**C*-Algebras with Weak (FN)**

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Let $A$ be a simple $C^*$-algebra of real rank zero and stable rank one. We show that, for any normal element $x \in A$ with

$$\lambda - x \in \overline{\text{Inv}_0(A)}$$

for all $\lambda \notin \text{sp}(x)$

there is a sequence of normal elements $x_n \in A$ with finite spectra such that

$$x = \lim_{n \to \infty} x_n.$$

We show this for more general $C^*$-algebras. We also point out that the condition that $A$ is simple cannot be removed.

1. INTRODUCTION

In recent years there have been significant developments in the study of classifying $C^*$-algebras of real rank zero (see [E] for a survey). A $C^*$-algebra $A$ is said to have real rank zero, if the set of selfadjoint elements with finite spectrum is dense in the set of selfadjoint elements of $A$ (FS). One observes that, if $A$ is a unital $C^*$-algebra of real rank zero with nontrivial $K_1(A)$, then not all unitaries in $A$ can be approximated by unitaries in $A$ with finite spectrum. In [P1] N. C. Phillips introduced the notion of weak (FU), i.e., a (unital) $C^*$-algebra $A$ has weak (FU), if every unitary in the connected component of the unitary group of $A$ containing the identity can be approximated by unitaries in $A$ with finite spectrum. Many $C^*$-algebras of real rank zero were proved to have weak (FU) (see [P1], [P2] and [GL]). It was later proved that every $C^*$-algebra of real rank zero has weak (FU) (see [Ln1]). But that is only the beginning of the story.

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A $C^*$-algebra $A$ is said to have (FN), if every normal element in $A$ can be approximated by normal elements in $A$ with finite spectrum. The question was raised in [Bl] whether every AF-algebra has (FN). It turned out to be a difficult question. It was shown in [Ln2] that, among other AF-algebras, every UHF-algebra has (FN). With the affirmative solution to the old problem whether a pair of almost commuting selfadjoint matrices is close to a pair of commuting selfadjoint matrices, one shows easily that, in fact, every AF-algebra has (FN). This is further generalized by P. Friis and M. Rørdam (see [FR]). It is shown in [FR] that a (unital) $C^*$-algebra of real rank zero with connected unitary group has (FN), if the set of normal elements is in the closure of invertible elements. In particular, a (unital) $C^*$-algebra of real rank zero and stable rank one with connected unitary group has (FN).

On the other hand, similar problems are also considered for $C^*$-algebras with disconnected unitary group. A $C^*$-algebra has weak (FN), if every normal element $x \in A$ with

$$\lambda - x \in \text{Inv}_0(\tilde{A}), \quad \text{for all} \quad \lambda \notin \text{sp}(x)$$

is the limit of a sequence of normal elements in $A$ with finite spectrum. The earliest result of this kind is the BDF-theory (see [BDF]) which says that the Calkin algebra has weak (FN). One should note that the Calkin algebra has real rank zero. In fact the Calkin algebra is purely infinite and simple. It has been shown in [Ln3] that every purely infinite simple $C^*$-algebra has weak (FN) and moreover every simple $C^*$-algebra of real rank zero, stable rank one with weakly unperforated $K_0$-group and with countably many extremal (up to scalar multiples) traces has weak (FN). It turns out that the above mentioned results are closely related to the problems of classifying of $C^*$-algebras of real rank zero as well as $C^*$-algebra extension theory.

Recall (3.1 in [FR]) that a unital $C^*$-algebra $A$ has (IR) if the set of invertible elements is dense in $R(A)$, the set of those elements $x \in A$ that for no ideal $I$ of $A$ is $x + I$ one-sided but not two-sided invertible in $A/I$. A non-unital $C^*$-algebra $A$ has (IR), if $\tilde{A}$, the $C^*$-algebra obtained by adjoining a unit to $A$ has (IR). From the definition, every $C^*$-algebra of stable rank one has (IR). It is also shown (Theorem 4.4 in [R]) that all unital purely infinite simple $C^*$-algebras have (IR).

In this short note we show that every simple $C^*$-algebra with real rank zero and (IR) has weak (FN). When $A$ is not simple, counterexamples that $C^*$-algebras of real rank zero and stable rank one have no weak (FN) can easily be obtained and are certainly known (see Remark).

The essential ingredients in the proof include
Lemma 1,  
(ii) For any normal element \( x \) with \( \dim(\text{sp}(x)) \leq 1 \) (i.e., the covering dimension of \( \text{sp}(x) \) is no more than 1) in a \( C^* \)-algebra \( A \) of real rank zero, if

\[
\lambda - x \in \text{Inv}_0(A) \quad \text{for all} \quad \lambda \notin \text{sp}(x)
\]

then \( x \) is approximated by normal elements in \( A \) with finite spectrum.

(iii) An almost normal element \( x \) (i.e. \( \| x^*x - xx^* \| \) is small) in a \( C^* \)-algebra of (IR) is close to a normal element (Theorem 4.4 in [FR]).

2. PROOF OF THE RESULT

Lemma 1 (cf. 3.12 in [Ln3]). For any \( \varepsilon > 0 \), there is an integer \( L \) that satisfies the following: for any unital \( C^* \)-algebra \( A \) of real rank zero, if \( x \in A \) is normal, \( \| x \| \leq 1 \) and

\[
\lambda - x \in \text{Inv}_0(A)
\]

for all \( \lambda \in F \), where \( \text{sp}(x) \subset F \subset \{ \lambda : |\lambda| \leq 2 \} \), then there are normal elements \( y \in M_L(A) \) and \( z \in M_{L+1}(A) \) with finite spectra such that

\[
\| x \oplus y - z \| < \varepsilon,
\]

and \( \text{sp}(y), \text{sp}(z) \subset F \).

Proof. This is a variation of [Ln3]. In 3.12 of [Ln3], let \( \delta = 0 \) and \( p = 1 \). The integer \( L \) in 3.12 of [Ln3] depends on \( F \). However, since \( \varepsilon \) is given, there are only finitely many compact subsets of the disk with center at origin and radius 2 which need to be considered.

Lemma 2. For any \( \varepsilon > 0 \), \( M > 0 \) and \( r > 0 \), there is a \( \delta = \delta(\varepsilon, M, r) > 0 \) satisfying the following: for any unital \( C^* \)-algebra \( A \) and a normal element \( x \in A \) with \( \| x \| \leq M \) and any element \( z \in A \) with

\[
\| x - z \| < \delta,
\]

we have

\[
\| (\lambda - x)^{-1}(\lambda - z) - 1 \| < \varepsilon
\]

if \( \text{dist}(\lambda, \text{sp}(x)) \geq r \).

Proof. Standard.
Lemma 3. Let $A$ be a unital non-elementary simple $C^*$-algebra of real rank zero and let $X$ be an one dimensional finite CW complex in the plane. Suppose that $\pi: K_1(C(X)) \to K_1(A)$ is a homomorphism. Then, for any non-zero projection $p \in A$, there is a normal element $x \in pAp$ with $\text{sp}(x) \subset X$ such that $\phi_x = \pi$, where $\phi: C(X) \to pAp$ is the monomorphism induced by $x$ and $\phi_x: K_1(C(X)) \to K_1(A)$ is the homomorphism induced by $\phi$.

Proof. Let $g_1, g_2, ..., g_n$ be the generators of $K_1(C(X))$ corresponding to the bounded connected components $A_1, A_2, ..., A_n$ of $C\setminus X$.

Suppose that $z_i = \pi(g_i), i = 1, 2, ..., n$. Let $p_1, p_2, ..., p_n$ be nonzero mutually orthogonal projections in $pAp$ (they exist because $A$ is a non-elementary simple $C^*$-algebra of real rank zero). There are unitaries $v_i \in p_iAp$, such that $[v_i] = z_i$ in $K_1(A)$. There is a continuous function $f_i: S^1 \to \partial A_i$, the boundary of $A_i$, for each $i$. Set

$$x = \sum_{i=1}^{n} f_i(v_i).$$

One checks that $x$ meets the requirements.

Lemma 4. If $A$ is a $C^*$-algebra with (IR), then $pAp$ has (IR) for every projection $p \in M(A)$, where $M(A)$ is the multiplier algebra of $A$.

Proof. The proof is exactly the same as that of 2.5 in [BP]. For completeness, we will give the proof for the case that $p \in A$. The general case follows the same way as in the proof of 2.5 in [BP]. We also note that, if $A$ is not unital, $\tilde{A}$ is unital and has (IR). So we may assume that $A$ is unital. Let $x \in R(pAp)$. Then $1 - p + x$ is in $R(A)$, so there is, for each $\varepsilon > 0$, an invertible element $y \in A$ such that

$$\|x + (1 - p)y\| < \varepsilon.$$  

With $b = (1 - p)y(1 - p)$ this means that

$$\|1 - p - b\| < \varepsilon.$$  

Assuming that $\varepsilon < 1$, it follows that $b$ is invertible in $(1 - p)A(1 - p)$. By Lemma 2.3 in [BP], this implies that

$$z = pyp - pyp(1 - p)b^{-1}(1 - p)y.$$
is invertible in $pAp$. We estimate that
\[ \|b^{-1}\| < (1 - \varepsilon)^{-1}. \]
Hence
\[ \|py(1 - p) b^{-1}(1 - p) yp\| < (1 - \varepsilon)^{-1} \|py(1 - p)\|^2 < (1 - \varepsilon)^{-1} \varepsilon^2. \]
Thus
\[ \|x - z\| < \|x - pyp\| + (1 - \varepsilon)^{-1} \varepsilon^2 < \varepsilon + (1 - \varepsilon)^{-1} \varepsilon^2. \]
This implies that $pAp$ has (IR).

**Lemma 5.** Let $A$ be a simple $C^*$-algebra of real rank zero and let $p_1, p_2, \ldots, p_n$ be nonzero projections in $A$. Then there exists a nonzero projection $e \in A$ with $e \leq p_1$ and partial isometries $v_1, v_2, \ldots, v_{n-1} \in A$ such that
\[ v_i v_i^* = e \quad \text{and} \quad v_i^* v_i \leq p_{i+1}, \quad i = 1, 2, \ldots, n-1. \]

**Proof.** This is certainly known. We give a proof for completeness. By induction, it suffices to show Lemma 5 for the case that $i = 2$. One way to show this is to apply Lemma 1.1 in [Zh2]. Take $p = p_1$ and $q = p_2$ in Lemma 1.1 in [Zh2]. Let $e$ be one of (nonzero) $r$, in Lemma 1.1 in [Zh2].

**Theorem.** Let $A$ be a simple $C^*$-algebra of real rank zero with (IR). Then $A$ has weak (FN), i.e., if $x \in A$ is a normal element with
\[ \lambda - x \in \text{Inv}_0(A) \quad \text{(or Inv}_0(\tilde{A}) \text{ if } A \text{ is not unital)} \]
for any $\lambda \notin \text{sp}(x)$, then there is a sequence of normal elements $x_n \in A$ with finite spectrum such that
\[ \lim_{n \to \infty} x_n = x \quad \text{in norm}. \]

**Proof.** We may assume that $A$ is not elementary. We may also assume that $|\lambda| \leq 1$. We first consider the case that $A$ is unital.

Let $1/2 > \varepsilon > 0$. With $r = \varepsilon/8$ and $M = 2$, let $\delta_1 = \delta(\varepsilon/8, M, r) > 0$ be as in the Lemma 2. By Theorem 4.4 in [FR], there exists $\delta_2 > 0$ such that for any $C^*$-algebra $D$ with (IR), any element $z \in D$ with $\|z\| \leq 2$ and
\[ \|z^* z - z z^*\| < \delta_2, \]
there exists a normal element $z_1 \in D$ such that 

$$\|z - z_1\| < \min(\varepsilon/8, \delta_1/2).$$

We are now going to apply Lemma 2 in [Ln2]. Note that, since $A$ has real rank zero, nonzero projections $p_k$ described in that lemma exist. Thus we obtain nonzero mutually orthogonal projections $p_1, p_2, \ldots, p_n \in A$ such that 

$$\left\| x - \sum_{i=1}^{n} \lambda_i p_i - pxp \right\| < \varepsilon/8 \quad \text{and} \quad \| px - xp \| < \min(\delta_2/2, \delta_1/2)$$

where $p = 1 - \sum_{i=1}^{n} p_i$, $\lambda_i \in \text{sp}(x)$, and $\{ \lambda_i \}$ is $\varepsilon/4$-dense in $\text{sp}(x)$. Note that, by Lemma 4, $pAp$ has (IR). By 4.4 in [FR] and the choice of $\delta_2$, there is a normal element $x' \in pAp$ such that 

$$\| px - x' \| < \min(\varepsilon/8, \delta_1/2).$$

Let $r = \varepsilon/8$. By Lemma 2, we may assume that $\text{sp}(x') \subset F_1$ where 

$$F_1 = \{ \lambda \in \mathbb{C} : \text{dist}(\lambda, \text{sp}(x)) < r \},$$

and

$$\lambda - x' \in \text{Inv}_d(pAp)$$

for all $\lambda \notin F_1$. By applying 3.4 in [FR], we may further assume that $\text{sp}(x')$ is contained in a one dimensional finite CW complex $F_2 \subset F_1$. Since $A$ is simple, by Lemma 5, there is a nonzero projection $q_1 \leq p$ such that 

$$q_1 \leq p_i, \quad i = 1, 2, \ldots, n.$$ 

Let $L$ be the integer in Lemma 2 associated with $\varepsilon/8$. By Theorem 1.1 in [Zh], there is a nonzero projection $q \leq q_1$ such that 

$$(L + 2)[q] \leq [q_1].$$

where $[q]$ and $[q_1]$ are equivalence classes of projections in matrices over $A$. This inequality means that there mutually orthogonal projections 

$$e_1, e_2, \ldots, e_{L+2} \leq q_1$$

such that every $e_i$ is equivalent to $q$. By Lemma 3 there is a normal element $x_2 \in qAp$ such that 

$$\lambda - (x' \oplus x_2) \in \text{Inv}_d((p \oplus q) \ M_d(A)(p \oplus q))$$
for all $\lambda \not\in F_2$. Since $F_2$ has dimension 1, it follows from 5.4 in [Ln3] that there is a normal element $z_1 \in (p \oplus q) M_2(A)(p \oplus q)$ with finite spectrum such that

$$\|x' \oplus x_2 - z_1\| < \epsilon/8.$$

Since

$$\lambda - x' \in \text{Inv}_0(p Ap)$$

for all $\lambda \not\in F_1$ and $A$ has real rank zero, by applying Lemma 2.3 in [Ln3], we conclude that

$$\lambda - x_2 \in \text{Inv}_0(q A q)$$

for all $\lambda \not\in F_1$. By Lemma 1, there is a normal element $x_3 \in M_{L+1}(q A q)$ with finite spectrum and there is a normal element $z_2 \in M_{L+1}(q A q)$ with finite spectrum such that

$$\|x_2 \oplus x_3 - z_2\| < \epsilon/8.$$

This implies that

$$\|p x p \oplus z_2 - z_1 \oplus x_3\| \leq \|p x p \oplus z_2 - x' \oplus z_2\| + \|x' \oplus z_2 - x' \oplus x_2 \oplus x_3\| + \|x' \oplus x_2 \oplus x_3 - z_1 \oplus x_3\|$$

$$< \epsilon/8 + \epsilon/8 + \epsilon/8 < 3\epsilon/8.$$

Note that $z_2 \in M_{L+1}(q A q)$. Write

$$z_2 = \sum_{k=1}^{m} \xi_k d_k,$$

where $\xi_k \in F_2$ and $d_k \in M_{L+1}(q A q)$ are mutually orthogonal projections. Since $r = \epsilon/8$, and $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ is $\epsilon/4$-dense in $X$, there are $\lambda_i$ such that $|\xi_k - \lambda_i| < 3\epsilon/8$. So we may assume that

$$\|p x p \oplus z'_2 - z_1 \oplus x_3\| < 6\epsilon/8,$$

where $z'_2 = \sum_{k=1}^{n} \xi_k d_k$ and where $d_k \in M_{L+1}(q A q)$ (some of the $d_k$ could be zero). Since

$$(L+2)[q] \leq [q_1] \leq [p_i]$$

for each $i$, we conclude that there is a partial isometry $v \in M_{L+1}(A)$ such that

$$v^* d_k v \leq p_k, \quad k = 1, 2, ..., n.$$
We may also assume that
\[ v^*v = \sum_{k=1}^{n} v^*d_k v \quad \text{and} \quad vv^* = \sum_{k=1}^{n} d_k. \]

Since \( \sum_{k=1}^{n} d_k \) is orthogonal to \( \sum_{k=1}^{n} v^*d_k v \),
\[ u = v + v^* + \left( 1 - \sum_{k=1}^{n} v^*d_k v - \sum_{k=1}^{n} v^*d_k \right) \]
is a unitary in \( M_{L_+}(A) \) such that
\[ u^*d_k u \leq p_k \quad k = 1, 2, \ldots, n, \quad up = pu = p \]
and
\[ u \left( \sum_{k=1}^{n} \lambda_k(p_k - u^*d_k u) \right) = \sum_{k=1}^{n} \lambda_k(p_k - u^*d_k u). \]

Note that
\[ u^*(p_{x\oplus z_1^*})u = \sum_{k=1}^{n} \lambda_k u^*d_k u \oplus p_{x\oplus z_1^*}. \]
Thus
\[
\| x - \left[ \sum_{k=1}^{n} \lambda_k(p_k - u^*d_k u) \oplus u^*(z_1 \oplus x_1) \right]u \|
\leq \| x - \left[ \sum_{k=1}^{n} \lambda_k p_k + p_{x\oplus z_1^*} \right] + \left[ \sum_{k=1}^{n} \lambda_k(p_k - u^*d_k u) + u^*(p_{x\oplus z_1^*})u \right] - \left[ \sum_{k=1}^{n} \lambda_k(p_k - u^*d_k u) + u^*(z_1 \oplus x_1) \right]u \|
\leq \epsilon/8 + 6\epsilon/8 < \epsilon.
\]

Now we consider the case that \( A \) is not unital. Then \( 0 \in \text{sp}(x) \). Let \( h \) be a continuous function defined on the unit disk \( D \) such that \( |h| \leq 1 \), \( h(\zeta) = \zeta \) if \( |\zeta| > \epsilon/2 \) and \( h(\zeta) = 0 \) if \( |\zeta| < \epsilon/4 \) with
\[ \| h(x) - x \| < \epsilon/2. \]

Now let \( p \) be the spectral projection of \( x \) in \( A^{**} \) corresponding to the open subset \( \{ \zeta \in D : |\zeta| > \epsilon/16 \} \) and \( q \) be the spectral projection of \( x \) in \( A^{**} \) corresponding to the closed subset \( \{ \zeta \in D : |\zeta| \geq \epsilon/8 \} \). Then \( p \) is open
projection and \( q \) is a closed projection in \( A^{**} \) and \( p \leq q \). Clearly \( q \) is compact (it is dominated by \( g(x) \) for some continuous function \( g \) vanishing at zero). It follows from [B2] that there is a projection \( e \in A \) such that \( q \leq e \leq p \). Since \( qh(x) = h(x)q = h(x), h(x) \in eAe \). Since
\[
\lambda - x \in \text{Inv}_d(\tilde{A}) \quad \text{for all} \quad \lambda \notin \text{sp}(x),
\]
every invertible element in \( B \), the \( C^* \)-subalgebra of \( \tilde{A} \) generated by \( x \) and the identity (of \( \tilde{A} \)) is in \( \text{Inv}_d(\tilde{A}) \). Thus, if \( \lambda \notin \text{sp}(h(x)) \),
\[
\lambda - h(x) \in \text{Inv}_d(\tilde{A}),
\]
since \( \lambda - h(x) \) is invertible in \( B \). By Lemma 2.3 in [Ln2], since \( A \) has real rank zero, we have
\[
\lambda e - h(x) \in \text{Inv}_d(eAe) \quad \text{for all} \quad \lambda \notin \text{sp}(h(x)).
\]
Now, by Lemma 4, \( eAe \) is a unital simple \( C^* \)-algebra of real rank zero with (1R). So the unital case can apply to \( h(x) \).

**Remark.** The condition of simplicity can not be removed. Let \( B = A \oplus A \), where \( A \) is a unital \( C^* \)-algebra of real rank zero and stable rank one with \( K_1(A) \neq 0 \). Let \( u \in A \) be a unitary such that \( [u] \neq 0 \) in \( K_1(A) \) and \( z \) is a normal element with spectrum being the whole unit disk \( D \). Set \( x = u \oplus z \). Then \( x \) is a normal element in \( B \) with \( \text{sp}(x) = D \). So, for any \( \lambda > 1 \), \( \lambda - x \in \text{Inv}_d(B) \). But it is clear that \( x \) can not possibly be approximated by normal elements with finite spectrum. More complicated examples can also be constructed.

**REFERENCES**


