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On Relations Between Detection and Estimation of Discrete Time Processes*

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It is shown that, for discrete-time processes, both the causal minimum variance estimate of an arbitrary random signal process corrupted by additive white Gaussian noise, and the associated error covariance matrix, may be obtained, by simple formulas, from the likelihood ratio which arises in the optimum detection of the same signal. As a consequence of this result, the optimum detector is amenable to a causal estimator–correlator type interpretation. An example is worked out to illustrate the relations obtained.

1. INTRODUCTION

The problems of detection, as well as estimation, of signals in the presence of noise have been studied extensively in the literature. However, very few results have been established concerning explicit relations between the processing procedures of detection and estimation. For continuous-time processes, Kailath (1969, 1970) obtained the likelihood ratio for the optimum detection of an arbitrary signal process corrupted by additive Gaussian noise as a *causal* estimator–correlator type operation (involving the Ito integral) and in (Kailath, 1968) obtained a converse relation. Esposito (1968) obtained related results for discrete-time processes; however, his analysis necessitated the use of the *noncausal* estimator. Some nontrivial differences between the discrete-time and continuous–time analyses, including the use of the noncausal estimator in Esposito's discrete-time analysis, have been pointed out by Kailath (1968).

In this context, the purpose here is to show that, for discrete-time processes, both the causal minimum variance estimate of an arbitrary signal process corrupted by additive white Gaussian noise and its associated error covariance

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matrix, may be obtained from the sequential likelihood ratio by means of simple formulas. The signal and noise processes need not be statistically independent; a certain kind of "one-sided" dependence is permitted. It is also shown that, as a consequence of these results, the likelihood ratio detector is amenable to a causal estimator–correlator type interpretation. The special case where the signal process is Gauss–Markov is worked out as an example and serves to illustrate the relations obtained.

2. Relations Involving Estimate and Estimation Error Covariance Matrix with Likelihood Ratio

Consider the following problem of deciding between two hypotheses H^1 and H^0:

$$\begin{aligned} H^1 &: \mathbf{z}_k = \mathbf{x}_k + \mathbf{v}_k , \\ H^0 &: \mathbf{z}_k = \mathbf{v}_k , \end{aligned} \tag{1}$$

where $\{\mathbf{v}_k\}$ is an n-dimensional vector, white zero-mean Gaussian noise process with covariance $E[\mathbf{v}_k \mathbf{v}_l^T] = \mathbf{R}_k \delta_{kl}$ and $\{\mathbf{x}_k\}$ is an arbitrary (not necessarily Gaussian) n-dimensional vector random process. $\{\mathbf{v}_k\}$ and $\{\mathbf{x}_k\}$ need not be mutually independent but only such that the present measurement noise is independent of present and past signal and past noise, i.e.,

$$f(\mathbf{v}_k \mid \mathbf{X}_k , \mathbf{V}_{k-1}) = f(\mathbf{v}_k), \tag{1a}$$

where $\mathbf{X}_k \triangleq \{\mathbf{x}_1 , \mathbf{x}_2 ,..., \mathbf{x}_k\}$ and $\mathbf{V}_k \triangleq \{\mathbf{v}_1 , \mathbf{v}_2 ,..., \mathbf{v}_k\}$.

It is well known that the Bayes optimum test for deciding between hypotheses H^1 and H^0 is the following:

$$\Lambda_k \underset{\text{choose } H^0}{\overset{\text{choose } H^1}{\gtrless}} \eta \tag{2}$$

where $\Lambda_k \triangleq f(\mathbf{Z}_k \mid H^1)/f(\mathbf{Z}_k \mid H^0)$ is the likelihood ratio, η is a threshold which depends upon the cost assignments and the *a priori* probabilities of the hypotheses, and $\mathbf{Z}_k$ is the observation sequence $\{\mathbf{z}_1 , \mathbf{z}_2 ,..., \mathbf{z}_k\}$.

Let $\hat{\mathbf{x}}_k(\mathbf{Z}_k)$ denote the causal minimum variance estimate of the signal $\mathbf{x}_k$ under the assumption that hypothesis H^1 is true, and $\mathbf{P}_k$ the associated estimation error covariance matrix. Then the main results of this paper are that

$\hat{\mathbf{x}}_k(\mathbf{Z}_k)$ and $\mathbf{P}_k$ may be obtained from the sequential likelihood ratio Λ_k by the following formulas:

$$\hat{\mathbf{x}}_k = \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} [\ln \Lambda_k], \tag{3a}$$

$$\mathbf{P}_k = \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} \left\{ \left[\frac{\partial}{\partial \mathbf{z}_k} \ln \Lambda_k \right]^T \right\} \mathbf{R}_k , \tag{3b}$$

where $\partial / \partial \mathbf{z}_k$ denotes the n-dimensional column vector of partial derivatives with respect to the components of $\mathbf{z}_k$. To prove (3a) and (3b), first note that Λ_k can be expressed as

$$\Lambda_k = \frac{f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^1)}{f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0)} \frac{f(\mathbf{Z}_{k-1} \mid H^1)}{f(\mathbf{Z}_{k-1} \mid H^0)}, \tag{4}$$

or

$$\Lambda_k = \Lambda_k' \cdot \Lambda_{k-1},$$

where

$$\Lambda_k' = \frac{f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^1)}{f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0)}.$$

From (1) and the fact that $\{\mathbf{v}_k\}$ is a "white" Gaussian sequence, we see that $f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0)$ is zero-mean Gaussian with covariance $\mathbf{R}_k$, that is,

$$f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0) = N_{\mathbf{z}_k}(\mathbf{0}, \mathbf{R}_k). \tag{5}$$

Also,

$$f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^1) = \int f(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1) f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k. \tag{6}$$

Now, from (1)[1],

$$f(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1) = f_{\mathbf{v}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1}(\mathbf{z}_k - \mathbf{x}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1). \tag{6a}$$

Since, under H^1, $\mathbf{Z}_{k-1}$ is a function of $\mathbf{X}_{k-1}$ and $\mathbf{V}_{k-1}$, then application of condition (1a) to the right-hand side of (6a) yields

$$\begin{aligned} f(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1) &= f_{\mathbf{v}_k}(\mathbf{z}_k - \mathbf{x}_k) \\ &= N_{\mathbf{z}_k}(\mathbf{x}_k, \mathbf{R}_k). \end{aligned} \tag{7}$$

[1] Where no confusion is liable to arise the conditional density of a random variable x given random variable y is specified as $f(x \mid y)$; otherwise, the more explicit notation $f_{x|y}(\alpha \mid \beta)$ is used where α and β are dummy variables.

Thus

$$\Lambda_k' = \frac{\int \exp[-\tfrac{1}{2}(\mathbf{z}_k - \mathbf{x}_k)^T \mathbf{R}_k^{-1}(\mathbf{z}_k - \mathbf{x}_k)] f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k}{\exp[-\tfrac{1}{2}\mathbf{z}_k^T \mathbf{R}_k^{-1}\mathbf{z}_k]} \tag{8}$$

$$= \int \exp[\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{z}_k - \tfrac{1}{2}\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{x}_k] f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k . \tag{9}$$

Taking the partial derivative of Λ_k' with respect to $\mathbf{z}_k$,

$$\frac{\partial \Lambda_k'}{\partial \mathbf{z}_k} = \mathbf{R}_k^{-1} \int \mathbf{x}_k \exp[\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{z}_k - \tfrac{1}{2}\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{x}_k] f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k . \tag{10}$$

The causal minimum variance estimator under the assumption that the signal process $\{\mathbf{x}_k\}$ is surely present in the observation interval, has the expression

$$\hat{\mathbf{x}}_k = \int \mathbf{x}_k f(\mathbf{x}_k \mid \mathbf{Z}_k, H^1)\, d\mathbf{x}_k , \tag{11}$$

or

$$\hat{\mathbf{x}}_k = \frac{\int \mathbf{x}_k f(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1) f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k}{f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^1)}, \tag{12}$$

or

$$\hat{\mathbf{x}}_k = \frac{\int \mathbf{x}_k f(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{Z}_{k-1}, H^1) f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k}{\Lambda_k' \cdot f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0)} . \tag{13}$$

Hence

$$\hat{\mathbf{x}}_k = \frac{\int \mathbf{x}_k \exp[\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{z}_k - \tfrac{1}{2}\mathbf{x}_k^T \mathbf{R}_k^{-1}\mathbf{x}_k] f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k}{\Lambda_k'} . \tag{14}$$

On comparing (14) with (10), we immediately deduce that

$$\hat{\mathbf{x}}_k = \frac{\mathbf{R}_k}{\Lambda_k'} \cdot \frac{\partial \Lambda_k'}{\partial \mathbf{z}_k} = \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} [\ln \Lambda_k'] \tag{15}$$

$$= \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} [\ln \Lambda_k - \ln \Lambda_{k-1}],$$

or

$$\hat{\mathbf{x}}_k = \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} [\ln \Lambda_k], \tag{3a}$$

which proves the first of the two relations. Equation (3a) is a stronger version of Esposito's result for the noncausal estimator [see Esposito (1968), Eq. (6)].

The estimation error covariance matrix $\mathbf{P}_k$ is defined by

$$\mathbf{P}_k = \int [\mathbf{x}_k - \hat{\mathbf{x}}_k][\mathbf{x}_k - \hat{\mathbf{x}}_k]^T f(\mathbf{x}_k \mid \mathbf{Z}_k, H^1)\, d\mathbf{x}_k , \tag{16}$$

or

$$\mathbf{P}_k = \int \mathbf{x}_k \mathbf{x}_k^T f(\mathbf{x}_k \mid \mathbf{Z}_k, H^1)\, d\mathbf{x}_k - \hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^T. \tag{17}$$

A second differentiation of the expression $[(\partial/\partial\mathbf{z}_k)(\ln \Lambda_k')]^T$ with respect to $\mathbf{z}_k$ gives

$$\frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}(\ln \Lambda_k')\right]^T\right\} = \frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{1}{\Lambda_k'}\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\right\} \tag{18}$$

$$= \frac{1}{\Lambda_k'}\frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\right\}$$

$$- \frac{1}{\Lambda_k'}\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k' \cdot \frac{1}{\Lambda_k'}\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T \tag{19}$$

or

$$\mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}(\ln \Lambda_k')\right]^T\right\}\mathbf{R}_k = \frac{1}{\Lambda_k'}\mathbf{R}_k\frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\right\}\mathbf{R}_k$$

$$- \frac{\mathbf{R}_k}{\Lambda_k'}\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k' \cdot \frac{1}{\Lambda_k'}\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\mathbf{R}_k \tag{20}$$

or

$$\mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}(\ln \Lambda_k')\right]^T\right\}\mathbf{R}_k = \frac{1}{\Lambda_k'}\mathbf{R}_k\frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\right\}\mathbf{R}_k - \hat{\mathbf{x}}_k\hat{\mathbf{x}}_k^T \tag{21}$$

as a consequence of (15). Partial differentiation of (10) with respect to $\mathbf{z}_k$ allows the first term on the right side of (21) to be written as

$$\frac{1}{\Lambda_k'}\mathbf{R}_k\frac{\partial}{\partial \mathbf{z}_k}\left\{\left[\frac{\partial}{\partial \mathbf{z}_k}\Lambda_k'\right]^T\right\}\mathbf{R}_k$$

$$= \frac{\int \mathbf{x}_k\mathbf{x}_k^T \exp[\mathbf{x}_k^T\mathbf{R}_k^{-1}\mathbf{z}_k - \tfrac{1}{2}\mathbf{x}_k^T\mathbf{R}_k^{-1}\mathbf{x}_k] f(\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1)\, d\mathbf{x}_k}{\Lambda_k'}. \tag{22}$$

Comparison of the right-hand sides of (11) and (14) immediately establishes that the right-hand side of (22) may be written as $\int \mathbf{x}_k\mathbf{x}_k^T f(\mathbf{x}_k \mid \mathbf{Z}_k, H^1)\, d\mathbf{x}_k$. Thus (21) simplifies to

$$\mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k} \left\{ \left[\frac{\partial}{\partial \mathbf{z}_k} (\ln \Lambda_k') \right]^T \right\} \mathbf{R}_k = \int \mathbf{x}_k\mathbf{x}_k^T f(\mathbf{x}_k \mid \mathbf{Z}_k, H^1)\, d\mathbf{x}_k - \hat{\mathbf{x}}_k\hat{\mathbf{x}}_k^T = \mathbf{P}_k \tag{23}$$

by virtue of (17).

The final result, Eq. (3b), follows on noting that

$$\frac{\partial}{\partial \mathbf{z}_k} (\ln \Lambda_k') = \frac{\partial}{\partial \mathbf{z}_k} (\ln \Lambda_k).$$

Additionally, it may be readily verified from (15) that

$$\ln \Lambda_k' = \int \mathbf{R}_k^{-1}\hat{\mathbf{x}}_k\, d\mathbf{z}_k + C_k(\mathbf{Z}_{k-1}), \tag{24}$$

where $C_k(\mathbf{Z}_{k-1})$ is not a function of $\mathbf{z}_k$. Now let $\mathbf{z}_{k_f}$ denote the last or final observation vector. The likelihood ratio $\Lambda_{k_f}(\mathbf{Z}_{k_f})$ based on all past data $\mathbf{Z}_{k_f}$, which is used in the decision rule (2) for choosing H^1 or H^0, has the expression

$$\ln \Lambda_{k_f} = \sum_{k=1}^{k_f} \ln \Lambda_k'. \tag{25}$$

Substitution of (24) into (25) yields

$$\ln \Lambda_{k_f} = \sum_{k=1}^{k_f} \int \mathbf{R}_k^{-1}\hat{\mathbf{x}}_k\, d\mathbf{z}_k + \sum_{k=1}^{k_f} C_k(\mathbf{Z}_{k-1}). \tag{26}$$

On integrating by parts, (26) can be written as

$$\ln \Lambda_{k_f} = \sum_{k=1}^{k_f} \mathbf{z}_k^T\mathbf{R}_k^{-1}\hat{\mathbf{x}}_k - \sum_{k=1}^{k_f} \int \mathbf{z}_k^T\mathbf{R}_k^{-1}\, d\hat{\mathbf{x}}_k + \sum_{k=1}^{k_f} C_k(\mathbf{Z}_{k-1}). \tag{27}$$

Thus the computation of the likelihood ratio is basically an estimator–correlator operation (represented by the first term in (27)) with additional operations for evaluating the bias terms (which are a function of the estimate and the data). Note that use of the causal estimator permits on-line computation of the likelihood ratio.

The actual evaluation of the term $\int \mathbf{z}_k^T\mathbf{R}_k^{-1}\, d\hat{\mathbf{x}}_k$ in (27) is difficult, in general, since $\hat{\mathbf{x}}_k$ is not usually a known analytic function of $\mathbf{z}_k$. For the special case of a Guass–Markov sequence $\{\mathbf{x}_k\}$, however, $\hat{\mathbf{x}}_k$ is an explicit linear function of $\mathbf{z}_k$ [see (44)] and the integral can be calculated.

3. An Example

To illustrate the relations (3a) and (3b), we consider here the special case where $\mathbf{x}_k$ evolves as a Gauss–Markov process according to the equation

$$\mathbf{x}_k = \boldsymbol{\phi}_{k,k-1}\mathbf{x}_{k-1} + \mathbf{w}_{k-1} \tag{28}$$

$\boldsymbol{\phi}_{k,k-1}$ is a $n \times n$ one-step transition matrix and $\{\mathbf{w}_k\}$ is a zero-mean white Gaussian process with $E[\mathbf{w}_k\mathbf{w}_l^T] = \mathbf{Q}_k\delta_{kl}$. $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are assumed to be mutually independent.

The causal minimum variance estimate of $\mathbf{x}_k$ under H^1 has been obtained in the literature by several methods (Kalman, 1960; Ho and Lee, 1964) and is given by the familiar Kalman estimator algorithms:

$$\hat{\mathbf{x}}_k = \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{N}_k[\mathbf{N}_k + \mathbf{R}_k]^{-1}[\mathbf{z}_k - \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1}], \tag{29}$$

$$\mathbf{N}_k = \operatorname{cov}\{\mathbf{x}_k \mid \mathbf{Z}_{k-1}, H^1\} = \boldsymbol{\phi}_{k,k-1}\mathbf{P}_k\boldsymbol{\phi}_{k,k-1}^T + \mathbf{Q}_k, \tag{30}$$

$$\mathbf{P}_k = \operatorname{cov}\{\mathbf{x}_k \mid \mathbf{Z}_k, H^1\} = \mathbf{N}_k - \mathbf{N}_k[\mathbf{N}_k + \mathbf{R}_k]^{-1}\mathbf{N}_k, \tag{31}$$

or equivalently

$$\mathbf{P}_k^{-1} = \mathbf{N}_k^{-1} + \mathbf{R}_k^{-1}. \tag{32}$$

We propose to rederive the Kalman filter algorithms (29)–(32) by computing the likelihood ratio and using (3a) and (3b). It may be easily shown (Ho and Lee, 1964)

$$f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^1) = N_{z_k}(\boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1}, \mathbf{N}_k + \mathbf{R}_k), \tag{33}$$

and

$$f(\mathbf{z}_k \mid \mathbf{Z}_{k-1}, H^0) = N_{z_k}(\mathbf{0}, \mathbf{R}_k).$$

Thus the logarithm of the likelihood ratio Λ_k is

$$\ln \Lambda_k = \ln \Lambda_k' + \ln \Lambda_{k-1},$$

or

$$\begin{aligned}\ln \Lambda_k = &-\tfrac{1}{2}\mathbf{z}_k^T[\mathbf{N}_k + \mathbf{R}_k]^{-1}\mathbf{z}_k + \hat{\mathbf{x}}_{k-1}^T\boldsymbol{\phi}_{k,k-1}^T[\mathbf{N}_k + \mathbf{R}_k]^{-1}\mathbf{z}_k \\ &-\tfrac{1}{2}\hat{\mathbf{x}}_{k-1}^T\boldsymbol{\phi}_{k,k-1}^T[\mathbf{N}_k + \mathbf{R}_k]^{-1}\boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} \\ &+\tfrac{1}{2}\mathbf{z}_k^T\mathbf{R}_k^{-1}\mathbf{z}_k + \text{terms which are not a function of } \mathbf{z}_k.\end{aligned} \tag{34}$$

Taking the partial derivative of (34) with respect to $\mathbf{z}_k$ gives

$$\frac{\partial}{\partial \mathbf{z}_k}[\ln \Lambda_k] = -[\mathbf{N}_k + \mathbf{R}_k]^{-1}\mathbf{z}_k + [\mathbf{N}_k + \mathbf{R}_k]^{-1}\boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{R}_k^{-1}\mathbf{z}_k \quad (35)$$

$$= [\mathbf{R}_k^{-1} - (\mathbf{N}_k + \mathbf{R}_k)^{-1}]\, \mathbf{z}_k + [\mathbf{N}_k + \mathbf{R}_k]^{-1}\boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} \,. \quad (36)$$

By the matrix inversion lemma (Sage, 1968),

$$(\mathbf{N}_k + \mathbf{R}_k)^{-1} = \mathbf{R}_k^{-1} - \mathbf{R}_k^{-1}\mathbf{T}_k\mathbf{R}_k^{-1}, \quad (37)$$

where

$$\mathbf{T}_k^{-1} = \mathbf{N}_k^{-1} + \mathbf{R}_k^{-1}. \quad (38)$$

Substitution of (37) into (36) yields

$$\frac{\partial}{\partial \mathbf{z}_k}[\ln \Lambda_k] = \mathbf{R}_k^{-1}\mathbf{T}_k\mathbf{R}_k^{-1}\mathbf{z}_k + [\mathbf{R}_k^{-1} - \mathbf{R}_k^{-1}\mathbf{T}_k\mathbf{R}_k^{-1}]\, \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} \,. \quad (39)$$

Therefore, according to our relation (3a),

$$\hat{\mathbf{x}}_k = \mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k}[\ln \Lambda_k] = \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{T}_k\mathbf{R}_k^{-1}[\mathbf{z}_k - \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1}]. \quad (40)$$

Furthermore, partial differentiation of (39) with respect to $\mathbf{z}_k$ and use of relation (3b) yields

$$\mathbf{R}_k \frac{\partial}{\partial \mathbf{z}_k}\left[\frac{\partial}{\partial \mathbf{z}_k} \ln \Lambda_k\right]^T \mathbf{R}_k = \mathbf{T}_k = \mathbf{P}_k \,, \quad (41)$$

where $\mathbf{P}_k = \operatorname{cov}\{\mathbf{x}_k \mid \mathbf{Z}_k \,, H^1\}$. Also, from (38) and (41),

$$\mathbf{P}_k\mathbf{R}_k^{-1} = \mathbf{I} - \mathbf{P}_k\mathbf{N}_k^{-1}, \quad (42)$$

where $\mathbf{I}$ is the identity matrix. Application of the matrix inversion lemma to (38) yields

$$\mathbf{P}_k = \mathbf{N}_k - \mathbf{N}_k[\mathbf{N}_k + \mathbf{R}_k]^{-1}\,\mathbf{N}_k \,. \quad (43)$$

Use of (42) and (43) allow (40) to be rewritten in the more familiar form:

$$\hat{\mathbf{x}}_k = \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{N}_k[\mathbf{N}_k + \mathbf{R}_k]^{-1}[\mathbf{z}_k - \boldsymbol{\phi}_{k,k-1}\hat{\mathbf{x}}_{k-1}], \quad (44)$$

where

$$\mathbf{N}_k = \boldsymbol{\phi}_{k,k-1}\mathbf{P}_{k-1}\boldsymbol{\phi}_{k,k-1}^T + \mathbf{Q}_k \,, \quad (45)$$

and

$$\mathbf{P}_k^{-1} = \mathbf{N}_k^{-1} + \mathbf{R}_k^{-1}. \tag{46}$$

Equations (44)–(46) are, however, precisely the Kalman filter algorithms (29)–(32).

Hence our relations (3a) and (3b) are verified for the special case of a Gauss–Markov signal process.

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