psi_0(1-x) - \psi_1 x\big) + \big(\big(k(x)-k(0)\big)u_x(x,t)\big)_x \\ &\quad + \int_0^t AT(t-s)\big(\big(k(x)-k(0)\big)u_x(x,s)\big)_x\,ds. \end{aligned}$$

Using semigroup properties, we obtain

$$u_t(x,t) = T(t)g''(x) + 2T(0)\big(\big(k(x)-k(0)\big)u_x(x,t)\big)_x - T(t)\big(\big(k(x)-k(0)\big)u_x(x,0)\big)_x.$$

Taking $x=0$ in the above identity, we get

$$u_t(0,t) = T(t)g''(0) + 2T(0)\big(k'(0)u_x(0,t)\big) - T(t)\big(k'(0)u_x(0,0)\big).$$

Since $u(0,t) = \psi_0$ we have $u_t(0,t) = 0$. Taking into account this and substituting $t=0$ yield

$$0 = g''(0) + k'(0)u_x(0,0).$$

Solving this equation for $k'(0)$ and substituting $u_x(0,0) = f(0)/k(0)$, we obtain the following explicit formula for the value $k'(0)$ of the first derivative $k'(x)$ of the unknown coefficient

$$k'(0) = -\frac{k(0)g''(0)}{f(0)}. \tag{14}$$

Under the determined values $k(0)$ and $k'(0)$, the set of admissible coefficients can be defined as follows:

$$\mathcal{K}_0 := \big\{k \in \mathcal{K}:\ k(0) = f(0)/\big(-\psi_0 + \psi_1 + z(0,0)\big),\ k'(0) = -k(0)g''(0)/f(0)\big\}.$$

The right-hand side of identity (12) defines explicitly *the semigroup representation of the input–output mapping* $\Phi[F]$ on the set of admissible source functions $\mathcal{F}$

$$\Phi[k](x) := k(0)\left(-\psi_0 + \psi_1 + z(0,t) + \int_0^t w(0,t-s,s)\,ds\right), \quad \forall t \in [0,T]. \tag{15}$$

The following lemma implies the relation between the coefficients $k_1(x), k_2(x) \in \mathcal{K}_0$ at $x = 0$ and the corresponding outputs $f_j(t) := k_j(0)u_x(0, t; k_j)$, $j = 1, 2$.

Lemma 1. *Let $u_1(x,t) = u(x,t;k_1)$ and $u_2(x,t) = u(x,t;k_2)$ be the solutions of the direct problem* (5) *corresponding to the admissible coefficients $k_1(x), k_2(x) \in \mathcal{K}$, $k_1(x) \neq k_2(x)$. Suppose that $f_j(t) = k_j(0)u_x(0,t;k_j)$, $j = 1,2$, are the corresponding outputs, and denote by $\Delta f(t) = f_1(t) - f_2(t)$, $\Delta w(x,t,s) = w^1(x,t,s) - w^2(x,t,s)$. If the condition*

$$k_1(0) = k_2(0) := k(0),$$

holds, then the outputs $f_j(t)$, $j = 1, 2$, satisfy the following integral identity:

$$\Delta f(\tau) = k(0)\left(\int_0^\tau \Delta w(0, \tau - s, s)\, ds\right) ds, \tag{16}$$

for each $\tau \in (0, T]$.

Proof. By using identity (12), the measured output data $f_j(t) := k_j(0)u_x(0,t;k_j)$, $j = 1, 2$, can be written as follows:

$$f_1(\tau) = k(0)\left(-\psi_0 + \psi_1 + z^1(0,\tau) + \int_0^t w^1(0, \tau - s, s)\, ds\right),$$

$$f_2(\tau) = k(0)\left(-\psi_0 + \psi_1 + z^2(0,\tau) + \int_0^t w^2(0, \tau - s, s)\, ds\right),$$

respectively. From identity (10) it is obvious that $z^1(0,\tau) = z^2(0,\tau)$ for each $\tau \in (0, T]$. Hence the difference of these formulas implies the desired result. □

This lemma with identity (11) implies the following.

Corollary 1. *Let the conditions of Lemma* 1 *hold. Then $f_1(t) = f_2(t)$, $\forall t \in [0, T]$, if and only if*

$$\left\langle \phi_n(x), \xi^1(x,t) - \xi^2(x,t) \right\rangle = 0, \quad \forall t \in (0, T],\ n = 0, 1, \ldots.$$

Since the null space of semigroup contains only zero function, i.e., $N(T) = \{0\}$, Corollary 1 states that $f_1 \equiv f_2$ if and only if $\xi^1(x,t) - \xi^2(x,t) = 0$ for all $(x,t) \in \Omega_T$. From the definition of $\xi(x,t)$, it implies that $k_1(x) = k_2(x)$ for all $x \in [0, 1]$.

Theorem 1. *Let conditions* (C1) *and* (C2) *hold. Assume that $\Phi[\cdot] : \mathcal{K}_0 \to C^1[0,T]$ be the input–output mapping defined by* (3) *and corresponding to the measured output $f(t) := k(0)u_x(0,t)$. Then the mapping $\Phi[k]$ has the distinguishability property in the class of admissible coefficients $\mathcal{K}_0$, i.e.,*

$$\Phi[k_1] \neq \Phi[k_2], \quad \forall k_1, k_2 \in \mathcal{K}_0,\ k_1(x) \neq k_2(x).$$

3. An analysis of the inverse problem with measured output data $h(t)$

Consider now the inverse problem with one measured output data $h(t)$ at $x = 1$. As in the previous section, let us arrange the equation in parabolic problem (1) as follows:

$$u_t(x,t) - k(1)u_{xx}(x,t) = \big(\big(k(x) - k(1)\big)u_x(x,t)\big)_x, \quad (x,t) \in \Omega_T.$$

Then the initial boundary value problem (1) can be rewritten in the following form:

$$\begin{aligned} &u_t(x,t) - k(1)u_{xx}(x,t) = \big(\big(k(x) - k(1)\big)u_x(x,t)\big)_x, \quad (x,t) \in \Omega_T, \\ &u(x,0) = g(x), \quad 0 < x < 1, \\ &u_x(0,t) = \psi_0, \quad u_x(1,t) = \psi_1, \quad 0 < t < T. \end{aligned} \tag{17}$$

In order to formulate the solution of the above parabolic problem in terms of semigroup, let us use the same function $v(x,t)$ in identity (6) which satisfies the following parabolic problem:

$$\begin{aligned} &v_t(x,t) + B\big[v(x,t)\big] = \big(\big(k(x) - k(1)\big)u_x(x,t)\big)_x, \quad (x,t) \in \Omega_T, \\ &v(x,0) = g(x) - \psi_0(1-x) - \psi_1 x, \quad 0 < x < 1, \\ &v(0,t) = 0, \quad v(1,t) = 0, \quad 0 < t < T. \end{aligned} \tag{18}$$

Here $B[.] := -k(1)\,d^2[.]/dx^2$ is a second order differential operator whose domain is $D_B = \{u \in C^2(0,1) \cap C^2[0,1]\colon u(0) = u(1) = 0\}$.

Denote by $S(t)$ the semigroup of linear operators generated by the operator B. As mentioned above, in order to construct semigroup $S(t)$ we need to know the eigenvalues and eigenfunctions of the infinitesimal generator B. Therefore we first consider the following eigenvalue problem:

$$\begin{aligned} &B\phi(x) = \lambda\phi(x), \\ &\phi(0) = 0, \qquad \phi(1) = 0. \end{aligned}$$

Then the eigenvalues of the above problem become $\lambda_n = k(1)n^2\pi^2$ for all $n = 0, 1, \ldots,$ and the corresponding eigenfunctions become $\phi_n(x) = \sqrt{2}\sin(n\pi x)$. Hence the semigroup $S(t)$ can be represented in the following form:

$$S(t)U(x,s) = \sum_{n=0}^{\infty} \big\langle \phi_n(.), U(.,s)\big\rangle e^{-\lambda_n t}\phi_n(x).$$

Using above representation of the semigroup $S(t)$ of the linear operators, we can define the null space of it as follows:

$$N(S) = \big\{U(x,s)\colon \big\langle \phi_n(x), U(x,s)\big\rangle = 0, \text{ for all } n = 1, 2, 3, \ldots\big\}.$$

The definition of the semigroup $S(t)$ above implies that the null space of it consists of only zero function, i.e., $N(S) = \{0\}$. As we mentioned in the previous section, this result plays very important role in the uniqueness of the unknown coefficient $k(x)$.

The unique solution of the initial value problem (18) in terms of semigroup $S(t)$ can be represented in the following form:

$$v(x,t) = S(t)v(x,0) + \int_0^t S(t-s)\big(\big(k(x) - k(1)\big)\big(v_x(x,t) + \psi_1 - \psi_0\big)\big)_x\, ds.$$

Hence by using the identity (6) the solution $u(x,t)$ of the parabolic problem (17) in terms of semigroup can be written in the following form:

$$\begin{aligned} u(x,t) = {} & \psi_0(1-x) + \psi_1 x + S(t)\big(u(x,0) - \psi_0(1-x) - \psi_1 x\big) \\ & + \int_0^t S(t-s)\big(\big(k(x) - k(1)\big)u_x(x,s)\big)_x\, ds. \end{aligned} \tag{19}$$

Defining the followings:

$$\chi(x,s) = \big(\big(k(x) - k(1)\big)u_x(x,s)\big)_x,$$

$$z_1(x,t) = \sum_{n=0}^{\infty} \big\langle \phi_n(.), \zeta(.)\big\rangle e^{-\lambda_n t}\phi_n'(x), \tag{20}$$

$$w_1(x,t,s) = \sum_{n=0}^{\infty} \big\langle \phi_n(.), \chi(.,s)\big\rangle e^{-\lambda_n t}\phi_n'(x). \tag{21}$$

The solution representation of the parabolic problem (17) can be rewritten in the following form:

$$u(x,t) = \psi_0(1-x) + \psi_1 x + S(t)\zeta + \int_0^t S(t-s)\chi(x,s)\,ds.$$

Now differentiating both sides of the above identity with respect to x and substituting $x = 1$ yield

$$u_x(1,t) = -\psi_0 + \psi_1 + z_1(1,t) + \int_0^t w_1(1,t-s,s)\,ds.$$

Taking into account the measured output data $k(1)u_x(1,t) = h(t)$, we get

$$h(t) = k(1)\left(-\psi_0 + \psi_1 + z_1(1,t) + \int_0^t w_1(1,t-s,s)\,ds\right). \tag{22}$$

Now we can determine the value $k(1)$ by using the overmeasured output data $h(t) = k(1)u_x(1,t)$. The identity $k(1) = h(t)/u_x(1,t)$ for all $t > 0$, can be rewritten in terms of semigroup in the following form:

$$k(1) = h(t)\Big/\left(-\psi_0 + \psi_1 + z_1(1,t) + \int_0^t w(1,t-s,s)\,ds\right).$$

Taking limit as $t \to 0$ in the above identity yields:

$$k(1) = h(0)/\big(-\psi_0 + \psi_1 + z_1(1,0)\big). \tag{23}$$

Differentiating both sides of identity (19) with respect to t, we get

$$u_t(x,t) = S(t)B\big(u(x,0) - \psi_0(1-x) - \psi_1 x\big) + \big(\big(k(x) - k(1)\big)u_x(x,t)\big)_x + \int_0^t BS(t-s)\big(\big(k(x)-k(1)\big)u_x(x,s)\big)_x\,ds.$$

Using semigroup properties, we obtain

$$u_t(x,t) = S(t)g''(x) + 2S(0)\big(\big(k(x)-k(1)\big)u_x(x,t)\big)_x - S(t)\big(\big(k(x)-k(1)\big)u_x(x,0)\big)_x.$$

Taking $x = 1$ in the above identity, we get

$$u_t(1,t) = S(t)g''(1) + 2S(0)\big(k'(1)u_x(1,t)\big) - S(t)\big(k'(1)u_x(1,0)\big).$$

Since $u(1,t) = \psi_1$ we have $u_t(1,t) = 0$ Taking into account this and substituting $t = 0$, we get

$$0 = g''(1) + k'(1)u_x(1,0).$$

Solving this equation for $k'(1)$ and substituting $u_x(1,0) = h(0)/k(1)$, we reach the following result:

$$k'(1) = -\frac{k(1)g''(1)}{h(0)}. \tag{24}$$

Then we can define the admissible set of diffusion coefficients as follows:

$$\mathcal{K}_1 := \big\{k \in \mathcal{K}\colon\ k(1) = h(0)/\big(-\psi_0 + \psi_1 + z(1,0)\big),\ k'(1) = -k(1)g''(1)/h(0)\big\}.$$

The right-hand side of identity (22) defines *the semigroup representation of the input–output mapping* $\Psi[k]$ on the set of admissible source functions $\mathcal{F}$

$$\Psi[k](t) := k(1)\left(-\psi_0 + \psi_1 + z_1(1,t) + \int_0^t w_1(1,t-s,s)\,ds\right), \quad \forall t \in [0,T]. \tag{25}$$

The following lemma implies the relation between the coefficients $k_1(x), k_2(x) \in \mathcal{K}_1$ at $x = 1$, and the corresponding outputs $h_j(t) := k_j(1)u_x(1, t; k_j),\ j = 1, 2$.

Lemma 2. *Let $u_1(x, t) = u(x, t; k_1)$ and $u_2(x, t) = u(x, t; k_2)$ be solutions of the direct problem* (17) *corresponding to the admissible coefficients $k_1(x), k_2(x) \in \mathcal{K}$, $k_1(x) \neq k_2(x)$. Suppose that $h_j(t) = u(1, t; k_j),\ j = 1, 2$, are the corresponding outputs and denote by $\Delta h(t) = h_1(t) - h_2(t)$, $\Delta w_1(x, t, s) = w_1^1(x, t, s) - w_1^2(x, t, s)$. If the condition*

$$k_1(1) = k_2(1) := k(1)$$

holds, then the outputs $h_j(t),\ j = 1, 2$, satisfy the following integral identity:

$$\Delta h(\tau) = k(1) \int_0^\tau \Delta w_1(1, \tau - s, s)\, ds, \tag{26}$$

for each $\tau \in [0, T]$.

Proof. By using identity (22), the measured output data $h_j(t) := k_j(1)u_x(1, t; k_j),\ j = 1, 2$, can be written as follows:

$$h_1(\tau) = k(1)\left(-\psi_0 + \psi_1 + z_1^1(1, \tau) + \int_0^t w_1^1(1, \tau - s, s)\, ds\right),$$

$$h_2(\tau) = k(1)\left(-\psi_0 + \psi_1 + z_1^2(1, \tau) + \int_0^t w_1^2(1, \tau - s, s)\, ds\right),$$

respectively. From identity (20), it is obvious that $z_1^1(1, \tau) = z_1^2(1, \tau)$ for each $\tau \in (0, T]$. Hence the difference of these formulas implies the desired result. □

This lemma with identity (21) implies the following conclusion.

Corollary 2. *Let conditions of Lemma* 2 *hold. Then $h_1(t) = h_2(t)$, $\forall t \in [0, T]$, if and only if*

$$\langle \phi_n(x), \chi^1(x, t) - \chi^2(x, t) \rangle = 0, \quad \forall t \in (0, T],\ n = 0, 1, \ldots,$$

hold. Then $h_1(t) = h_2(t)$, $\forall t \in [0, T]$.

Since the null space of it consists of only zero function, i.e., $N(S) = \{0\}$, Corollary 2 states that $h_1 \equiv h_2$ if and only if $\chi^1(x, t) - \chi^2(x, t) = 0$ for all $(x, t) \in \Omega_T$. From the definition of $\chi(x, t)$, it implies that $k_1(x) = k_2(x)$ for all $x \in (0, 1]$.

Theorem 2. *Let conditions* (C1) *and* (C2) *hold. Assume that $\Psi[\cdot] : \mathcal{K}_1 \to C^1[0, T]$ be the input–output mapping defined by* (3) *and corresponding to the measured output $h(t) := k(1)u_x(1, t)$. Then the mapping $\Psi[k]$ has the distinguishability property in the class of admissible coefficients $\mathcal{K}_1$, i.e.,*

$$\Psi[k_1] \neq \Psi[k_2], \quad \forall k_1, k_2 \in \mathcal{K}_1,\ k_1(x) \neq k_2(x).$$

4. The inverse problem with two Neumann measured output data

Consider now the inverse problem (1)–(2) with two measured output data $f(t)$ and $h(t)$. As shown before, having these two data, the values $k(0)$ as well as $k(1)$ can be defined by the above explicit formulaes. Based on this result, let us define now the set of admissible coefficients $\mathcal{K}_2$ as an intersection

$$\mathcal{K}_2 := \mathcal{K}_0 \cap \mathcal{K}_1 = \big\{k \in \mathcal{K}\colon\ k(0) = f(0)/\big(-\psi_0 + \psi_1 + z(0, 0)\big),\ k(1) = h(0)/\big(-\psi_0 + \psi_1 + z(1, 0)\big),$$
$$k'(0) = -k(0)g''(0)/f(0),\ k'(1) = -k(1)g''(1)/h(0)\big\}.$$

On this set both input–output mapping $\Phi[k]$ and $\Psi[k]$ have distinguishability property.

Corollary 3. *The input–output mappings $\Phi[\cdot]:\mathcal{K}_2\to C^1[0,T]$ and $\Psi[\cdot]:\mathcal{K}_2\to C^1[0,T]$ distinguish any two functions $k_1(x)\neq k_2(x)$ from the set $\mathcal{K}_2$, i.e.,*

$$\Phi[k_1]\neq\Phi[k_2],\quad \Psi[k_1]\neq\Psi[k_2],\quad \forall k_1(x),k_2(x)\in\mathcal{K}_2,\ k_1(x)\neq k_2(x).$$

5. Local representations of the diffusion coefficient $k(x)$ near $x=0$ and $x=1$

This section deals with the local analysis of the unknown coefficient $k(x)$ in order to obtain its representations in the neighborhood of $x=0$ and $x=1$. The semigroup representation of the solution is used here as an effective tool for the determination of the unknown coefficient $k=k(x)$. We determine the diffusion coefficient $k(x)$ first numerically.

Let us introduce the uniform mesh $w_h=\{x_i\in[0,1]:\ x_0=0,\ x_i=x_{i-1}+h,\ i=1,\dots,N;\ h=1/N\}$ and denote $k_i=k(x_i)$, $i=1,\dots,N$. Then the piecewise linear approximation of the coefficient $k(x)$ is as follows:

$$k_h(x)=\sum_{i=1}^{N}k_i\xi_i(x), \tag{27}$$

here $\xi_i(x)$ is the continuous, piecewise linear Lagrange basic functions; $\xi_i(x_j)=\delta_{ij}$ [6].

Having the values $k_0=k(0)$ and $k'(0)$ of the diffusion coefficient $k(x)$ and its derivative at $x=0$, we can approximately determine its value at $x=h$ for small enough $h>0$ as follows:

$$k_1\cong k_0+k'(x_0)h,$$

where $h=\frac{1}{N}$ and N is the number of meshes.

To determine the value $k_2=k(x_2)$ of the diffusion coefficient $k(x)$, we rearrange the parabolic equation as follows:

$$u_t(x,t)-k(x_1)u_{xx}(x,t)=\big(\big(k(x)-k(x_1)\big)u_x(x,t)\big)_x,\quad (x,t)\in\Omega_T.$$

Then the initial boundary value problem (1) can be rewritten in the following form:

$$\begin{aligned}
&u_t(x,t)+A_2u(x,t)=\big(\big(k(x)-k(x_1)\big)u_x(x,t)\big)_x,\quad (x,t)\in\Omega_T,\\
&u(x,0)=g(x),\quad 0<x<1,\\
&u_x(0,t)=\psi_0,\quad u_x(1,t)=\psi_1,\quad 0<t<T,
\end{aligned}$$

where $A_2[.]=-k(x_1)d^2[.]/dx^2$ and $T_2(t)$ is the analytic semigroup.

Differentiating both sides of this identity with respect to t, we get

$$\begin{aligned}
u_t(x,t)={}&T_2(t)A_2\big(u(x,0)-\psi_0(1-x)-\psi_1x\big)+\big(\big(k(x)-k(x_1)\big)u_x(x,t)\big)_x\\
&+\int_0^t A_2T_2(t-s)\big(\big(k(x)-k(x_1)\big)u_x(x,s)\big)_x\,ds.
\end{aligned}$$

Using semigroup properties, we obtain

$$u_t(x,t)=T_2(t)g''(x)+2T_2(0)\big(\big(k(x)-k(x_1)\big)u_x(x,t)\big)_x-T_2(t)\big(\big(k(x)-k(x_1)\big)u_x(x,0)\big)_x.$$

At $x=x_1$, we have

$$u_t(x_1,t)=T_2(t)g''(x_1)+2T_2(0)k'(x_1)u_x(x_1,t)-T_2(t)k'(x_1)u_x(x_1,t).$$

Furthermore using the data of $\psi_0=u(0,t)$ and $f(t)=k(0)u_x(0,t)$ at the point $(0,t)$, we can determine the approximate value of $u(h,t)$ for small enough h as follows:

$$u(x_1,t)=u(0,t)+u_x(0,t)h,\qquad u(x_1,t)=\psi_0+\frac{f(t)}{k(0)}h.$$

Last equation implies that $u_t(x_1,t)=\frac{f'(t)}{k(0)}h$. Here we assume that $k(0)\neq0$. Substituting this in (28) at $x=x_1$

$$\frac{f'(t)}{k(0)}h = T_2(t)g''(x_1) + 2T_2(0)k'(x_1)u_x(x_1,t) - T_2(t)k'(x_1)u_x(x_1,t).$$

At $t = 0$, we have

$$f'(0)h = k(0)g''(x_1) + \big(k'(x_1)k(0)u_x(x_1,0)\big).$$

Using here the data $u_x(x_1,0) = g'(x_1)$, finally we obtain

$$k'(x_1) = \frac{f'(0)h - k(0)g''(x_1)}{k(0)g'(x_1)}.$$

This identity allow us to determine the approximate value of the $k(x)$ at $x = x_2$

$$k(x_2) \cong k(x_1) + k'(x_1)h, \qquad k(x_2) \cong k(x_1) + \frac{f'(0)h - k(0)g''(x_1)}{k(0)g'(x_1)}h.$$

Continuing this procedure recursively, at the ith step we get

$$k'(x_i) \cong \frac{f'(0)h - k(0)g''(x_i)}{k(0)g'(x_i)}, \quad \forall i \in 1,\dots,N.$$

Hence the approximate value of the unknown coefficient $k(x)$ at $x = x_i$ can be calculated as follows:

$$k(x_i) \cong k(x_{i-1}) + \frac{f'(0)h - k(0)g''(x_i)}{k(0)g'(x_i)}h.$$

More generally this identity can be rewritten in the following form:

$$k(x_i) \cong k(0) + \sum_{i=1}^{n} \frac{f'(0)h - k(0)g''(x_{i-1})}{k(0)g'(x_{i-1})}h. \tag{28}$$

This shows also influence of the Cauchy data $g(x)$ to the diffusion coefficient on the neighborhood of $x = 0$.

By the similar way we can derive the representation formula for $k(x)$ in the neighborhood of $x = 1$ with the assumption $k(1) \neq 0$. Then the general form of this identity becomes

$$k(x_{N-i}) \cong k(1) - \sum_{i=1}^{n} \frac{h'(0)h - k(1)g''(x_{N-i})}{k(1)g'(x_{N-i})}h. \tag{29}$$

Remark 1. In some physical situations the flux attains maximum (or minimum) value at the initial moment $t = 0$. In this case we have $f'(0) = 0$. Using this in (29) and taking limit as $N \to \infty$, we get

$$k(x) \cong k(0) - \int_0^x \frac{g''(s)}{g'(s)}\,ds.$$

The same consideration are valid for the right flux $h(t)$. If $h'(0) = 0$, then the limit case of the representation (30) has the following form:

$$k(x) \cong k(1) + \int_x^1 \frac{g''(s)}{g'(s)}\,ds.$$

Formulas (29) and (30) can sequentially determine the approximate values of $k_i = k(x_i)$, having those values and using (27) we can restore the piecewise linear approximation $k_h(x)$ of the unknown coefficient $k(x)$.

On the other hand the local representation formulas (29), (30) for the unknown coefficient $k(x)$ in the neighborhoods of the endpoints $x = 0$ and $x = 1$, that the first and second derivatives of the function $g(x)$ play here an important role. If the function $g'(x)$ takes values close 0 at any point $x \in [0,1]$, then the coefficient $k(x)$ blows up at that point.

6. Conclusion

The aim of this study was to analyze distinguishability properties of the input–output mappings $\Phi[\cdot] : \mathcal{K}_2 \to C^1[0,T]$ and $\Psi[\cdot] : \mathcal{K}_2 \to C^1[0,T]$, which are naturally determined by the measured output data. In this paper we show that if the null spaces of the semigroups $T(t)$ and $S(t)$ include only zero function then the corresponding input–output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$ are distinguishable.

This study shows that boundary conditions and the region on which the problem is defined play an important role on the distinguishability of the input–output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$ since these key elements determine the structure of the semigroup $T(t)$ of linear operators and its null space.

The similarities among the formulas for $k(0)$, $k'(0)$, $k(1)$, $k'(1)$ give insight us that we can find the values $k(x_i)$ and $k'(x_i)$ for all $x_i \in (0,1)$ by the following formulas:

$$k(x_i) = f_i(0)/\big(-\psi_0 + \psi_1 + z(0,0)\big), \tag{30}$$

$$k'(x_i) = -k(x_i)g''(x_i)/f_i(0), \tag{31}$$

where $f_i(t) = k(x_i)u_x(x_i,t)$. This implies that more measured output data f_i, more information about unknown coefficient $k(x)$. Here the only difficulty we encounter here is the determination of the functions f_i. In numerical calculations we can approximate the these functions by using the following Taylor expansions $k(x_i) = k(x_{i-1}) + k'(x_{i-1})(x_i - x_{i-1})$ and $u_x(x_i,t) = u_x(x_{i-1},t) + u_{xx}(x_{i-1},t)$, successively. Hence we approximately determine f_i as $f_i(t) = (k(x_{i-1}) + k'(x_{i-1})(x_i - x_{i-1}))(u_x(x_{i-1},t) + u_{xx}(x_{i-1},t))$.

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