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On Some General Lacunary Interpolation Problems

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The object of this paper is to give a unified and complete treatment of these related cases of lacunary interpolation whose special cases can be found scattered in the literature. We consider only the case when the nodes are the zeros of $\pi_n(x)$ whose derivative is $-n(n-1)\,P_{n-1}(x)$, the Legendre polynomial of degree $n-1$. We find the fundamental polynomials and give the convergence results. © 1996 Academic Press, Inc.

Let

$$(-1 \leqslant)\, x_n < x_{n-1} < \cdots < x_1 (\leqslant 1) \tag{1}$$

be an arbitrary system of nodes and $0 = m_0 < m_1 < m_2 < \cdots < m_{q-1}$ an arbitrary sequence of integers. The $(0, m_1, ..., m_{q-1})$ interpolation on the nodes (1) means that we are looking for a polynomial of degree at most $nq-1$ whose m_ith derivatives $(i = 0, ..., q-1)$ at the nodes (1) are equal to given values. If this problem is uniquely solvable for any set of given data, then we say that the problem is *regular*. In this case there exist fundamental polynomials $L_{\nu, m_i}(x)$ of degree at most $nq-1$ such that

$$L^{(m_i)}_{\nu, m_j}(x_k) = \delta_{ij}\delta_{\nu k} \qquad (k, \nu = 1, ..., n;\ i, j = 0, ..., q-1).$$

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A necessary condition for the regularity is known to be the Polya condition viz. $m_j < nj$ $(j = 1, 2, ..., q-1)$ (see [6, Section 1.4]).

Usually, it is a difficult problem to find these polynomials. However, if we assume that $L_{\nu, m_i}(x)$ $(\nu = 1, ..., n;\ i = p+1, ..., q-1)$ are known, where p is defined by the inequalities $m_p < q \leqslant m_{p+1}$, then the remaining fundamental functions can be easily obtained by the formulae

$$L_{\nu, m_j}(x) = H_{\nu, m_j, q}(x) - \sum_{s=p+1}^{q-1} \sum_{k=1}^{n} H^{(m_s)}_{\nu, m_j, q}(x_k)\, L_{k, m_s}(x),$$
$$(j = 0, 1, ..., p). \tag{2}$$

Here $H_{\nu, m_j, q}(x)$ are the corresponding fundamental functions of $(0, 1, ..., q-1)$ Hermite interpolation (of degree $nq-1$). In this context two problems arise. First, how do we find the Hermite interpolating polynomials? The answer to this question is easy:

$$H_{\nu, m_j, q}(x) = \frac{l_\nu^q(x)}{m_j!} \sum_{s=0}^{q-m_j-1} \frac{[l_\nu^{-q}(x)]^{(s)}_{x_\nu}}{s!} (x - x_\nu)^{s+m_j}$$
$$(\nu = 1, ..., n;\ j = 0, ..., p), \tag{3}$$

where $l_\nu(x)$ is the νth fundamental function of Lagrange interpolation based on the nodes (1) (see [11, Lemma 1]).

The second problem, i.e., to find $L_{\nu, m_i}(x)$ $(\nu = 1, ..., n;\ i = p+1, ..., q)$ is more difficult. In some cases it is possible to find the last fundamental functions $L_{\nu, m_{q-1}}(x)$ $(\nu = 1, ..., n)$. Then in order to be able to find the other fundamental functions by using (2), we must have one of the following three situations:

(a) $(0, 1, ..., r-2, r)$ interpolation $(r \geqslant 2)$,

(b) $(0, 1, ..., r-3, r)$ interpolation $(r \geqslant 3)$,

(c) $(0, 1, ..., r-3, r-1, r)$ interpolation $(r \geqslant 3)$.

The object of this paper is to investigate these case of lacunary interpolation when (1) consists of the roots of

$$\pi_n(x) = -n(n-1) \int_{-1}^{x} P_{n-1}(t)\, dt, \tag{4}$$

the integral of the Legendre polynomial $P_{n-1}(t)$. Let us mention that fundamental functions of lacunary interpolation in this generality have not been investigated so far, except in Gonska [5] where Case (a) was considered, but without explicitly presenting the fundamental polynomials. We will show how to estimate the Hermite polynomials on the roots of (4), as well as the last fundamental functions $L_{\nu,r}(x)$ $(\nu=1, ..., n)$. In order to find the polynomials $L_{\nu,r}(x)$ $(\nu=1, ..., n)$, it is convenient to consider the so-called modified lacunary interpolation problem where the requirement of presenting the rth derivative at ± 1 is replaced by presenting some lower order derivatives at ± 1. Thus, in the modified problem in Case (a) we present the $(r-1)$th derivative at ± 1 so that instead of $L_{1,r}(x)$ and $L_{n,r}(x)$ we obtain $L_{1,r-1}(x)$ and $L_{n,r-1}(x)$. In cases (b) and (c), the $(r-2)$th derivatives at ± 1 are presented and this gives rise to the polynomials $L_{1,r-2}(x)$ and $L_{n,r-2}(x)$.

Case (a). $L^{(i)}_{\nu,j}(x_k)=\delta_{ij}\delta_{\nu k}$, where

$$\begin{aligned} &0 \leqslant i \leqslant r-2 \quad \text{and} \quad 1 \leqslant k \leqslant n;\\ &\text{or} \quad i=r-1 \quad \text{and} \quad k=1, n;\\ &\text{or} \quad i=r \quad \text{and} \quad 2 \leqslant k \leqslant n-1, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &0 \leqslant j \leqslant r-2 \quad \text{and} \quad 1 \leqslant \nu \leqslant n;\\ &\text{or} \quad j=r-1 \quad \text{and} \quad \nu=1, n;\\ &\text{or} \quad j=r \quad \text{and} \quad 2 \leqslant \nu \leqslant n-1. \end{aligned}$$

Case (b).

$$\begin{aligned} &0 \leqslant i \leqslant r-3 \quad \text{and} \quad 1 \leqslant k \leqslant n;\\ &\text{or} \quad i=r-2 \quad \text{and} \quad k=1, n;\\ &\text{or} \quad i=r \quad \text{and} \quad 2 \leqslant k \leqslant n-1, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &0 \leqslant j \leqslant r-3 \quad \text{and} \quad 1 \leqslant \nu \leqslant n;\\ &\text{or} \quad j=r-2 \quad \text{and} \quad \nu=1, n;\\ &\text{or} \quad j=r \quad \text{nad} \quad 2 \leqslant \nu \leqslant n-1. \end{aligned}$$

Case (c).

$$\begin{aligned} &i=0, ..., r-3, r-1 \quad \text{and} \quad 1 \leqslant k \leqslant n;\\ &\text{or} \quad i=r-2 \quad \text{nad} \quad k=1, n;\\ &\text{or} \quad i=r \quad \text{and} \quad 2 \leqslant k \leqslant n-1, \end{aligned} \tag{7}$$

and

$$j=0, ..., r-3, r-1 \quad \text{and} \quad 1 \leqslant v \leqslant n;$$
$$\text{or} \quad j=r-2 \quad \text{and} \quad v=1, n;$$
$$\text{or} \quad j=r \quad \text{and} \quad 2 \leqslant v \leqslant n-1.$$

Case (a) was investigated by Balázs–Turán [3], Freud [4], and Vértesi [14] for $r=2$, Saxena and Sharma [8, 9] for $r=3$, Saxena [10] for $r=4$; Case (b) by Akhlaghi, Chak, and Sharma [2], Szabados and Varmo [12] for $r=3$; and Case (c) by Akhlaghi, Chak, and Sharma [1] for $r=3$.

2. THE CASE OF $(0, ..., r-2, r)$ INTERPOLATION

THEOREM 1. *In case n is even, the modified $(0, ..., r-2, r)$ interpolation on the roots of (4) is regular, and the last fundamental function is given by*

$$L_{v,r}(x)=\frac{(1-x_v^2)\,\pi_n^{r-1}(x)}{r!\,\pi_n'(x_v)^r}\left[\int_{-1}^{x}\frac{P_{n-1}'(t)}{t-x_v}\,dt-\frac{P_{n-1}(x)+1}{2}\int_{-1}^{1}\frac{P_{n-1}'(t)}{t-x_v}\,dt\right]$$
$$(2 \leqslant v \leqslant n-1).$$

Proof. For regularity, it is sufficient to show that if $Q(x) \in \Pi_{nr-1}$ satisfies $Q^{(i)}(x_k)=0$ for indices i, k in case (a), then $Q(x) \equiv 0$. Evidently, $Q(x)=\pi_n^{r-1}(x)\,q(x)$ with $q \in \Pi_{n-1}$, and $q(\pm 1)=0$ by (5). A short calculation yields

$$Q^{(r)}(x_k)=r!\,\pi_n'(x_k)^{r-1}\,q'(x_k)=0 \qquad (2 \leqslant k \leqslant n-2), \tag{8}$$

whence $q'(x)=cP_{n-1}'(x)$, i.e., $q(x)=c[P_{n-1}(x)-P_{n-1}(-1)]$ and $q(1)=2c=0$, $c=0$ (n even!). This shows $Q(x) \equiv 0$.

To find $L_{v,r}(x)$, let $L_{v,r}(x)=\pi_n^{r-1}(x)\,q_v(x)$ with $q_v \in \Pi_{n-1}$, $q_v(\pm 1)=0$. Then instead of (8) we have

$$L_{v,r}^{(r)}(x_k)=r!\,\pi_n'(x_k)^{r-1}\,q_v'(x_k)=\delta_{vk} \qquad (2 \leqslant k \leqslant n-2),$$

whence

$$r!\,\pi_n'(x_v)^{r-1}\,q_v'(x_k)=\frac{1}{P_{n-1}''(x_k)}\left(\frac{P_{n-1}'(x)}{x-x_v}\right)_{x=x_k} \qquad (2 \leqslant k \leqslant n-1).$$

This implies

$$\frac{r!\,\pi_n'(x_v)^r}{1-x_v^2}\,q_v'(x)=\frac{P_{n-1}'(x)}{x-x_v}+cP_{n-1}'(x),$$

whence by integration and observing $q_\nu(-1)=0$ we get

$$\frac{r!\,\pi_n'(x_\nu)^r}{1-x_\nu^2}\,q_\nu(x)=\int_{-1}^{x}\frac{P_{n-1}'(t)}{t-x_\nu}\,dt+c[P_{n-1}(x)+1].$$

Hence by $q_\nu(1)=0$ we have $c=-\frac{1}{2}\int_{-1}^{1}(P_{n-1}'(t)/t-x_\nu)\,dt$ and the theorem is proved. ▮

We now turn to determining the other fundamental functions.

THEOREM 2. *Under the conditions of Theorem* 1 *we have*

$$L_{\nu,j}(x)=\frac{l_\nu^r(x)}{j!}\sum_{s=0}^{r-j-1}\frac{[l_\nu^{-r}(x)]_{x_\nu}^{(s)}}{s!}(x-x_\nu)^{s+j}+(1-\delta_{\nu1}-\delta_{\nu n})$$
$$\times\binom{r}{j}[l_\nu^{-r}(x)]_{x_\nu}^{(r-j)}L_{\nu,r}(x)-\frac{r!}{j!\,\pi_n'(x_\nu)^r}$$
$$\times\sum_{\substack{k=2\\k\neq\nu}}^{n-1}\pi_n'(x_k)^r L_{k,r}(x)\sum_{s=0}^{r-j-1}\frac{[l_\nu^{-r}(x)]_{x_\nu}^{(s)}}{s!\,(x_k-x_\nu)^{r-s-j}}$$

$(0\leqslant j\leqslant r-2$ *and* $1\leqslant \nu\leqslant n$, *or* $j=r-1$ *and* $\nu=1, n)$.

Proof. The $(0, 1, ..., r-1)$ Hermite interpolating polynomial can be expressed in the form (see (3))

$$H_{\nu,j,r}(x)=\frac{l_\nu^r(x)}{j!}\sum_{s=0}^{r-j-1}\frac{[l_\nu^{-r}(x)]_{x_\nu}^{(s)}}{s!}(x-x_\nu)^{s+j}. \tag{9}$$

In order to apply (9) we have to calculate $H_{\nu,j,r}^{(r)}(x_k)$ $(2\leqslant k\leqslant n-1)$. To simplify calculations we write

$$l_\nu^r(x)(x-x_\nu)^{s+j}=\left(\frac{x_n'(x_k)}{\pi_n'(x_\nu)}\right)^r\frac{l_k^r(x)}{(x-x_\nu)^{r-s-j}}(x-x_k)^r,$$

whence

$$[l_\nu^r(x)(x-x_\nu)^{s+j}]_{x_k}^{(r)}=\frac{r!}{(x_k-x_\nu)^{r-s-j}}\left(\frac{\pi_n'(x_k)}{\pi_n'(x_\nu)}\right)^r \qquad (k\neq\nu).$$

On the other hand,

$$[l_\nu^r(x)(x-x_\nu)^{s+j}]_{x_\nu}^{(r)}=\frac{r!}{(r-s-j)!}[l_\nu^r(x)]_{x_\nu}^{(r-s-j)}.$$

Substituting these values in (9) we obtain

$$H^{(r)}_{\nu, j, r}(x_k) = \frac{1}{j!} \sum_{s=0}^{r-j-1} \frac{[l_\nu^{-r}(x)]^{(s)}_{x_\nu}}{s!} \cdot \frac{r!}{(x_k - x_\nu)^{r-s-j}} \left(\frac{\pi'_n(x_k)}{\pi'_n(x_\nu)}\right)^r$$

$$= \frac{r!}{j!} \left(\frac{\pi'_n(x_k)}{\pi'_n(x_\nu)}\right)^r \sum_{s=0}^{r-j-1} \frac{[l_\nu^{-r}(x)]^{(s)}_{x_\nu}}{s!\,(x_k - x_\nu)^{r-s-j}} \qquad (k \neq \nu)$$

and by Leibniz's rule

$$H^{(r)}_{\nu, j, r}(x_\nu) = \frac{1}{j!} \sum_{s=0}^{r-j-1} \frac{[l_\nu^{-r}(x)]^{(s)}_{x_\nu}}{s!} \cdot \frac{r!}{(r-s-j)!} [l_\nu^r(x)]^{(r-s-j)}_{x_\nu}$$

$$= \binom{r}{j} \sum_{s=0}^{r-j-1} \binom{r-j}{s} [l_\nu^{-r}(x)]^{(s)}_{x_\nu} [l_\nu^r(x)]^{(r-j-s)}_{x_\nu}$$

$$= -\binom{r}{j} [l_\nu^{-r}(x)]^{(r-j)}_{x_\nu}.$$

This proves Theorem 2. ∎

3. THE CASE OF $(0, ..., r-3, r)$ INTERPOLATION

THEOREM 3. *For all n's, the modified* $(0, ..., r-3, r)$ *interpolation on the roots of* (4) *is regular, and the last fundamental function is given by*

$$L_{\nu, r}(x) = -\frac{6(1 - x_\nu^2)\, \pi_n(x)^{r-2}\, n(n-1)}{r!\, \pi'_n(x_\nu)^r} \times \sum_{k=2}^{n-1} \frac{(2k-1)\, \pi_k(x)\, P'_{k-1}(x_\nu)}{k(k-1)\, \lambda_k} \qquad (2 \leqslant \nu \leqslant n-1), \tag{10}$$

where

$$\lambda_{kn} = (r-2)\, n(n-1) + 3k(k-1) \qquad (2 \leqslant k \leqslant n-1).$$

Proof. As in the proof of Theorem 1, we assume that for a polynomial $Q(x) \in \Pi_{(r-1)n-1}$ we have $Q^{(i)}(x_k) = 0$, where the indices i, k satisfy conditions of Case (b). Let $Q(x) = \pi_n^{r-2}(x)\, q(x)$, where $q(x) \in \Pi_{n-1}$ and $q(\pm 1) = 0$. We get (since $l'_k(x_k) = 0$)

$$Q^{(r)}(x_k) = \frac{r!\, \pi'_n(x_k)^{r-2}}{2} [q''(x_k) + (r-2)\, l''_k(x_k)\, q(x_k)] = 0 \qquad (2 \leqslant k \leqslant n-1), \tag{11}$$

that is,

$$3(1-x_k^2)\, q''(x_k) - (r-2)\, n(n-1)\, q(x_k) = 0 \qquad (2 \leqslant k \leqslant n-1).$$

This together with $q(\pm 1) = 0$ implies

$$3(1-x^2)\, q''(x) - (r-2)\, n(n-1)\, q(x) = 0. \tag{12}$$

Now set $q(x) = \sum_{k=2}^{n-1} a_k \pi_k(x)$, then (12) takes the form

$$-\sum_{k=2}^{n-1} [(r-2)\, n(n-1) + 3k(k-1)]\, a_k \pi_k(x) = -\sum_{k=2}^{n-1} \lambda_{kn} a_k \pi_k(x) = 0.$$

Hence $a_k = 0$ $(2 \leqslant k \leqslant n-1)$, i.e., $q(x) \equiv 0$. This proves the regularity.

Now if we look for the last fundamental functions in the form $L_{v,r}(x) = \pi_n^{r-2}(x)\, q_v(x)$ with $q_v \in \Pi_{n-1}$, $q_v(\pm 1) = 0$, then instead of (11) we will have

$$\frac{r!\, \pi_n'(x_k)^{r-2}}{2} [q_v''(x_k) + (r-2)\, l_k''(x_k)\, q_v(x_k)] = \delta_{kv} \qquad (2 \leqslant k \leqslant n-1),$$

since in this case $2 \leqslant v \leqslant n-1$. Hence,

$$\begin{aligned} 3(1-x_k^2)\, q_v''(x_k) - (r-2)\, n(n-1)\, q_v(x_k) \\ = \frac{6(1-x_v^2)}{r!\, \pi_n'(x_v)^{r-2}}\, l_v(x_k) \qquad (2 \leqslant k \leqslant n-1) \end{aligned}$$

and by $q_v(\pm 1) = 0$ this can be extended to $k = 1$ and n. Thus,

$$3(1-x^2)\, q_v''(x) - (r-2)\, n(n-1)\, q_v(x) = \frac{6(1-x_v^2)}{r!\, \pi_n'(x_v)^{r-2}}\, l_v(x).$$

Again, we set $q_v(x) = \sum_{k=2}^{n-1} a_k \pi_k(x)$ and use the decomposition

$$l_v(x) = \frac{n(n-1)}{\pi_n'(x_v)^2} \sum_{k=2}^{n-1} \frac{2k-1}{k(k-1)}\, \pi_k(x)\, P_{k-1}'(x_v) \tag{13}$$

(see, e.g., Akhlaghi, Chak, and Sharma [1, (3.4a)]) to get

$$-\sum_{k=2}^{n-1} \lambda_{kn} a_k \pi_k(x) = \frac{6(1-x_v^2)\, n(n-1)}{r!\, \pi_n'(x_v)^r} \sum_{k=2}^{n-1} \frac{2k-1}{k(k-1)}\, \pi_k(x)\, P_{k-1}'(x_v).$$

Hence,

$$a_k = -\frac{6(1-x_\nu^2)\, n(n-1)(2k-1)\, P'_{k-1}(x_\nu)}{r!\, \lambda_{kn} \pi'_n(x_\nu)^r\, k(k-1)} \qquad (2 \leqslant k \leqslant n-1),$$

whence (10) follows. ▮

THEOREM 4. *Under the conditions of Theorem 3 we have*

$$\begin{aligned} L_{\nu,j}(x) = {} & \frac{l_\nu^{r-1}(x)}{j!} \sum_{s=0}^{r-j-2} \frac{[l_\nu^{1-r}(x)]_{x_\nu}^{(s)}}{s!} (x-x_\nu)^{s+j} + (1-\delta_{\nu 1}-\delta_{\nu n}) \\ & \times \binom{r}{j} [l_\nu^{1-r}(x)]_{x_\nu}^{(r-j)} L_{\nu,r}(x) - \frac{r!}{j!\, \pi'_n(x_\nu)^{r-1}} \sum_{\substack{k=2 \\ k \neq \nu}}^{n-1} \pi'_n(x_k)^r \\ & \times L_{k,r}(x) \sum_{s=0}^{r-j-2} \frac{(r-j-s-1)[l_\nu^{1-r}(x)]_{x_\nu}^{(r-1)}}{s!(x_k-x_\nu)^{r-j-s}} \\ & (0 \leqslant j \leqslant r-3 \text{ and } 1 \leqslant \nu \leqslant n; \text{ or } j = r-2 \text{ and } \nu = 1, n). \end{aligned}$$

In the proof we use

$$H_{\nu,j,r-1}(x) = \frac{l_\nu^{r-1}(x)}{j!} \sum_{s=0}^{r-j-2} \frac{[l_\nu^{1-r}(x)]_{x_\nu}^{(s)}}{s!} (x-x_\nu)^{s+j},$$

instead of (9). Since the calculations are analogous, we omit the details.

4. THE CASE OF $(0, ..., r-3, r-1, r)$ INTERPOLATION

THEOREM 5. *For all $n \geqslant 2$, the modified $(0, ..., r-3, r-1, r)$ interpolation on the roots of (4) is uniquely solvable, and the last fundamental function is given by*

$$L_{\nu,r}(x) = \frac{6n(n-1)(1-x_\nu^2)\, \pi_n^{r-2}(x)}{r!\, \pi'_n(x_\nu)^{r+1}} \sum_{k=2}^{n-1} \frac{(2k-1)\, s_{kn}(x)\, P'_{k-1}(x_\nu)}{k(k-1)\, \lambda_{kn}} \qquad (2 \leqslant \nu \leqslant n-1), \tag{14}$$

where

$$s_{kn}(x) = \pi_n(x)\, \pi'_k(x) - \pi'_n(x)\, \pi_k(x) \qquad (2 \leqslant k \leqslant n-1) \tag{15}$$

and

$$\lambda_{kn} = (r+1)\, n(n-1) - 3k(k-1) \qquad (2 \leqslant k \leqslant n-1).$$

Proof. Let $Q(x)=\pi_n^{r-2}(x)\, q(x)$, $q\in\Pi_{2n-1}$, satisfy the homogeneous interpolatory conditions. $Q^{(r-2)}(\pm 1)=0$ implies $q(\pm 1)=0$, while

$$\begin{aligned} Q^{(r-1)}(x_k) &= [\pi_n^{r-2}(x)]_{x_k}^{(r-1)}\, q(x_k)+(r-1)[\pi_n^{r-2}(x)]_{x_k}^{(r-2)}\, q'(x_k) \\ &= \pi_n'(x_k)^{r-2}\, \{[(x-x_k)^{r-2}\, l_k^{r-2}(x)]_{x_k}^{(r-1)}\, q(x_k) \\ &\quad +(r-1)[(x-x_k)^{r-2}\, l_k^{(r-2)}(x)]_{x_k}^{(r-2)}\, q'(k)\} \\ &= \pi_n'(x_k)^{r-2}\, (r-1)!\, q'(x_k)=0 \qquad (1\leqslant k\leqslant n) \end{aligned}$$

yields $q'(x_k)=0$ $(1\leqslant k\leqslant n)$ (since $l_k'(x_k)=0$). These conditions imply the existence of a polynomial $S(x)\in\Pi_{n-1}$ such that

$$q(x)=\pi_n(x)\, S'(x)-\pi_n'(x)\, S(x) \tag{16}$$

(see the proof of Theorem 1 in [1]). Among others, this implies $q\in\Pi_{2n-2}$ and $S(\pm 1)=0$.

Now using the conditions on the rth derivative and recalling that $q'(x_k)=0$, $k=1, ..., n$, we get

$$\begin{aligned} Q^{(r)}(x_k) &= [\pi_n^{r-2}(x)]_{x_k}^{(r)}\, q(x_k)+\binom{r}{2}\, [\pi_n^{r-2}(x)]_{x_k}^{(r-2)}\, q''(x_k) \\ &= \pi_n'(x_k)^{r-2}\left\{[(x-x_k)^{r-2}\, l_k^{r-2}(x)]_{x_k}^{(r)}\, q(x_k)+\frac{1}{2}\, r!\, q''(x_k)\right\} \\ &= \pi_n'(x_k)^{r-2}\left\{\binom{r}{2}\, (r-2)!\, (r-2)\, l_k''(x_k)\, q(x_k)+\frac{1}{2}\, r!\, q''(x_k)\right\} \\ &= \frac{r!}{2}\, \pi_n'(x_k)^{r-2}\left[-\frac{(r-2)\, n(n-1)}{3(1-x_k^2)}\, q(x_k)+q''(x_k)\right]=0 \\ &\qquad (2\leqslant k\leqslant n-1), \end{aligned}$$

since $l_k''(x_k)=-n(n-1)/3(1-x_k^2)$. That is,

$$3(1-x_k^2)\, q''(x_k)-(r-2)\, n(n-1)\, q(x_k)=0 \qquad (2\leqslant k\leqslant n-1). \tag{17}$$

Substituting here

$$q(x_k)=-\pi_n'(x_k)\, S(x_k), \qquad q''(x_k)=\pi_n'(x_k)\, S''(x_k)-\pi_n'''(x_k)\, S(x_k)$$

which we get from (16), we obtain

$$\begin{aligned} &3(1-x_k^2)\, \pi_n'(x_k)\, S''(x_k)-[3(1-x_k^2)\, \pi_n'''(x_k)-(r-2)\, n(n-1) \\ &\qquad \pi_n'(x_k)]\, q(x_k)=0 \qquad (2\leqslant k\leqslant n-1). \end{aligned}$$

Since $\pi_n'''(x_k) = -(n(n-1)/(1-x_k^2))\, \pi_n'(x_k)$, we get

$$3(1-x_k^2)\, S''(x_k) + (r+1)\, n(n-1)\, S(x_k) = 0 \qquad (2 \leqslant k \leqslant n-1). \tag{18}$$

Because of $S(\pm 1) = 0$, this extends to $k = 1$ and n. Since $S \in \Pi_{n-1}$, we conclude that

$$3(1-x^2)\, S''(x) + (r+1)\, n(n-1)\, S(x) = 0.$$

Using the representation $S(x) = \sum_{k=2}^{n-1} a_k \pi_k(x)$ we obtain

$$\sum_{j=2}^{n-1} [(r+1)\, n(n-1) - 3k(k-1)]\, a_k \pi_k(x) = \sum_{k=2}^{n-1} \lambda_k a_k \pi_k(x) = 0,$$

whence $a_k = 0$ $(2 \leqslant k \leqslant n-1$, $S(x) = q(x) = Q(x) \equiv 0$ and the regularity if proved.

In order to find the last fundamental functions we write $L_{\nu, r}(x) = \pi_n^{r-2}(x)\, q_\nu(x)$ and get, instead of (17),

$$3(1-x_k^2)\, q_\nu''(x_k) - (r-2)\, n(n-1)\, q_\nu(x_k)$$
$$= \frac{6(1-x_\nu^2)}{r!\, \pi_n'(x_\nu)^{r-2}}\, l_\nu(x_k) \qquad (2 \leqslant k \leqslant n-1).$$

Introducing $S_\nu(x)$ by

$$q_\nu(x) = \pi_n(x)\, S_\nu'(x) - \pi_n'(x) - \pi_n'(x)\, S_\nu(x), \tag{19}$$

instead of (16) we obtain

$$3(1-x_k^2)\, S_\nu''(x_k) + (r+1)\, n(n-1)\, S_\nu(x_k) = \frac{6(1-x_\nu^2)}{r!\, \pi_n'(x_\nu)^{r-1}}\, \delta_{\nu k} \qquad (1 \leqslant k \leqslant n)$$

(compare (18)). Hence,

$$3(1-x^2)\, S_\nu''(x) + (r+1)\, n(n-1)\, S_\nu(x) = \frac{6(1-x_\nu^2)}{r!\, \pi_n'(x_\nu)^{r-1}}\, l_\nu(x).$$

Using the representation $S_\nu(x) = \sum_{k=2}^{n-1} a_k \pi_k(x)$ and the formula (13) we obtain

$$a_k = \frac{6(1-x_\nu^2)\, n(n-1)(2k-1)\, P_{k-1}'(x_\nu)}{r!\, \pi_n'(x_\nu)^{r+1}\, k(k-1)\, \lambda_{kn}} \qquad (2 \leqslant k \leqslant n-1),$$

$$S_\nu(x) = \frac{6(1-x_\nu^2)\, n(n-1)}{r!\, \pi_n'(x_\nu)^{r+1}} \sum_{k=2}^{n-1} \frac{(2k-1)\, \pi_k(x)\, P_{k-1}'(x_\nu)}{k(k-1)\, \lambda_{kn}},$$

$$q_\nu(x) = \frac{6(1-x_\nu^2)\, n(n-1)}{r!\, \pi_n'(x_\nu)^{r+1}} \sum_{k=2}^{n-1} \frac{(2k-1)\, s_{kn}(x)\, P_{k-1}'(x_\nu)}{k(k-1)\, \lambda_{kn}}.$$

This together with (19) proves the theorem. ▮

THEOREM 6. *Under the conditions of Theorem 5 we have*

$$L_{\nu,j}(x) = H_{\nu,j,r}(x) + (1-\delta_{1\nu}-\delta_{n\nu}) \binom{r}{j} [l_\nu^{-r}(x)]_{x_\nu}^{(r-j)} L_{\nu,r}(x)$$

$$- \frac{r!}{j!\, \pi_n'(x_\nu)^r} \sum_{\substack{k=2 \\ k \neq \nu}}^{n-1} \pi_n'(x_k)^r L_{k,r}(x) \sum_{s=0}^{r-j-1} \frac{[l_\nu^{-r}(x)]_{x_\nu}^{(s)}}{s!(x_k - x_\nu)^{r-s-j}} \qquad (19a)$$

$(0 \leqslant j \leqslant r-3$ *and* $1 \leqslant \nu \leqslant n$; *or* $j = r-2$ *and* $\nu = 1, n)$, *where*

$$H_{\nu,j,r}(x) = \frac{l_\nu^r(x)}{j!} \sum_{s=0}^{r-j-1} \frac{[l_\nu^{-r}(x)]_{x_\nu}^{(s)}}{s!} (x-x_\nu)^{s+j}.$$

Formally, this is the same formula as given by Theorem 2, but of course the $L_{\nu,r}(x)$ here are given by Theorem 5. We omit the obvious proof.

5. ESTIMATING THE HERMITE INTERPOLATING POLYNOMIALS

Once we have the fundamental functions, the next natural question is to estimate them. For this purpose we need estimates for the Hermite interpolating polynomials appearing in Theorems 2, 4, and 6. Let us introduce the notations

$$\Delta_n(x) = \frac{\sin t}{n} + \frac{1}{n^2}, \qquad x = \cos t, \quad x_k = \cos t_k \quad (1 \leqslant k \leqslant n). \qquad (20)$$

LEMMA 1. *For the roots of* (4) *we have*

$$[l_\nu^m(x)]_{x_\nu}^{(j)} = \begin{cases} O(\Delta_n(x_\nu)^{-j}) & \text{if } j \text{ is even} \\ O(n^{-2}\Delta_n(x_\nu)^{-j-1}) & \text{if } j \text{ is odd,} \end{cases} \qquad (20a)$$

where m is an arbitrary fixed integer and the "O" signs refer to $n \to \infty$.

Proof. First we estimate $l_\nu^{(j)}(x_\nu)$. Evidently,

$$l_\nu^{(j)}(x_\nu)=\frac{\pi_n^{(j+1)}(x_\nu)}{(j+1)\,\pi_n'(x_\nu)} \qquad (j=0, 1, ...). \tag{21}$$

In order to calculate higher derivatives of $\pi_n(x)$ at x_ν we start from the differential equation

$$(1-x^2)\,\pi_n''(x)=-n(n-1)\,\pi_n(x)$$

and differentiate $(j-1)$-times:

$$(1-x^2)\,\pi_n^{(j+1)}(x)-2(j-1)\,x\pi_n^{(j)}(x)-(j-1)(j-2)\,\pi_n^{(j-1)}(x)$$
$$=-n(n-1)\,\pi_n^{(j-1)}(x).$$

Hence

$$\pi_n^{(j+1)}(x_\nu)=\frac{2(j-1)\,x_\nu}{1-x_\nu^2}\,\pi_n^{(j)}(x_\nu)-\frac{n(n-1)-(j-1)(j-2)}{1-x_\nu^2}\,\pi_n^{(j-1)}(x_\nu). \tag{22}$$

Since $\pi_n''(x_\nu)=0$ $(2\leqslant\nu\leqslant n-1)$, we obtain

$$\pi_n^{(2j)}(x_\nu)=O\left(\frac{n^{2(j-1)}\pi_n'(x_\nu)}{\sin^{2j}t_\nu}\right), \qquad \pi_n^{(2j+1)}(x_\nu)=O\left(\frac{n^{2j}\pi_n'(x_\nu)}{\sin^{2j}t_\nu}\right)$$
$$(2\leqslant\nu\leqslant n-1) \tag{23}$$

which can be proved by induction and easily seen from (22) and $1/\sin t_\nu=O(n)$. Also notice that by $\|\pi_n'\|=n(n-1)$ and Markov's inequality $\pi_n^{(j)}(\pm 1)=O(n^{2j})$ $(j=1, 2, ...)$. This together with (23), using the notation (20) yields

$$\pi_n^{(2j)}(x_\nu)=O(n^{-2}\Delta_n(x_\nu)^{-2j}\,\pi_n'(x_\nu)), \qquad \pi_n^{(2j+1)}(x_\nu)=O(\Delta_n(x_\nu)^{-2j}\,\pi_n'(x_\nu))$$
$$(1\leqslant\nu\leqslant n).$$

Hence and by (21)

$$l_\nu^{(2j)}(x_\nu)=O(\Delta_n(x_\nu)^{-2j}), \qquad l_\nu^{(2j-1)}(x_\nu)=O(n^{-2}\Delta_n(x_\nu)^{-2j})$$
$$(1\leqslant\nu\leqslant n; j=1, 2, ...). \tag{24}$$

LEMMA 2. *For the Hermite polynomial* (9) *we have*

$$|H_{v,j,r}(x)| = \begin{cases} O\left(\dfrac{|\pi_n(x)|^r}{n^r \Delta_n(x_v)^{r/2-j-1}\,|x-x_v|}\right) \\ \quad \text{if} \quad r-j \text{ is odd}, \\ O\left(\dfrac{|\pi_n(x)|^r}{n^r \Delta_n(x_v)^{r/2j-1}\,|x-x_v|}\left(\dfrac{\Delta_n(x_v)}{|x-x_v|}+\dfrac{1}{n^2\Delta_n(x_v)}\right)\right) \\ \quad \text{if} \quad r-j \text{ is even} \end{cases} \tag{25}$$

for $|t-t_v|>c/n$, *and*

$$|H_{v,j,r}(x)| = O(|x-x_v|^j) \qquad \text{for} \quad |t-t_v| \leqslant c/n. \tag{26}$$

Here $c>0$ *is a properly chosen absolute constant,* $v=1, ..., n$ *and* $j=0, 1, ..., r-1$.

Proof. If the constant c in the lemma is large enough, then $\Delta_n(x_v) \leqslant |x-x_v|$ whenever $|t-t_v|>c/n$. Thus applying Lemma 1, in case of $r-j$ odd, the last term in the sum (9) will dominate, and it can be estimated as $O(|x-x_v|^{r-1}/\Delta_n(x_v)^{r-j-1})$. Hence,

$$\begin{aligned} |H_{v,j,r}(x)| &= O\left(|l_v^r(x)|\,\frac{|x-x_v|^{r-1}}{\Delta_n(x_v)^{r-j-1}}\right) \\ &= O\left(\frac{|\pi_n(x)|^r}{|\pi_n'(x_v)|^r\,|x-x_v|\,\Delta_n(x_v)^{r-j-1}}\right). \end{aligned}$$

Here, using the well-known estimate

$$|\pi_n'(x_v)| \geqslant c_1 n \Delta_n(x_v)^{-1/2} \qquad (v=1, ..., n) \tag{27}$$

(see Lemmas 12.3 and 12.5 in [6]), we obtain the first relation in (25).

Now if $r-j$ is even then we use the second estimate in Lemma 1 (with $r-j-1$ for j), to majorize the last term in the sum (9). Of course, in this case the rest of the sum in (9) is majorized by the last term there, where we can apply the first estimate in Lemma 1 (with $r-j-2$ for j). These result in the second estimate in (25).

Finally, if $|t-t_v| \leqslant c/n$ then we use $|l_v(x)| \leqslant 1$ and $|x-x_v| = O(\Delta_n(x_v))$ to get (26). (In this case $r-j$ even does not yield a better estimate.)

6. ESTIMATING THE LAST FUNDAMENTAL FUNCTIONS

It is obvious from the given representation of the fundamental polynomials that besides the estimates obtained for the Hermite polynomials in the previous section, we have to estimate the last fundamental functions. For the special values of r, this has been done before. For example, Balázs and Turán [3] proved

$$\sum_{\nu=2}^{n-1} |L_{\nu,2}(x)| = O(n^{-1}) \qquad \text{(Case (a), } r=2\text{);}$$

Saxenna and Sharma [9] proved

$$\sum_{\nu=2}^{n-1} |L_{\nu,3}(x)| = O(n^{-2}) \qquad \text{(Case (a), } r=3\text{);}$$

Szabados and Varma [12] showed

$$\sum_{\nu=2}^{n-1} \frac{|L_{\nu,3}(x)|}{\sin^3 t_\nu} = O\left(\frac{\log n}{n^3}\right) \qquad \text{(Case (b), } r=3\text{).}$$

No estimate has been given in Case (c). Therefore, as a model example, we shall estimate the last fundamental functions of modified (0, 2, 3) interpolation (Case (c), $r=3$).

THEOREM 7. *For the modified* (0, 2, 3) *interpolation we have*

$$|L_{\nu,3}(x)| = \begin{cases} O\left(\dfrac{\sin^2 t_\nu}{n^5 \sin(|t-t_\nu|/2)} + \dfrac{\sin^3 t_\nu}{n^5 \sin^2((t-t_\nu)/2)}\right) & \text{if } |t-t_\nu| > \dfrac{c}{n} \\ O\left(\dfrac{\sin^3 t_\nu}{n^3}\right) & \text{if } |t-t_\nu| \leqslant \dfrac{c}{n}, \end{cases}$$

where $c>0$ is an absolute constant.

COROLLARY. $\sum_{\nu=2}^{n-1} (|L_{\nu,3}(x)|/\Delta_n(x_\nu)^3) = O(1)$.

Proof.

$$\begin{aligned} \sum_{\nu=2}^{n-1} \frac{|L_{\nu,3}(x)|}{(\Delta_n(x_\nu))^3} &\leqslant c \sum_{|t-t_\nu| \geqslant c/n} \frac{n^3}{\sin^3 t_\nu} \left(\frac{\sin^2 t_\nu}{n^5 \sin |t-t_\nu|/2} + \frac{\sin^3 t_\nu}{n^5 \sin^2((t-t_\nu)/2)}\right) \\ &\quad + c\,\frac{n^3}{\sin^3 t_\nu} \cdot \frac{\sin^3 t_\nu}{n^3} \\ &\leqslant \frac{c}{n^2} \sum_{|t-t_\nu| \geqslant c/n} \frac{1}{\sin t_\nu \sin(|t-t_\nu|/2)} + \frac{c}{n^2} \\ &\quad \times \sum_{|t-t_\nu| \geqslant c/n} \frac{1}{\sin^2((t-t_\nu)/2)} + c \end{aligned}$$

$$\leqslant \frac{c}{n^2} \sum \frac{1}{(v/n)(|i-v|/n)} + \frac{c}{n^2} \sum \frac{1}{(i-v)^2/n^2} + 1$$
$$= O(1). \quad \blacksquare$$

Proof of Theorem 7. We shall first prove the case when $|t - t_v| > c/n$. From Theorem 5 we get

$$L_{v,3}(x) = \frac{n(n-1)(1-x_v^2)\,\pi_n(x)}{\pi_n'(x_v)^4} \sum_{k=2}^{n-1} \frac{(2k-1)\,s_{kn}(x)\,P_{k-1}'(x_v)}{k(k-1)\,\lambda_{kn}} \qquad (2 \leqslant v \leqslant n-1), \tag{29}$$

where

$$\lambda_{kn} = 4n(n-1) - 3k(k-1) \qquad (k = 2, 3, ...).$$

Using Abel transform with the factors $1/\lambda_{kn}$ we get

$$\sum_{k=2}^{n-1} \frac{(2k-1)\,s_{kn}(x)\,P_{k-1}'(x_v)}{k(k-1)\,\lambda_{kn}} = \frac{1}{\lambda_{n-1,n}} \sum_{k=2}^{n-1} \frac{(2k-1)\,s_{kn}(x)\,P_{k-1}'(x_v)}{k(k-1)} + \sum_{k=3}^{n-1} \left(\frac{1}{\lambda_{k-1,n}} - \frac{1}{\lambda_{k,n}}\right) \sum_{j=2}^{k-1} \frac{(2j-1)\,s_{jn}(x)\,P_{j-1}'(x_v)}{j(j-1)} := A + B. \tag{30}$$

Here

$$A = \frac{\pi_n'(x_v)^3\, l_v^2(x)}{\lambda_{n-1,n} n(n-1)}. \tag{31}$$

This can be easily proved from (13) and its differential form

$$\sum_{k=2}^{n-1} \frac{(2k-1)\,\pi_k'(x)\,P_{k-1}'(x_v)}{k(k-1)} = \frac{\pi_n'(x_v)^2\, l_v'(x)}{n(n-1)}$$

on recalling that $s_{kn}(x) = \pi_n(x)\,\pi_k'(x) - \pi_n'(x)\,\pi_k(x)$ and using the identity

$$\pi_n(x)\, l_v'(x) - \pi_n'(x)\, l_v(x) = -\pi_n'(x_v)\, l_v^2(x).$$

(See [12, formula (19)].) Since by $|\pi_n(x)| = O(n^{1/2} \sin^{1/2} t)$ and (27) and

$$\max(\sin t, \sin t_v) \leqslant 2 \sin \frac{t + t_v}{2},$$

we get

$$|l_\nu(x)| = O\left(\frac{n^{1/2}\sin^{1/2} t \sin^{1/2} t_\nu}{n^{3/2}\,|x - x_\nu|}\right) = O\left(\frac{\sin^{1/4} t_\nu}{n \sin(|t - t_\nu|/2) \sin^{1/4} t}\right),$$

(31) yields

$$A = O\left(\frac{n^{9/2}}{n^4 \sin t_\nu n^2((t - t_\nu)/2) \sin^{1/2} t}\right)$$
$$= O\left(\frac{1}{n^{3/2} \sin t_\nu \sin^2((t - t_\nu/2) \sin^{1/2} t}\right). \qquad (32)$$

As for B, we get by (15)

$$B = -6 \sum_{k=3}^{n-1} \frac{k-1}{\lambda_{k-1,n}\lambda_{kn}} \sum_{j=1}^{k-1} \frac{(2j-1)\, s_{jn}(x)\, P'_{j-1}(x_\nu)}{j(j-1)}$$
$$= 6\pi_n(x) \sum_{k=2}^{n-1} \frac{k-1}{\lambda_{k-1,n}\lambda_{kn}} \sum_{j=1}^{k-1} (2j-1)\, P_{j-1}(x)\, P'_{j-1}(x_\nu)$$
$$-6\pi'_n(x)(1-x^2) \sum_{k=2}^{n-1} \frac{k-1}{\lambda_{k-1,n}\lambda_{kn}} \sum_{j=1}^{k-1} \frac{2j-1}{j(j-1)} P'_{j-1}(x)\, P'_{j-1}(x_\nu)$$
$$:= 6\pi_n(x)\, B_1 - 6\pi'_n(x)(1-x^2)\, B_2.$$

Here we settle the estimate of B_2 first: Since $|P_n(x)| = O(1/\sqrt{n \sin t})$ (Sansone [7]), and $\pi'|_n(x) = n(n-1)\, P_{n-1}(x)$, we have $\pi'_n(x)(1-x^2) = O(n^{3/2} \sin^{3/2} t)$. In order to estimate B_2, we use the method of Lemma 4 of [12] and apply Abel-transformation again to the double summation in B_2. In Lemma 3 of [12], it has been shown that

$$\sum_{s=2}^{m} (2s+1)\, P'_s(x)\, P_s(x_\nu) = O\left(\frac{m}{\sin^{3/2} t \sin^{1/2} t_\nu \sin(|t - t_\nu|/2)}\right). \qquad (32a)$$

Using this estimate and the method of Lemma 4 of [12], we get

$$\pi'_n(x)(1-x^2)\, B_2 = O\left(n^{3/2} \sin^{3/2} t \frac{1}{n^3 \sin^{3/2} t \sin^{3/2} t_\nu}\right.$$
$$\left.\times\left(\frac{1}{|x - x_\nu|} + \frac{\sin^{1/2} t_\nu}{\sin^{1/2} t \sin^2((t - t_\nu)/2)}\right)\right)$$

$$= O\left(\frac{1}{n^{3/2}\sin^2 t_\nu \sin^{1/2} t \sin(|t - t_\nu/2)} + \frac{1}{n^{3/2}\sin^{1/2} t \sin t_\nu \sin^2((t-t_\nu)/2)}\right)$$

which is the same as the estimate for A.

For B_1 we use the Christoffel–Darboux formula for the inner sum in the form

$$(k-1)\frac{P_{k-1}(x)\,P_{k-2}(y) - P_{k-1}(y)\,P_{k-2}(x)}{x-y} = \sum_{j=2}^{k-1}(2j-1)\,P_{j-1}(x)\,P_{j-1}(y),$$

differentiate both sides with respect to y and set x_ν for y. Then we have

$$B_1 = \sum_{k=2}^{n-1}\frac{(k-1)^2}{\lambda_{k-1}\lambda_k}\left[\frac{P_{k-1}(x)\,P'_{k-2}(x_\nu) - P_{k-2}(x)\,P'_{k-1}(x_\nu)}{x - x_\nu} - \frac{P_{k-1}(x)\,P_{k-2}(x_\nu) - P_{k-2}(x)\,P_{k-1}(x_\nu)}{(x-x_\nu)^2}\right]$$

$$:= \frac{C_1}{x-x_\nu} + \frac{C_2}{x-x_\nu}. \tag{33}$$

Here, in estimating C_1 we use the relation

$$\frac{(k-2)^2}{\lambda_{k-2}\lambda_{k-1}} - \frac{(k-1)^2}{\lambda_{k-1}\lambda_k} = \frac{8k}{n^4} + O\left(\frac{1}{n^4}\right) \tag{33a}$$

and that

$$P'_{k-2}(x_\nu) = P'_k(x_\nu) - (2k-1)\,P_{k-1}(x_\nu)$$

(cf. Szegő [13, (4.7.29)]) to get

$$C_1 = \sum_{k=2}^{n-1}\frac{(k-1)^2}{\lambda_{k-1}\lambda_k}\left[P_{k-1}(x)\,P'_k(x_\nu) - P_{k-2}(x)\,P'_{k-1}(x_\nu)\right] - \sum_{k=1}^{n-1}\frac{(k-1)^2}{\lambda_{k-1}\lambda_k}(2k-1)\,P_{k-1}(x)\,P_{k-1}(x_\nu)$$

$$:= D_1 - D_2.$$

Applying again the Abel transformation to D_1, we get

$$D_1 = \frac{(n-1)^2}{\lambda_{n-1}\lambda_n} \sum_{k=3}^{n-1} [P_{k-1}(x_\nu)\, P'_k(x_\nu) - P_{k-2}(x)\, P'_{k-1}(x_\nu)]$$
$$+ \sum_{k=3}^{n-1} \left(\frac{(k-2)^2}{\lambda_{k-2}\lambda_{k-1}} - \frac{(k-1)^2}{\lambda_{k-1}\lambda_k} \right) \sum_{j=2}^{k-1} [P_{j-1}(x)\, P'_j(x_\nu)$$
$$- P_{j-2}(x)\, P'_{j-1}(x_\nu)].$$

Using the fact that $P'_{n-1}(x_\nu) = 0$ and that $P_k(x) = O(k^{-1/2} \sin^{-1/2} t)$, $P'_k(x) = O(k^{1/2} \sin^{-3/2} t)$, we see that the first sum to the right in D_1 is

$$\frac{(n-1)^2}{\lambda_{n-1}\lambda_n} \sum_{k=3}^{n-1} [P_{k-1}(x)\, P'_k(x_\nu) - P_{k-2}(x)\, P'_{k-1}(x_\nu)]$$
$$= \frac{(n-1)^2}{\lambda_{n-1}\lambda_n} P_1(x)\, P'_2(x_\nu) = O(n^{-2}).$$

The second sum on the right in D_1 on unsing (33a) becomes

$$O(n^{-4}) \sum_{n=3}^{n-1} k\, |P_{k-1}(x)\, P'_k(x_\nu)| = O\left(\frac{1}{n^2 \sin^{1/2} t \sin^{3/2} t_\nu} \right).$$

In order to estimate D_2 we again apply Abel transformation to D_2 and thereby obtain

$$D_2 = \frac{(n-2)^2}{\lambda_{n-2}\lambda_{n-1}} \sum_{k=1}^{n-1} (2k-1)\, P_{k-1}(x)\, P_{k-1}(x_\nu) + \sum_{k=2}^{n-1} \left\{ \frac{(k-2)^2}{\lambda_{k-2}\lambda_{k-1}} - \frac{(k-1)^2}{\lambda_{k-1}\lambda_k} \right\}$$
$$\times \left[(k-1)\, \frac{P_{k-2}(x)\, P_{k-1}(x_\nu) - P_{k-1}(x)\, P_{k-2}(x_\nu)}{x - x_\nu} \right],$$

where we have used the Darboux–Christoffel formula in the second summation. This yields in view of (33a),

$$D_2 = O(n^{-2}) \sum_{k=2}^{n-1} (2k-1)\, P_{k-1}(x)\, P_{k-1}(x_\nu) + O(n^{-4})$$
$$\times \sum_{k=2}^{n-1} k^2\, \frac{P_{k-2}(x)\, P_{k-1}(x_\nu) - P_{k-1}(x)\, P_{k-2}(x_\nu)}{x - x_\nu}.$$

From Sansone [7, pp. 178–179], we know that

$$P'_{n-2}(x) - P'_n(x) = -(2n-1)\, P_n(x) = \frac{2n-1}{n(n-1)}\, \pi'_n(x)$$

which gives

$$P_{n-2}(x) - P_n(x) = \frac{2n-1}{n(n-1)}\, \pi_n(x). \qquad (*)$$

Also,

$$P_{n-2}(x) = xP_{n-1}(x) + \frac{1}{n-1}\, \pi_n(x), \qquad P_{n-2}(x_v) = x_v P_{n-1}(x_v).$$

Then we can get from the Christoffel–Darboux formula and the above relations that

$$\sum_{k=1}^{n-1} (2k-1)\, P_{k-1}(x)\, P_{k-1}(x_v)$$

$$= (n-1)\, \frac{P_{n-2}(x)\, P_{n-1}(x_v) - P_{n-1}(x)\, P_{n-2}(x_v)}{x - x_v}$$

$$= -P_{n-1}(x_v) \left[\frac{\pi_n(x)}{x - x_v} + (n-1)\, P_{n-1}(x) \right]$$

$$= O\left(\frac{1}{\sin^{1/2} t \sin^{1/2} t_v \sin |t - t_v|/2} \right),$$

where we have used the estimates for $P_{n-1}(x)$, $P_{n-1}(x_v)$, and $\pi_n(x)$. It remains to estimate S_2, where

$$S_2 = O(n^{-4}) \sum_{k=2}^{n-1} k^2\, \frac{P_{k-2}(x)\, P_{k-1}(x_v) - P_{k-1}(x)\, P_{k-2}(x_v)}{x - x_v}$$

$$= O\left(\frac{n^{-4}}{x - x_v} \right) \left[\sum_{k=2}^{n-1} k^2 (P_k(x)\, P_{k-1}(x_v) - P_{k-1}(x)\, P_{k-2}(x_v)) \right.$$

$$\left. + \sum_{k=2}^{n-1} \frac{k(2k-1)}{k-1}\, \pi_k(x)\, P_{k-1}(x_v) \right]$$

which is easily obtained on using ($*$). Again using the Abel-transform on the sum $\sum_{k=2}^{n-1} k^2(P_k(x)\, P_{k-1}(x_v) - P_{k-1}(x)\, P_{k-2}(x_v))$ and estimates on the second summations, we get a little effort that

$$S_2 = O\left(\frac{1}{n^3 \sin^{1/2} t \sin^{1/2} t_v \sin((t-t_v)/2) \sin((t+t_v)/2)}\right)$$

$$= O\left(\frac{1}{n^3 \sin t \sin t_v \sin((t-t_v)/2)}\right).$$

Thus we have

$$D_2 = O\left(\frac{1}{n^2 \sin^{1/2} t \sin^{1/2} t_v \sin(|t-t_v|/2)} + \frac{1}{n^3 \sin t \sin t_v \sin(|t-t_v|/2)}\right).$$

Since C_2 in (33) is

$$\sum_{k=3}^{n-1} \frac{(k-1)^2}{\lambda_{k-1}\lambda_k} \cdot \frac{P_{k-2}(x)\, P_{k-1}(x_v) - P_{k-1}(x)\, P_{k-2}(x_v)}{x - x_v}$$

which is the sum in S_2, it follows that

$$C_2 = O\left(\frac{1}{n^2 \sin^{1/2} t \sin^{1/2} t_v \sin(|t-t_v|/2)}\right).$$

Collecting these estimates, we get

$$B = O\left(|\pi_n(x)|\, \frac{D_1 + D_2 + C_2}{|x - x_v|} + |\pi_n'(x)|(1-x^2)\, B_2\right)$$

$$= O\left(\frac{1}{n^{3/2} \sin^{1/2} t \sin^2 t_v \sin(|t-t_v|/2)} + \frac{1}{n^{3/2} \sin^{1/2} t \sin t_v \sin^2((t-t_v)/2)}\right). \tag{34}$$

Thus we obtain from (29), (30), (32), and (34)

$$L_{v,3}(x) = O\left(\frac{\sin^4 t_v \sin^{1/2} t}{n^{7/2}}\right)\left(\frac{1}{n^{3/2} \sin^{1/2} t \sin^2 t_v \sin(|t-t_v|/2)}\right.$$

$$\left. + \frac{1}{n^{3/2} \sin^{1/2} t \sin t_v \sin^2((t-t_v/2)}\right)$$

$$= O\left(\frac{\sin^2 t_v}{n^5 \sin(|t-t_v|/2)} + \frac{\sin^3 t_v}{n^5 \sin^2((t-t_v)/2)}\right)$$

which proves the first estimate in (28).

The second estimate comes from (29), by estimating the terms of the sum individually, and using the estimates

$$|s_{kn}(x) = O(n^{3/2}k^{1/2}), \qquad \sin t = O(\sin t_\nu) \qquad (2 \leqslant \nu \leqslant n-1).$$

Thus when $|t - t_\nu| \leqslant c/n$, we have

$$\frac{n(n-1)(1-x_\nu^2)\,\pi_n(x)}{(\pi_n'(x_\nu))^4} = O\left(\frac{\sin^{9/2} t_\nu}{n^{7/2}}\right)$$

and

$$\left|\sum_{k=2}^{n-1} \frac{(2k-1)\, s_{kn}(x)\, P_{k-1}'(x_\nu)}{k(k-1)\,\lambda_{kn}}\right| \leqslant O\left(\frac{n^{1/2}}{\sin^{3/2} t_\nu}\right).$$

Combining them we get the required estimate when $|t - t_\nu| \leqslant c/n$. This completes the proof. ▮

THEOREM 8. *For the modified* (0, 2, 3) *interpolation we have*

$$\sum_{\nu=1}^{n} \frac{|L_{\nu,j}(x)|}{\Delta_n(x_\nu)^j} = O(\log n) \qquad (j = 0 \text{ or } 2).$$

Proof. From Theorems 6–7 and Lemmas 1–2 we get for $|t - t_\nu| \geqslant c/n$ and $2 \leqslant \nu \leqslant n-1$,

$$\begin{aligned}
L_{\nu,j}(x) = O\Bigg[&\frac{\sin^{3/2} t}{n^{3/2}\Delta_n(x_\nu)^{(1/2)-j}\,|x - x_\nu|} + n^{-2}\Delta_n(x_\nu)^{j-4} \\
&\times\left(\frac{\sin^2 t_\nu}{n^5 \sin(|t - t_\nu|/2)} + \frac{\sin^3 t_\nu}{n^5 \sin^2(t - t_\nu)/2}\right) \\
&+ \Delta_n(x_\nu)^{3/2}\left\{\sum_{k\neq\nu} \Delta_n(x_k)^{-3/2}\left(\frac{\sin^2 t_k}{n^5 \sin(|t - t_k|/2)}\right.\right. \\
&\left.\left.+ \frac{\sin^3 t_k}{n^5 \sin^2(t - t_k)/2}\right)\cdot\frac{\Delta_n(x_\nu)^{j-2}}{|x_k - x_\nu|}\right\}\Bigg] \\
= O\Bigg[&\frac{\sin^{1/2} t \sin^{j-(1/2)} t_\nu}{n^{j+1} \sin(|t - t_\nu|/2)} + \frac{\sin^{j-2} t_\nu}{n^{j+3} \sin(|t - t_\nu|/2)} + \frac{\sin^{j-1} t_\nu}{n^{j+3} \sin^2((t - t_\nu)/2)} \\
&+ \frac{\sin^{j-(1/2)} t_\nu}{n^{j+3}} \sum_{k\neq\nu}\left(\frac{\sin^{1/2} t_k}{\sin(|t - t_k|/2)} + \frac{\sin^{3/2} t_k}{\sin^2((t - t_k)/2)}\right)\frac{1}{|x_k - x_\nu|}\Bigg]
\end{aligned}$$

$$= O\left[\frac{\sin^{j-(1/2)} t_\nu}{n^{j+1} \sin^{1/2}(|t-t_\nu|/2)} + \frac{\sin^j t_\nu}{n^{j+1} \sin(|t-t_\nu|/2)}\right.$$

$$+ \frac{\sin^{j-1} t_\nu}{n^{j+3}} \sum_{\substack{k \neq \nu \\ |t-t_k| \geqslant c/n}} 1 \Big/ \left(\sin \frac{|t-t_k|}{2} \sin \frac{|t_k - t_\nu|}{2}\right) + \frac{\sin^{j-(1/2)} t_\nu}{n^{j+3}}$$

$$\times \sum_{\substack{k \neq \nu \\ |t-t_k| \geqslant c/n}} \left(1 \Big/ \sin^2 \frac{|t-t_k|}{2} \sin^{1/2} \frac{|t_\nu - t_k|}{2}\right)$$

$$\left. + \left(\sin^{1/2} t_\nu \Big/ \sin^2 \frac{t-t_k}{2} \sin \frac{|t_k - t_\nu|}{2}\right)\right], \tag{35}$$

that is,

$$\frac{|L_{\nu,j}(x)|}{\sin^j t_\nu} = O\left[\frac{1}{n^{j+1}}\left(1 \Big/ \sin^{1/2} t_\nu \sin^{1/2} \frac{|t-t_\nu|}{2} + 1 \Big/ \sin \frac{|t-t_\nu|}{2} + \frac{1}{\sin t_\nu}\right)\right.$$

$$+ \frac{1}{n^{j+3}} \sum_{\substack{k \neq \nu \\ |t-t_k| \geqslant c/n}} \left(1 \Big/ \sin^2 \frac{t-t_k}{2} \sin^{1/2} t_\nu \sin^{1/2} \frac{|t-t_k|}{2}\right.$$

$$\left.\left. + 1 \Big/ \sin^2 \frac{t-t_k}{2} \sin \frac{|t_k - t_\nu|}{2}\right)\right]$$

$$= O\left[\frac{1}{n^{j+1}}\left(\frac{1}{\sin t_\nu} + 1 \Big/ \sin \frac{|t-t_\nu|}{2}\right) + \frac{1}{n^{j+3}}\right.$$

$$\left. \times \sum_{\substack{k \neq \nu \\ |t-t_k| \geqslant c/n}} \left(1 \Big/ \sin^2 \frac{t-t_k}{2} \sin \frac{|t_k - t_\nu|}{2}\right)\right].$$

Summing up for ν fields

$$\sum_{\nu=2}^{n-1} \frac{|L_{\nu,j}(x)|}{\sin^j t_\nu} = O\left(\frac{\log n}{n^j}\right) + O\left(\frac{\log n}{n^{j+2}}\right) \sum_{\substack{k=2 \\ |t-t_k| \geqslant c/n}}^{n-1} \left(1 \Big/ \sin^2 \frac{t-t_k}{2}\right)$$

$$= O\left(\frac{\log n}{n^j}\right) \qquad \left(2 \leqslant \nu \leqslant n-1,\ |t-t_\nu| \geqslant \frac{c}{n}\right). \tag{36}$$

If $|t-t_\nu| \leqslant c/n$, then $\sin t = O(\sin t_\nu)$ and using (26) and (28), we get

$$L_{\nu,j}(x) = O\left\{|x-x_\nu|^j + (\Delta_n(x_\nu))^{j-3} \frac{\sin^3 t_\nu}{n^3} + \sin^{3/2} t_\nu \sum_{k \neq \nu} \sin^{-3/2} t_k\right.$$

$$\left. \times \left(\frac{\sin^2 t_k}{n^5 \sin(|t-t_k|/2)} + \frac{\sin^3 t_k}{n^5 \sin^2((t-t_k/2))}\right) \frac{(\Delta(x_\nu))^{j-2}}{|x_k - x_\nu|}\right\}$$

$$= O\left\{(\Delta_n(x_v))^j + \frac{\sin^{j-(1/2)} t_v}{n^{j+3}} \sum_{k \neq v} \left(\frac{\sin^{1/2} t_k}{\sin(|t-t_k|/2)} + \frac{\sin^{3/2} t_k}{\sin^2((t-t_k)/2)}\right)\right.$$
$$\left.\times\left(1\Big/\sin\frac{|t_k - t_v|}{2}\sin\frac{t_v+t_k}{2}\right)\right\}.$$

Note that $|x - x_v| = 2\sin(|t-t_v|/2)\sin((t+t_v)/2) \leqslant (c/n)\sin t_v \leqslant \Delta_n(x_v)$ when $|t - t_v| \leqslant c/n$. Then we have

$$\frac{L_{v,j}(x)}{(\Delta_n(x_v))^j} = O(1) + O\left(\frac{1}{n^3}\right) \sum_{k \neq v}\left(1\Big/\sin^{1/2} t_v \sin^{1/2} t_k \sin^2 \frac{|t-t_k|}{2}\right.$$
$$\left. + \frac{\sin^{1/2} t_k - \sin^{1/2} t_v}{\sin^{1/2} t_v \sin^3((t-t_k)/2)} + 1\Big/\sin^3\frac{t_k - t_v}{n}\right)$$
$$= O(1) + O\left(\frac{1}{n^3}\right)\left\{\sum_{k\neq v}\left(1\Big/\frac{v^{1/2}}{n^{1/2}}\cdot\frac{k^{1/2}}{n^{1/2}}\frac{|v-k|^2}{n^2}\right)\right.$$
$$\left. + \sum_{k\neq v}\left(1\Big/\frac{v^{1/2}}{n^{1/2}}\frac{|k-v|^2}{n^{5/2}}\right) + \sum\left(1\Big/\frac{|k-v|^3}{n^3}\right)\right\}$$
$$= O(1).$$

In the above considerations, we had to exclude $v=1$ and n. In this respect we note that

$$|L_{v,j}(x)| = O(n^{-2j}) \qquad (v = 1 \text{ or } n, j = 0 \text{ or } 2). \tag{37}$$

Namely, in this case (35) takes the form (say, $v=1$)

$$L_{1,j}(x) = O\left[\frac{n^2\sin^2 t}{n^{2j+2}(1-x)} + \frac{1}{n^6}\sum_{k=2}^{n-1}\frac{n^{9/2}}{\sin^{3/2} t_k}\right.$$
$$\left.\times\left(\frac{\sin^2 t_k}{n^5\sin(|t-t_k|/2)} + \frac{\sin^3 t_k}{n^5\sin^2((t-t_k)/2)}\right)\frac{n^{4-2j}}{(1-x_k)}\right]$$
$$= O\left[\frac{1}{n^{2j}} + \frac{1}{n^{2j+(5/2)}}\sum_{k=2}^{n-1}\left(1\Big/\sin^{3/2} t_k \sin\frac{|t-t_k|}{2}\right.\right.$$
$$\left.\left. + 1\Big/\sin^{1/2} t_k \sin^2\frac{t-t_k}{2}\right)\right]$$
$$= O\left(\frac{1}{n^{2j}}\right).$$

This, together with (36), proves the theorem. ∎

Finally, we have

THEOREM 9. *For the modified* (0, 2, 3) *interpolation we have*

$$\|L_{1,1}(x)\| = \|L_{n,1}(x)\| = O(n^{-2}). \tag{37a}$$

Proof. Let $v = 1$, and use again Theorem 6:

$$\begin{aligned}
L_{1,1}(x) &= H_{1,1,3}(x) + O\left[\frac{1}{n^6}\sum_{k=2}^{n-1}\frac{n^{9/2}}{\sin^{3/2} t_k}\left(\frac{\sin^2 t_k}{n^5 \sin(|t-t_k|/2)}\right.\right.\\
&\qquad \left.\left.+\frac{\sin^3 t_k}{n^5 \sin^2((t-t_k)/2)}\right)\left(\frac{1}{(1-x_k)^2}+\frac{n^2}{1-x_k}\right)\right]\\
&= O\left[\frac{n^4 \sin^4 t}{n^6(1-xc)^2}+\frac{n^2 \sin^2 t}{n^4(1-x)}+\frac{1}{n^{9/2}}\right.\\
&\qquad \left.\times \sum_{k=2}^{n-1}\left(1\Big/\sin^{3/2} t_k \sin\frac{|t-t_k|}{2}+1\Big/\sin^{1/2} t_k \sin^2\frac{t-t_k}{2}\right)\right]\\
&= O\left(\frac{1}{n^2}\right).
\end{aligned}$$

This proves (37a). ∎

7. CONVERGENCE THEOREM FOR MODIFIED (0, 2, 3) INTERPOLATION

Based on the estimates of the previous section, we are now able to prove a convergence theorem. Define the linear operator

$$R_n(f, x) = \sum_{v=1}^{n} f(x_k)\, L_{v,0}(x).$$

THEOREM 16. *For any* $f(x) \in C[-1, 1]$ *we have*

$$\|f(x) - R_n(f, x)\| = O\left(\omega\left(f, \frac{1}{n}\right)\log n\right),$$

where w is the modulus of continuity.

Proof. It is well known that there exist polynomials $p_n \in \Pi_n$ such that

$$\|f - p_n\| = O(\omega(f, 1/n))$$

and

$$|p_n^{(i)}(x)| = O(\Delta_n(x)^{-i})\, \omega(f, 1/n) \qquad (i = 1, 2, ...) \tag{38}$$

(see Dzyadik [4a, Theorem 3, p. 362, Lemma 1, p. 267]). Now evidently,

$$\begin{aligned}\|f - R_n(f)\| &\leqslant \|f - p_n\| + \|p_n - R_n(p_n)\| + \|R_n(p_n - f)\| \\ &= O(\omega(f, 1/n)) \left\| \sum_{\nu=1}^{n} |L_{\nu,0}(x)| \right\| + \|p_n - R_n(p_n)\| \\ &= O(\omega(f, 1/n) \log n) + \|p_n - R_n(p_n)\|\end{aligned}$$

by Theorem 8 ($j = 0$). Since the modified (0, 2, 3) interpolation is uniquely determined,

$$\begin{aligned}p_n(x) - R_n(p_n, x) &= p_n'(1)\, L_{1,1}(x) + p_n'(-1)\, L_{n,1}(x) \\ &\quad + \sum_{\nu=1}^{n} p_n''(x_\nu)\, L_{\nu,2}(x) + \sum_{\nu=2}^{n-1} p_n'''(x_\nu)\, L_{\nu,3}(x).\end{aligned}$$

Here, using (38), Theorems 7–9 (and the corollary to Theorem 7) we get

$$\begin{aligned}\|p_n - RE_n(p_n)\| &= O(n^{-2})\, n^2 \omega\left(f, \frac{1}{n}\right) + O\left(\omega\left(f, \frac{1}{n}\right)\right) \sum_{\nu=1}^{n} \frac{|L_{\nu,2}(x)|}{\Delta_n(x_\nu)^2} \\ &\quad + O\left(\omega\left(f, \frac{1}{n}\right)\right) \sum_{\nu=2}^{n-1} \frac{|L_{\nu,3}(x)|}{\Delta_n(x_\nu)^3} = O\left(\omega\left(f, \frac{1}{n}\right) \log n\right).\end{aligned}$$

Thus Theorem 9 is proved. ∎

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