ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 517–528
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Existence of positive solutions for singular boundary value problem on time scales

Zhao-Cai Hao [a,b,∗,1], Ti-Jun Xiao [a,2], Jin Liang [a,2]

[a] *Department of Mathematics, University of Science and Technology of China Hefei, Anhui 230026, People's Republic of China*

[b] *Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China*

Received 8 January 2005

Available online 28 February 2006

Submitted by A.C. Peterson

Abstract

This paper deals with a class of boundary value problem of singular differential equations on time scales. The conditions we used here differ from those in the majority of papers as we know. An existence theorem of positive solutions is established by using the Krasnosel'skii fixed point theorem and an example is given to illustrate it.

Keywords: Singular; Positive solutions; Time scales; Boundary value problem

1. Introduction and preliminaries

In this paper we are concerned with the following boundary value problem (BVP in short) of nonlinear dynamic equation

$$\begin{cases} [\varphi(t)x^{\Delta}(t)]^{\Delta} + \lambda m(t) f(t, x(\sigma(t))) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\Delta}(a) = 0, \\ \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0. \end{cases} \tag{1.1}$$

[∗] Corresponding author.
E-mail addresses: zchjal@163.com (Z.-C. Hao), xiaotj@ustc.edu.cn (T.-J. Xiao), jliang@ustc.edu.cn (J. Liang).
[1] The author acknowledges support from USTC (KD2004002), NSFC (10471075) and NSF of Shandong Province (Y2003A01).
[2] The author acknowledges support from CAS, EMC and NSFC.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.01.083

To understand this so-called dynamic equation on a time scale (measure chain), and so to foster an easy and convenient reading of this paper we present some definitions and notations as follows which are common in the recent literature. Our source for this background material are the papers [1–7,10–13,16–18,20–22].

Let $\mathbb{T}$ be a time scale, which is a nonempty closed subset of $\mathbb{R}$, the set of real numbers, with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. An alternative terminology for time scale is a measure chain.

We make the basic assumption that $a < b$ are points in $\mathbb{T}$ throughout this paper.

Definition 1.1. Define the interval in $\mathbb{T}$

$$[a,b] := \{t \in \mathbb{T}:\ a \leqslant t \leqslant b\}.$$

Open intervals and half-open intervals etc. are defined accordingly.

Definition 1.2. A measure chain may or may not be connected, so we define the forward jump operator and backward jump operator σ, ρ by

$$\sigma(t) := \inf\{\tau > t:\ \tau \in \mathbb{T}\} \in \mathbb{T},$$
$$\rho(t) := \sup\{\tau < t:\ \tau \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$ with $t < \sup \mathbb{T}$.

In this definition, we put $\inf \phi = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \phi = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m), where ϕ denotes the empty set. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say that t is left scattered. Points that are right scattered and left scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense are called dense. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k := \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k := \mathbb{T}$.

Now we consider a function $x : \mathbb{T} \to \mathbb{R}$ and define so-called delta (or Hilger) derivative of x at a point $t \in \mathbb{T}^k$.

Definition 1.3. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^k$. Then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big\| \big[x\big(\sigma(t)\big) - x(s)\big] - x^{\Delta}(t)\big[\sigma(t) - s\big] \big\| \leqslant \varepsilon \big|\sigma(t) - s\big|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the delta (or Hilger) derivative of $x(t)$ at $t \in \mathbb{T}$. The derivative can also be defined in terms of a limit as follows:

$$x^{\Delta}(t) := \lim_{s \to t,\, \sigma(s) \neq t} \frac{x(\sigma(s)) - x(t)}{\sigma(s) - t} = \lim_{s \to t,\, \sigma(t) \neq s} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}.$$

The second derivative of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t)$.

It is obvious that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}^k$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the set of integers, then it follows that

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t).$$

In particular, if $\mathbb{T} = \mathbb{R}$, we find $\sigma(s) = s$ for all $s \in \mathbb{T}$. Thus $x^{\Delta}(t)$ reduces to the usual derivative

$$x'(t) = \lim_{s \to t} \frac{x(s) - x(t)}{s - t}.$$

Definition 1.4. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. In this case we define the integral of f by

$$\int_a^t f(\tau)\,\Delta\tau = F(t) - F(a).$$

The class of continuous functions on $\mathbb{T}$ is too small for notion of an integral via antiderivatives. The introducing of an rd-continuous function turns out to be appropriate.

Definition 1.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

S. Hilger [17, p. 2688, line 8] tells us that, for the usual time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, rd-continuity coincides with continuity. S. Hilger [16] proves also that every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative. Further property of this integral can be seen in M. Bohner and G.Sh. Guseinov [4]. We have also the following two formulas from M. Bohner and A. Peterson [5, Theorems 1.29 and 1.30]:

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \bigl(\sigma(t) - t\bigr) f(t), \tag{1.2}$$

where $f : \mathbb{T} \to \mathbb{R}$ is an arbitrary function and $t \in \mathbb{T}$;

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t \in [a,b)} [\sigma(t) - t] f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{t \in [b,a)} [\sigma(t) - t] f(t), & \text{if } a > b, \end{cases} \tag{1.3}$$

where $f \in \mathbb{C}_{\mathrm{rd}}[a,b]$ and $[a,b]$ consists of only isolated points. Formulas (1.2) and (1.3) will function in the following Example 3.1.

Definition 1.6. By a positive solution of the BVP (1.1), we understand a function $x(t)$ which is positive on $(a, \sigma(b))$, satisfying equation $[\varphi(t)x^{\Delta}(t)]^{\Delta} + \lambda m(t) f(t, x(\sigma(t))) = 0$ and boundary conditions $\alpha x(a) - \beta x^{\Delta}(a) = 0$ and $\gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0$.

The theory of measure chains was introduced and developed by Aulbach and Hilger [3] in 1988. It has been created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. An introduction to this subject is given in M. Bohner and A. Peterson [5] and

V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan [20]. Some other papers on this topic are L.H. Erbe and S. Hilger [10] and S. Hilger [16]. Now, its development is still in its infancy, yet as inroads are made, interest is gathering steam. Recently, much attention is attracted by questions of existence of positive solutions to boundary value problem for differential equations on measure chains. For significant works along this line, see, e.g., [2,5–9,12,13,18,22].

The BVP (1.1) when $\varphi(t) \equiv 1$, $t \in [a, \sigma(b)]$, and $m(t) \in C[a, \sigma(b)]$ has been investigated in many papers. For example, C.J. Chyan and J. Henderson [6], L.H. Erbe and A. Peterson [13], L.H. Erbe and H.Y. Wang [14], J. Henderson and H.Y. Wang [16], C.H. Hong and C.C. Yeb [18], W.C. Lian, W.F. Wong and C.C. Yeh [21]. Most recently, J. Liang, T.J. Xiao, Z.C. Hao [22] studied the BVP (1.1) for general $\varphi(t)$ and $m(t)$, which may be singular at $t = a$ and/or $t = \sigma(b)$.

Stimulated by these works, in this paper, we will consider the BVP (1.1), where $\varphi(t) > 0$ on $[a, \sigma^2(b)]$ such that both the delta derivative of $\varphi(t)$ and the integral $\int_a^{\sigma(b)} \frac{1}{\varphi(\tau)}\,\Delta\tau$ exist (which is well defined due to the basic properties of the forward jump operator σ and the integral given by Definition 1.4), $m(\cdot)$ and $f(\cdot,\cdot)$ are given functions, $\alpha, \beta, \gamma, \delta \geqslant 0$ such that

$$d := \frac{\gamma\beta}{\varphi(a)} + \frac{\alpha\delta}{\varphi(\sigma(b))} + \alpha\gamma \int_a^{\sigma(b)} \frac{1}{\varphi(\tau)}\,\Delta\tau > 0, \tag{1.4}$$

and

$$\delta \geqslant \gamma\bigl[\sigma^2(b) - \sigma(b)\bigr]. \tag{1.5}$$

Write

$$\max f_\infty := \lim_{u\to\infty} \max_{t\in[a,\sigma(b)]} \frac{f(t,u)}{u}, \qquad \min f_\infty := \lim_{u\to\infty} \min_{t\in[a,\sigma(b)]} \frac{f(t,u)}{u},$$
$$\max f_0 := \lim_{u\to 0^+} \max_{t\in[a,\sigma(b)]} \frac{f(t,u)}{u}, \qquad \min f_0 := \lim_{u\to 0^+} \min_{t\in[a,\sigma(b)]} \frac{f(t,u)}{u}.$$

Then, as usual, the function f in the BVP (1.1) is called superlinear if $\max f_0 = 0$ and $\min f_\infty = \infty$ and it is called sublinear when $\max f_\infty = 0$ and $\min f_0 = \infty$. In papers [6,13–15,18,21,22] the function f is assumed to be superlinear or sublinear (see [13,14]), or at least one of the following two assumptions holds (see [6,15,18,21,22]):

$$\text{(i)}\quad \max f_\infty \in (0,\infty) \text{ and/or } \min f_0 \in (0,\infty); \tag{1.6}$$

$$\text{(ii)}\quad \min f_\infty \in (0,\infty) \text{ and/or } \max f_0 \in (0,\infty). \tag{1.7}$$

Therefore, the conditions on f required in this paper are different from those in papers [6,13–15,18,21,22] since f may not be superlinear (sublinear) and may not satisfy conditions (1.6) or (1.7) (see Remark 3.2). With suitable growth and limit conditions (see assumptions (H_1) and (H_4)), we will obtain the existence of positive solutions of the BVP (1.1).

We organize this paper as follows. In Section 2, starting with some preliminary results from recent literature, we then state and prove our main result Theorem 2.2, an existence theorem of positive solutions established by using the Krasnosel'skii fixed point theorem (see [19]). An example is given to illustrate the theorem in Section 3.

2. Main result

In this section we start with some preliminary results from recent literature. We assume that the set $[a, \sigma(b)]$ is such that both

$$\xi := \min\left\{\tau \in \mathbb{T}\colon\ \tau \geqslant \frac{\sigma(b)+3a}{4}\right\} \quad \text{and} \quad \omega := \max\left\{\tau \in \mathbb{T}\colon\ \tau \leqslant \frac{3\sigma(b)+a}{4}\right\}$$

exist and satisfy

$$\frac{\sigma(b)+3a}{4} \leqslant \xi < \omega \leqslant \frac{3\sigma(b)+a}{4}.$$

We also assume that

$$\text{If} \quad \sigma(\omega) = b \quad \text{and} \quad \delta = 0, \quad \text{then} \quad \sigma(\omega) < \sigma(b). \tag{2.1}$$

Let $G(t,s)$ be the *Green* function of the following BVP:

$$\begin{cases} [\varphi(t)x^{\Delta}(t)]^{\Delta} = 0, & t \in [a,b], \\ \alpha x(a) - \beta x^{\Delta}(a) = 0, & \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0. \end{cases}$$

From Erbe and Peterson [13] we know,

$$G(t,s) = \begin{cases} \frac{1}{d} u(t) v(\sigma(s)), & t \leqslant s, \\ \frac{1}{d} u(\sigma(s)) v(t), & \sigma(s) \leqslant t, \end{cases}$$

where d is given by (1.4) and

$$u(t) = \alpha \int_a^t \frac{1}{\varphi(\tau)}\,\Delta\tau + \frac{\beta}{\varphi(a)}, \qquad v(t) = \gamma \int_t^{\sigma(b)} \frac{1}{\varphi(\tau)}\,\Delta\tau + \frac{\delta}{\varphi(\sigma(b))}. \tag{2.2}$$

Then the monotonicity of functions u and v implies that

$$0 \leqslant G(t,s) \leqslant G\big(\sigma(s), s\big), \quad (t,s) \in \big[a, \sigma(b)\big] \times [a,b].$$

Moreover, (1.5) implies that

$$G\big(\sigma^2(b), s\big) \geqslant 0, \quad s \in [a,b].$$

Consequently,

$$0 \leqslant G(t,s) \leqslant G\big(\sigma(s), s\big), \quad (t,s) \in \big[a, \sigma^2(b)\big] \times [a,b], \tag{2.3}$$

$$G(t,s) \geqslant kG\big(\sigma(s), s\big), \quad (t,s) \in \left[\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)+a}{4}\right] \times [a,b], \tag{2.4}$$

where

$$k = \min\left\{\frac{u(\frac{\sigma(b)+3a}{4})}{u(\sigma(b))}, \frac{v(\frac{3\sigma(b)+a}{4})}{v(\sigma(a))}\right\} > 0.$$

As in [22], we set

$$l_1 := \min_{s\in[a,b]} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}, \qquad k_1 := \min\{k, l_1\}. \tag{2.5}$$

(2.1) implies that the constant $l_1 > 0$. So $k_1 > 0$.

Throughout this paper, we let

$$\mathbb{E} = \mathbb{C}_{\mathrm{rd}}\big[a, \sigma^2(b)\big].$$

Then $\mathbb{E}$ is a Banach space with the norm

$$\|x\| = \max_{t\in[a,\sigma^2(b)]} \big|x(t)\big|.$$

Define a set $\mathbb{P} \subset \mathbb{E}$ by

$$\mathbb{P} = \Big\{x \in \mathbb{E}\colon\ x \geqslant 0 \text{ on } \big[a, \sigma^2(b)\big] \text{ and } \min_{t\in[\xi,\sigma(\omega)]} x(t) \geqslant k_1\|x\|\Big\}, \tag{2.6}$$

where k_1 is given by (2.5). Clearly, $\mathbb{P}$ is a cone.

The following fixed point theorem will help us to obtain our main result.

Lemma 2.1. (See [19].) *Let X be a Banach space, and let P be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $A : P \cap (\overline{\Omega}_2\backslash\Omega_1) \to P$ be a completely continuous operator such that, either*

$$\text{(i)}\quad \|Ax\| \leqslant \|x\|,\quad x \in P \cap \partial\Omega_1,\qquad \|Ax\| \geqslant \|x\|,\quad x \in P \cap \partial\Omega_2,$$

or

$$\text{(ii)}\quad \|Ax\| \geqslant \|x\|,\quad x \in P \cap \partial\Omega_1,\qquad \|Ax\| \leqslant \|x\|,\quad x \in P \cap \partial\Omega_2.$$

Then A has a fixed point in $P \cap (\overline{\Omega}_2\backslash\Omega_1)$.

We now present our main result.

Theorem 2.2. *Assume that*

(H_1) *$f \in C([a, \sigma(b)] \times [0,\infty), (0,\infty))$ and there exist a constant L and a function F, which is integrable on $[a, \sigma(b)]$, satisfying*

$$f^2(t,s) \leqslant F(t),\quad (t,s) \in \big[a, \sigma(b)\big] \times [L, \infty).$$

(H_2) *$m(\cdot) : (a, \sigma(b)) \to [0,\infty)$ is rd-continuous and may be singular at $t = a$ and/or $t = \sigma(b)$.*

(H_3) *Both $0 < \int_\xi^\omega G(\sigma(s), s)m(s)\Delta s$ and $\int_a^{\sigma(b)} G(\sigma(s), s)m(s)\Delta s < +\infty$ hold.*

(H_4) *There exist constants r and R, $[r, R] \subseteqq [\xi, \omega]$, satisfying*

$$\varliminf_{s\to 0} \min_{t\in[r,R]} \frac{m(t)f(t,s)}{s} = +\infty.$$

Then, for any $\lambda \in (0,\infty)$, the BVP (1.1) has at least one positive solution.

Proof. For any $\lambda \in (0,\infty)$, we define an operator A_λ by

$$A_\lambda x(t) = \lambda \int\limits_a^{\sigma(b)} G(t,s)m(s)f\big(s, x\big(\sigma(s)\big)\big)\Delta s,\quad t \in \big[a, \sigma^2(b)\big], \tag{2.7}$$

in view of (2.3) and (H_3). [22, Lemmas 2.2 and 2.3] help us to obtain the following two assertions:

Assertion (i): A solution of the operator equation $x(t) = A_\lambda x(t)$ is a solution of the BVP (1.1), where A_λ is given by (2.7).
Assertion (ii): $A_\lambda\colon \mathbb{P} \to \mathbb{P}$ is completely continuous.

We now turn our attention to the existence of positive solutions of the BVP (1.1). There are three steps.

Step 1. We will show that

$$\lim_{y\to\infty} \frac{\Phi_1(y)}{y} = 0, \tag{2.8}$$

where

$$\Phi_1(y) := \sup_{x\in\partial\mathbb{P}(y)} \max_{t\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)m(s)f\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\,\Delta s,$$

$$\mathbb{P}(y) = \bigl\{x\in\mathbb{P}\colon\ \|x\| \leqslant y\bigr\}.$$

Set

$$\Psi(x) := \bigl\{s\in\bigl[a,\sigma(b)\bigr]x\bigl(\sigma(s)\bigr) \leqslant L\bigr\},$$

$$M_1 := \left\{\int_a^{\sigma(b)} \bigl[G\bigl(\sigma(s),s\bigr)m(s)\bigr]^2\,\Delta s\right\}^{\frac12},$$

$$M_2 := \max_{(s,x)\in[a,\sigma(b)]\times[0,L]} f^2(s,x), \qquad M_3 := \int_a^{\sigma(b)} F(s)\,\Delta s.$$

Then for any $t\in[a,\sigma^2(b)]$, $x\in\partial\mathbb{P}(y)$, we deduce that

$$\begin{aligned}
&\int_a^{\sigma(b)} G(t,s)m(s)f\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\,\Delta s \\
&\quad\leqslant \int_a^{\sigma(b)} G\bigl(\sigma(s),s\bigr)m(s)f\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\,\Delta s \\
&\quad\leqslant \left\{\int_a^{\sigma(b)} \bigl[G\bigl(\sigma(s),s\bigr)m(s)\bigr]^2\,\Delta s\right\}^{\frac12} \left\{\int_a^{\sigma(b)} f^2\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\,\Delta s\right\}^{\frac12} \\
&\quad= M_1\left\{\int_{[a,\sigma(b))-\Psi(x)} f^2\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\Delta s + \int_{\Psi(x)} f^2\bigl(s,x\bigl(\sigma(s)\bigr)\bigr)\,\Delta s\right\}^{\frac12} \\
&\quad\leqslant M_1\left\{\int_a^{\sigma(b)} F(s)\,\Delta s + \int_{\Psi(x)} \max_{(s,u)\in[a,\sigma(b)]\times[0,L]} f^2(s,u)\,\Delta s\right\}^{\frac12}
\end{aligned}$$

$$\leqslant M_1\big[M_2\big(\sigma(b)-a\big)+M_3\big]^{\frac{1}{2}}.$$

Consequently,

$$\lim_{y\to\infty}\frac{\Phi_1(y)}{y}\leqslant\lim_{y\to\infty}\frac{M_1[M_2(\sigma(b)-a)+M_3]^{\frac{1}{2}}}{y}=0.$$

Step 2. We now claim that

$$\lim_{y\to 0}\frac{\Phi_2(y)}{y}=+\infty, \tag{2.9}$$

where

$$\Phi_2(y):=\inf_{x\in\partial\mathbb{P}(y)}\max_{t\in[a,\sigma^2(b)]}\int_a^{\sigma(b)}G(t,s)m(s)f\big(s,x\big(\sigma(s)\big)\big)\,\Delta s.$$

In fact, hypothesis (H_4) tells us that, for any $n\in\mathbb{N}$, there exists $\varepsilon_n>0$ such that $\varepsilon_n\to 0$ and

$$m(t)f(t,s)\geqslant ns,\quad (t,s)\in[r,R]\times(0,\varepsilon_n).$$

For any $x\in\mathbb{P}$, $0<\|x\|\leqslant\varepsilon_n$ implies

$$0<x(t)\leqslant\varepsilon_n,\quad t\in[r,R].$$

Hence we infer

$$\frac{m(t)f(t,x)}{x}\geqslant n,\quad x\in\mathbb{T},\ 0<\|x\|\leqslant\varepsilon_n,\ t\in[r,R]. \tag{2.10}$$

Then, for any $y\in(0,\varepsilon_n)$, we find, in view of (2.4) and (2.10),

$$\begin{aligned}
\frac{\Phi_2(y)}{y}&=\frac{1}{y}\inf_{x\in\partial\mathbb{P}(y)}\max_{t\in[a,\sigma^2(b)]}\int_a^{\sigma(b)}G(t,s)m(s)f\big(s,x\big(\sigma(s)\big)\big)\,\Delta s\\
&\geqslant\frac{1}{y}\inf_{x\in\partial\mathbb{P}(y)}\max_{t\in[\frac{\sigma(b)+3a}{4},\frac{3\sigma(b)+a}{4}]}\int_a^{\sigma(b)}G(t,s)m(s)f\big(s,x\big(\sigma(s)\big)\big)\,\Delta s\\
&\geqslant\frac{1}{y}\inf_{x\in\partial\mathbb{P}(y)}\int_a^{\sigma(b)}kG\big(\sigma(s),s\big)m(s)f\big(s,x\big(\sigma(s)\big)\big)\,\Delta s\\
&\geqslant\frac{k}{y}M_4\inf_{x\in\partial\mathbb{P}(y)}\int_r^{R}m(s)f\big(s,x\big(\sigma(s)\big)\big)\,\Delta s\\
&\geqslant\frac{k}{y}M_4\inf_{x\in\partial\mathbb{P}(y)}\min_{s\in[r,R]}x\big(\sigma(s)\big)\int_r^{R}\frac{m(s)f(s,x(\sigma(s)))}{x(\sigma(s))}\,\Delta s\\
&\geqslant\frac{k}{y}M_4\inf_{x\in\partial\mathbb{P}(y)}\min_{s\in[\xi,\omega]}x\big(\sigma(s)\big)\int_r^{R}\frac{m(s)f(s,x(\sigma(s)))}{x(\sigma(s))}\,\Delta s
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{k}{y} M_4 \inf_{x\in\partial\mathbb{P}(y)} k_1\|x\| \int_r^R \frac{m(s) f(s, x(\sigma(s)))}{x(\sigma(s))}\,\Delta s \\
&= kk_1 M_4 \inf_{x\in\partial\mathbb{P}(y)} \int_r^R \frac{m(s) f(s, x(\sigma(s)))}{x(\sigma(s))}\,\Delta s \\
&\geqslant kk_1 M_4 [R-r] n,
\end{aligned}
\tag{2.11}
$$

where

$$M_4 := \min_{s\in[r,R]} G\big(\sigma(s), s\big).$$

Obviously, (2.11) implies (2.9).

Step 3. Let us prove the existence of positive solutions of the BVP (1.1). For any $\lambda \in (0,\infty)$, we know from (2.7) that

$$\lambda\Phi_1(y) = \sup_{x\in\partial\mathbb{P}(y)} \|A_\lambda x\|, \qquad \lambda\Phi_2(y) = \inf_{x\in\partial\mathbb{P}(y)} \|A_\lambda x\|.$$

Noting that (2.8) and (2.9) imply that there exist $N_1 \gg N_2$ satisfying

$$\frac{\lambda\Phi_1(N_1)}{N_1} \leqslant 1, \qquad \frac{\lambda\Phi_2(N_2)}{N_2} \geqslant 1.$$

Thus it follows that

$$\|A_\lambda x\| \leqslant \|x\|, \quad x \in \partial\mathbb{P}(N_1), \qquad \|A_\lambda x\| \geqslant \|x\|, \quad x \in \partial\mathbb{P}(N_2). \tag{2.12}$$

By virtue of (2.12), the above assertions (i) and (ii) and Lemma 2.1, we infer that there exists $x^* \in \mathbb{P}$ such that $A_\lambda x^* = x^*$ and $N_2 \leqslant \|x^*\| \leqslant N_1$. Clearly, x^* is a positive solution of the BVP (1.1). This completes the proof. □

3. An example

To illustrate our main result Theorem 2.2, we present the following example.

Example 3.1. Set

$$\mathbb{T} = \{0\} \cup \left\{\frac{t}{8}\colon t \in \mathbb{N}\right\}.$$

Then for any $\lambda \in (0,\infty)$, the following BVP

$$
\begin{cases}
\left[\dfrac{x^\Delta(t)}{1+t}\right]^\Delta + \lambda\left[\displaystyle\int_{\sigma(t)}^{\frac{9}{8}} (1+\tau)\,\Delta\tau\right]^{-1} f\big(t, x\big(\sigma(t)\big)\big) = 0, \\
x(0) = x\left(\sigma\left(\dfrac{7}{8}\right)\right) + x^\Delta\left(\sigma\left(\dfrac{7}{8}\right)\right) = 0
\end{cases}
\tag{3.1}
$$

has at least one positive solution, where

$$f(t,x)=\begin{cases}\max\{t-\frac{3}{8},\ \frac{3}{8}-t\}, & (t,x)\in[0,1]\times[0,\frac{7}{8}],\\ \max\{t-\frac{3}{8},\frac{3}{8}-t,x-\frac{7}{8}\}, & (t,x)\in[0,1]\times[\frac{7}{8},3],\\ \frac{17}{8}, & (t,x)\in[0,1]\times[3,\infty).\end{cases}$$

In fact, we write

$$a=0,\qquad b=\frac{7}{8},\qquad \varphi(t)=\frac{1}{1+t},\qquad m(t)=\left[\int_{\sigma(t)}^{\frac{9}{8}}(1+\tau)\,\Delta\tau\right]^{-1},$$
$$\alpha=\gamma=\delta=1,\qquad \beta=0.$$

It is obvious that $f(\cdot,\cdot)\in C([0,1]\times[0,\infty),[0,\infty))$. Choose

$$L=3,\quad F(t)=\frac{289}{64}.$$

Then

$$\int_0^1 F(t)\Delta t=\frac{289}{64}. \tag{3.2}$$

So, f and F satisfy (H_1) in Theorem 2.2.

We know easily that $m(t)\in\mathbb{C}_{\mathrm{rd}}(0,1)$ and $m(\cdot)$ is singular at $t=\sigma(\frac{7}{8})=1$. Thus (3.1) is a singular BVP on [0, 1]. This implies condition (H_2) in Theorem 2.2.

Furthermore, we compute

$$\sigma(b)=1,\qquad \xi=\frac{1}{4},\qquad \omega=\frac{3}{4},$$

$$d=\frac{\gamma\beta}{\varphi(a)}+\frac{\alpha\delta}{\varphi(\sigma(b))}+\alpha\gamma\int_a^{\sigma(b)}\frac{1}{\varphi(\tau)}\Delta\tau=2+\sum_{t\in[0,1)}\big[\sigma(t)-t\big](1+t)=\frac{55}{16}, \tag{3.3}$$

$$\int_\xi^\omega G\big(\sigma(s),s\big)m(s)\,\Delta s$$
$$=\frac{16}{55}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{\sigma(s)}(1+\tau)\,\Delta\tau\left(\int_{\sigma(s)}^{1}(1+\tau)\,\Delta\tau+2\right)\left[\int_{\sigma(s)}^{\frac{9}{8}}(1+\tau)\Delta\tau\right]^{-1}\Delta s>0, \tag{3.4}$$

$$\int_a^{\sigma(b)}G\big(\sigma(s),s\big)m(s)\,\Delta s$$
$$=\int_0^{\frac{7}{8}}G\big(\sigma(s),s\big)m(s)\,\Delta s+\int_{\frac{7}{8}}^{\sigma(\frac{7}{8})}G\big(\sigma(s),s\big)m(s)\,\Delta s$$

$$= \int_0^{\frac{7}{8}} G\big(\sigma(s), s\big)m(s)\Delta s + \frac{1}{8}\left[\int_{\sigma(\frac{7}{8})}^{\frac{9}{8}} (1+\tau)\Delta\tau\right]^{-1} G\left(\sigma\left(\frac{7}{8}\right), \frac{7}{8}\right)$$

$$= \int_0^{\frac{7}{8}} G\big(\sigma(s), s\big)m(s)\Delta s + \frac{1}{2}G\left(\sigma\left(\frac{7}{8}\right), \frac{7}{8}\right) < +\infty. \tag{3.5}$$

We employed the formulas (1.2) and (1.3) in (3.2)–(3.5) in the above calculations. Now, (3.3) implies (1.4) and (3.4)–(3.5) tell us that the condition (H_3) holds also.

At last, we choose

$$r = \frac{4}{8}, \qquad R = \frac{5}{8}.$$

Then

$$\varliminf_{s\to 0} \min_{t\in[r,R]} \frac{m(t)f(t,s)}{s} = \varliminf_{s\to 0} \frac{1}{s} \min_{t\in[\frac{4}{8},\frac{5}{8}]} \frac{\max\{\frac{3}{8}-t, t-\frac{3}{8}\}}{\int_{\sigma(t)}^{\frac{9}{8}}(1+\tau)\Delta\tau} = +\infty. \tag{3.6}$$

Hence condition (H_4) of Theorem 2.2 is fulfilled. So Theorem 2.2 gives us the desired conclusion.

We conclude this paper by the following remark.

Remark 3.2. In Example 3.1, it is clear that

$$\lim_{x\to\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x} = 0, \qquad \lim_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x} = 0,$$
$$\lim_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = 0, \qquad \lim_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x} = \infty.$$

So the function f in Example 3.1 does not satisfy either superlinear (sublinear) conditions, or conditions (1.6) and (1.7).

Acknowledgments

The authors are grateful to the anonymous referees for their helpful suggestions and comments.

References

[1] E. Akin, Cauchy functions for dynamic equation on a measure chain, J. Math. Anal. Appl. 267 (2002) 97–115.
[2] D.R. Anderson, Eigenvalue intervals for a two-point boundary value problem on a measure chain, J. Comput. Appl. Math. 141 (2002) 57–64.
[3] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, in: Nonlinear Dynamics and Quantum Dynamical Systems, in: Math. Res., vol. 59, Akademie Verlag, Berlin, 1990, pp. 9–20.
[4] M. Bohner, G.Sh. Guseinov, Improper integrals on time scales, Dynam. Syst. Appl. 12 (2003) 45–65.
[5] M. Bohner, A. Peterson (Eds.), Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[6] C.J. Chyan, J. Henderson, Eigenvalue problem for nonlinear differential equations on a measure chain, J. Math. Anal. Appl. 245 (2000) 547–559.
[7] J.J. DaCunha, J.M. Davis, P.K. Singh, Existence results for singular three point boundary value problem on time scales, J. Math. Anal. Appl. 295 (2004) 378–391.

[8] J.M. Davis, J. Henderson, K.R. Prasad, W. Yin, Eigenvalue intervals for nonlinear right focal problems, Appl. Anal. 74 (2000) 215–231.
[9] P.W. Eloe, J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problems, Differential Equations Dynam. Systems 6 (1998) 309–317.
[10] L.H. Erbe, S. Hilger, Sturmian theory on measure chains, Differential Equations Dynam. Systems 1 (1993) 223–244.
[11] L.H. Erbe, A. Peterson, Green's functions and comparison theorems for differential equation on measure chains, Dyn. Contin. Discrete Impuls. Syst. 6 (1999) 121–137.
[12] L.H. Erbe, A. Peterson, Eigenvalue conditions and positive solutions, J. Difference Equ. Appl. 6 (2000) 165–191.
[13] L.H. Erbe, A. Peterson, Positive solutions for nonlinear differential equation on a measure chain, Math. Comput. Modelling 32 (2000) 571–585.
[14] L.H. Erbe, H.Y. Wang, On the existence of positive solutions of ordinary difference equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.
[15] J. Henderson, H.Y. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259.
[16] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[17] S. Hilger, Differential and difference calculus—unified, Nonlinear Anal. 30 (1997) 2683–2694.
[18] C.H. Hong, C.C. Yeh, Positive solutions for eigenvalue problems on a measure chain, Nonlinear Anal. 51 (2002) 499–507.
[19] M.A. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.
[20] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamical Systems on Measure Chains, Kluwer Academic, Boston, 1996.
[21] W.C. Lian, W.F. Wong, C.C. Yeh, On the existence of positive solutions of nonlinear differential equations, Proc. Amer. Math. Soc. 124 (1996) 1117–1126.
[22] J. Liang, T.J. Xiao, Z.C. Hao, Positive solutions of singular differential equations on measure chains, Comput. Math. Appl. 49 (2005) 651–663.