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2A-orbifold construction and the baby-monster vertex operator superalgebra

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To the memory of my dear Taro

Abstract

In this article we give a new proof of the determination of the full automorphism group of the baby-monster vertex operator superalgebra based on a theory of simple current extensions. As a corollary, we also prove that the $\mathbb{Z}_2$-orbifold construction with respect to a 2A-involution of the Monster applied to the moonshine vertex operator algebra $V^\natural$ yields $V^\natural$ itself again.

1. Introduction

The famous moonshine vertex operator algebra $V^\natural$ constructed by Frenkel–Lepowsky–Muerman [FLM] is the first example of the $\mathbb{Z}_2$-orbifold construction of a holomorphic vertex operator algebra (VOA). Let us explain a $\mathbb{Z}_2$-orbifold construction briefly. Let V be a holomorphic vertex operator algebra and σ an involutive automorphism on V. Then the fixed point subalgebra $V^{\langle\sigma\rangle}$ is a simple vertex operator algebra. It is shown in [DLM] that there is a unique irreducible σ-twisted V-module M and we have a decomposition

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$M = M^0 \oplus M^1$ into a direct sum of irreducible $V^{\langle\sigma\rangle}$-modules such that M^0 has an integral top weight. A $\mathbb{Z}_2$-orbifold construction with respect to $\sigma \in \mathrm{Aut}(V)$ refers to a construction of a $\mathbb{Z}_2$-graded extension $W = V^{\langle\sigma\rangle} \oplus M^0$ of the fixed point subalgebra $V^{\langle\sigma\rangle}$ and it is expected to be a holomorphic vertex operator algebra.

In FLM's construction, we take V to be the lattice vertex operator algebra V_Λ associated to the Leech lattice Λ and the involution σ is a natural lifting $\theta \in \mathrm{Aut}(V_\Lambda)$ of the (-1)-isometry on Λ. Denote by $V_\Lambda = V_\Lambda^+ \oplus V_\Lambda^-$ the eigenspace decomposition such that θ acts on $V_\Lambda^\pm$ as ± 1, respectively. Let V_Λ^T be the unique irreducible θ-twisted V_Λ-module. Then there is a decomposition $V_\Lambda^T = (V_\Lambda^T)^+ \oplus (V_\Lambda^T)^-$ such that the top weight of $(V_\Lambda^T)^+$ is integral. Then the moonshine vertex operator algebra is defined by $V^\natural := V_\Lambda^+ \oplus (V_\Lambda^T)^+$ and it is proved in [FLM] that $V^\natural$ forms a $\mathbb{Z}_2$-graded extension of V_Λ^+. It is also proved in [FLM] that the full automorphism group of the moonshine vertex operator algebra is the Monster sporadic finite simple group $\mathbb{M}$ by using Griess' result [G].

In the Monster, there are two conjugacy classes of involutions, the 2A-conjugacy class and the 2B-conjugacy class (cf. [ATLAS]). One can explicitly see the action of a 2B-involution on $V^\natural$ by FLM's construction. But it is difficult to realize the action of a 2A-involution on $V^\natural$ before Miyamoto. In [M1], Miyamoto opened a way to study the action of 2A-involutions of the Monster on the moonshine VOA by using a sub VOA isomorphic to the unitary Virasoro VOA $L(1/2, 0)$. Let us recall the definition of Miyamoto involutions. Let V be a simple VOA and $e \in V_2$ be a vector such that the sub VOA $\mathrm{Vir}(e)$ generated by e is isomorphic to the Virasoro VOA $L(1/2, 0)$. Such a vector e is called a conformal vector with central charge $1/2$. Since V as a $\mathrm{Vir}(e)$-module is completely reducible, we have a decomposition

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$, $h = 0, 1/2, 1/16$, denotes a sum of all irreducible $\mathrm{Vir}(e)$-submodules isomorphic to $L(1/2, h)$. Then one can define a linear isomorphism τ_e on V by

$$\tau_e := 1 \quad \text{on } V_e(0) \oplus V_e(1/2), \qquad -1 \quad \text{on } V_e(1/16).$$

It is proved in [M1] that τ_e defines an involution of a VOA V if $V_e(1/16) \neq 0$. This involution is often called the Miyamoto involution of τ-type. On the fixed point subalgebra $V^{\langle\tau_e\rangle}$, one can define another automorphism by

$$\sigma_e := 1 \quad \text{on } V_e(0), \qquad -1 \quad \text{on } V_e(1/2).$$

This involution is called the Miyamoto involution of σ-type. It is shown in [C] and [M1] that in the moonshine VOA every Miyamoto involution τ_e defines a 2A-involution of the Monster and the correspondence between conformal vectors and 2A-involutions is one-to-one. Therefore, in the study of 2A-involutions, it is very important to study conformal vectors with central charge $1/2$. Along this idea, C.H. Lam, H. Yamada and the author obtained an interesting achievement on 2A-involutions of the Monster in [LYY].

The main purpose of this paper is to study the $\mathbb{Z}_2$-orbifold construction of $V^\natural$ with respect to the Miyamoto involution and to prove that the 2A-orbifold construction applied to $V^\natural$ yields $V^\natural$ itself again. Since a 2A-involution of the Monster is uniquely determined by a conformal vector e of $V^\natural$ with central charge $1/2$, we have to study the commutant subalgebra of $\mathrm{Vir}(e)$ together with $\mathrm{Vir}(e)$ in order to describe the 2A-orbifold construction. For a simple VOA V and a conformal vector e of V with central charge $1/2$, set the space of highest weight vectors by $T_e(h) := \{v \in V \mid L^e(0)v = hv\}$ for $h = 0, 1/2, 1/16$, where we expand $Y(e,z) = \sum_{n\in\mathbb{Z}} L^e(n)z^{-n-2}$. Then we have decompositions $V_e(h) = L(1/2,h) \otimes T_e(h)$ and the commutant subalgebra $T_e(0)$ acts on $T_e(h)$ for $h = 0, 1/2, 1/16$. Like $L(1/2,0)$ has a $\mathbb{Z}_2$-graded extension $L(1/2,0) \oplus L(1/2,1/2)$, we can introduce a vertex operator superalgebra (SVOA) structure on $T_e(0) \oplus T_e(1/2)$ and its $\mathbb{Z}_2$-twisted module structure on $T_e(1/16)$. It is easy to see that the one point stabilizer $C_{\mathrm{Aut}(V)}(e) = \{\rho \in \mathrm{Aut}(V) \mid \rho e = e\}$ naturally acts on the space of highest weight vectors $T_e(h)$. If we take $V = V^\natural$, then $C_{\mathrm{Aut}(V^\natural)}(e)$ is isomorphic to the 2-fold central extension $\langle \tau_e \rangle \cdot \mathbb{B}$ of the baby-monster sporadic finite simple group $\mathbb{B}$. Therefore, the SVOA $T_e^\natural(0) \oplus T_e^\natural(1/2)$, where we have set $V_e^\natural(h) = L(1/2,h) \otimes T_e^\natural(h)$ for $h = 0, 1/2, 1/16$, affords a natural action of $\mathbb{B}$. Motivated by this fact, Höhn first studied this SVOA in [Hö1] and he called it the *baby-monster SVOA*. Following him, we write $VB^0 := T_e^\natural(0)$, $VB^1 := T_e^\natural(1/2)$ and $VB := T_e^\natural(0) \oplus T_e^\natural(1/2)$. It is proved in [Hö2] that the full automorphism group of the even part VB^0 of VB is exactly isomorphic to the baby-monster $\mathbb{B}$. In this paper, we give a quite different proof of $\mathrm{Aut}(VB^0) \simeq \mathbb{B}$ based on a theory of simple current extensions.

In my recent work [Y1,Y2], a theory of simple current extensions of vertex operator algebras was developed and many useful results were obtained. Using the theory, we determine the automorphism group of the commutant subalgebra $T_e(0)$ as follows:

Theorem 1. *Let V be a holomorphic VOA and $e \in V$ a conformal vector with central charge $1/2$. Suppose the following*:

(a) $V_e(h) \neq 0$ *for* $h = 0, 1/2, 1/16$,
(b) $V_e(0)$ *and* $T_e(0)$ *are rational C_2-cofinite VOAs of CFT-type,*
(c) $V_e(1/16)$ *is a simple current $V^{\langle \tau_e \rangle}$-module,*
(d) $T_e(1/2)$ *is a simple current $T_e(0)$-module,*
(e) $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$ *is a simple group or an odd group.*

Then

(1) $\mathrm{Aut}(T_e(0)) = C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$.
(2) *The irreducible $T_e(0)$-modules are given by $T_e(0)$, $T_e(1/2)$ and $T_e(1/16)$.*
(3) *The τ_e-orbifold construction applied to V yields V itself again.*

The assumptions (c) and (d) in the theorem above seem to be rather restrictive. However, we prove that all the assumptions above hold if V is the moonshine VOA. Applying Theorem 1 to $V^\natural$, we obtain the following main theorem of this paper.

Theorem 2. *Let $VB = VB^0 \oplus VB^1$ be the commutant superalgebra obtained from $V^\natural$.*

(1) $\mathrm{Aut}(VB^0) = \mathbb{B}$ *and* $\mathrm{Aut}(VB) = 2 \times \mathbb{B}$.

(2) *There are exactly three inequivalent irreducible VB^0-modules, VB^0, VB^1 and $VB_T := T_e^\natural(1/16)$.*
(3) *The fusion rules for VB^0-modules are as follows*:

$$VB^1 \times VB^1 = VB^0, \qquad VB^1 \times VB_T = VB_T, \qquad VB_T \times VB_T = VB^0 + VB^1.$$

This theorem has the following corollaries.

Corollary 1. *The irreducible* 2*A-twisted* $V^\natural$*-module has a shape*

$$L(1/2, 1/2) \otimes VB^0 \oplus L(1/2, 0) \otimes VB^1 \oplus L(1/2, 1/16) \otimes VB_T.$$

Corollary 2. *For any conformal vector $e \in V^\natural$ with central charge* $1/2$*, there is no* $\rho \in \mathrm{Aut}(V^\natural)$ *such that* $\rho(V_e^\natural(h)) = V_e^\natural(h)$ *for* $h = 0, 1/2, 1/16$ *and* $\rho|_{(V^\natural)^{\langle\tau_e\rangle}} = \sigma_e$.

Corollary 3. *The* 2*A-orbifold construction applied to the moonshine VOA* $V^\natural$ *yields* $V^\natural$ *itself again.*

At the end of this paper, we give characters of VB^0-modules and their modular transformation laws. Surprisingly, we find that the fusion algebra and the modular transformation laws for the baby-monster VOA is canonically isomorphic to those of the Ising model $L(1/2, 0)$.

Notation. For a VOA V and a subgroup G of $\mathrm{Aut}(V)$, we denote by V^G the G-fixed subalgebra of V. For a V-module M and an automorphism $\tau \in \mathrm{Aut}(V)$, we denote the τ-conjugate module of M by M^τ. We denote the (restricted) dual module of M by M^*, and M is called *self-dual* if $M^* \simeq M$. For V-modules M^1 and M^2, we denote their fusion product by $M^1 \boxtimes_V M^2$. For a linear binary code D of length n and its element $\alpha = (\alpha_1, \ldots, \alpha_n) \in D$, we define $\mathrm{Supp}(\alpha) := \{i \mid \alpha_i \neq 0\}$.

2. Commutant superalgebra and its automorphisms

We denote by $L(c, h)$ the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. It is shown in [FZ] that $L(c, 0)$ has a structure of a simple VOA.

2.1. Ising model

We realize an SVOA $L(1/2, 0) \oplus L(1/2, 1/2)$ by using one free fermionic field. Let $\mathfrak{A}_\psi$ be a $\mathbb{C}$-algebra generated by $\{\psi_{n+1/2} \mid n \in \mathbb{Z}\}$ with the relation $[\psi_r, \psi_s]_+ := \psi_r\psi_s + \psi_s\psi_r = \delta_{r+s,0}$, $r, s \in \mathbb{Z} + 1/2$. Let $\mathfrak{A}_\psi^+$ to be the subalgebra of $\mathfrak{A}_\psi$ generated by $\{\psi_r \mid r > 0\}$ and let $\mathbb{C}|0\rangle$ be a trivial $\mathfrak{A}_\psi^+$-module. Consider the induced module

$$\mathfrak{M} := \mathrm{Ind}_{\mathfrak{A}_\psi^+}^{\mathfrak{A}_\psi} \mathbb{C}|0\rangle = \mathfrak{A}_\psi \otimes_{\mathfrak{A}_\psi^+} \mathbb{C}|0\rangle.$$

It is well known (cf. [KR]) that $\mathfrak{M}$ affords an action of the Virasoro algebra with central charge $1/2$ and $\mathfrak{M} \simeq L(1/2,0) \oplus L(1/2,1/2)$ as a Virasoro-module. Consider the generating series $\psi(z) := \sum_{n\in\mathbb{Z}} \psi_{n+1/2} z^{-n-1}$. It is also well known (cf. [K]) that the space $\mathfrak{M}$, with the standard $\mathbb{Z}_2$-grading, has a unique structure of a simple vertex operator superalgebra with the vacuum $\mathbb{1} = |0\rangle$ such that $Y_{\mathfrak{M}}(\psi_{-1/2}|0\rangle, z) = \psi(z)$.

Similarly, we can realize $L(1/2,1/16)$ as follows. Let $\mathfrak{A}_\phi$ be a $\mathbb{C}$-algebra generated by $\{\phi_m \mid m \in \mathbb{Z}\}$ with the relation $[\phi_m,\phi_n]_+ = \delta_{m+n,0}$, $m,n\in\mathbb{Z}$. Let $\mathfrak{A}_\phi^+$ be a subalgebra of $\mathfrak{A}_\phi$ generated by $\{\phi_m \mid m>0\}$ and let $\mathbb{C}|\frac{1}{16}\rangle$ be a trivial $\mathfrak{A}_\phi^+$-module. Consider the induced module

$$\mathfrak{N} := \mathrm{Ind}_{\mathfrak{A}_\phi^+}^{\mathfrak{A}_\phi} \mathbb{C}\big|\tfrac{1}{16}\big\rangle = \mathfrak{A}_\phi \otimes_{\mathfrak{A}_\phi^+} \mathbb{C}\big|\tfrac{1}{16}\big\rangle.$$

It is well known (cf. [KR]) that $\mathfrak{N}$ affords an action of the Virasoro algebra with central charge $1/2$. Set $v_{1/16}^{\pm} := (\sqrt{2}\phi_0 \pm 1)|\frac{1}{16}\rangle$. Then $v_{1/16}^{\pm}$ are highest weight vectors for the Virasoro algebra and we have a decomposition $\mathfrak{N} = \mathfrak{N}^+ \oplus \mathfrak{N}^-$, where $\mathfrak{N}^{\pm}$ are $\mathfrak{A}_\phi$-submodules generated by $v_{1/16}^{\pm}$, respectively, and $\mathfrak{N}^{\pm} \simeq L(1/2,1/16)$ as Virasoro-modules. The generating series $\phi(z) := \sum_{n\in\mathbb{Z}} \phi_n z^{-n-1/2}$ uniquely defines a $\mathbb{Z}_2$-twisted $\mathfrak{M}$-module structure on $\mathfrak{N}$ such that the vertex operator of $\psi_{-1/2}|0\rangle$ is given as $Y_{\mathfrak{N}}(\psi_{-1/2}|0\rangle, z) = \phi(z)$. We can also verify that $\mathfrak{N}^{\pm}$ are inequivalent irreducible $\mathbb{Z}_2$-twisted $\mathfrak{M}$-submodules (cf. [LLY]). This explicit construction will be used in the proof of Theorem 2.2.

2.2. *Miyamoto involution*

Let $(V, Y_V(\cdot,z), \mathbb{1}, \omega)$ be a VOA. A vector $e\in V$ is called a *conformal vector* if coefficients of its vertex operator $Y_V(e,z) = \sum_{n\in\mathbb{Z}} e_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L^e(n) z^{-n-2}$ generate a representation of the Virasoro algebra on V:

$$\big[L^e(m), L^e(n)\big] = (m-n)L^e(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c_e.$$

The scalar c_e is called the *central charge* of e. We denote by $\mathrm{Vir}(e)$ the sub VOA generated by e. If $\mathrm{Vir}(e)$ is a rational VOA, then e is called a *rational conformal vector*. A decomposition $\omega = e + (\omega - e)$ is called *orthogonal* if both e and $\omega - e$ are conformal vectors and their vertex operators are component-wisely mutually commutative.

Now assume that $e\in V$ is a rational conformal vector with central charge $1/2$. Then $\mathrm{Vir}(e)$ is isomorphic to $L(1/2,0)$ and has three irreducible representations $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$ (cf. [DMZ]). As $\mathrm{Vir}(e)$ acts on V semisimply, we can decompose V into a direct sum of irreducible $\mathrm{Vir}(e)$-modules as follows:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$, $h\in\{0,1/2,1/16\}$, denotes the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L(1/2,h)$. By the fusion rules for $L(1/2,0)$-modules (cf. [DMZ]), we have the following theorem.

Theorem 2.1 [M1].

(1) *The linear map* $\tau_e := 1$ *on* $V_e(0) \oplus V_e(1/2)$, -1 *on* $V_e(1/16)$ *defines an involutive automorphism on a VOA* V.
(2) *On the sub VOA* $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$, *the linear map* $\sigma_e := 1$ *on* $V_e(0)$, -1 *on* $V_e(1/2)$ *defines an involutive automorphism.*

The involutions $\tau_e \in \mathrm{Aut}(V)$ and $\sigma_e \in \mathrm{Aut}(V^{\langle \tau_e \rangle})$ are called *Miyamoto involutions*.

2.3. Commutant superalgebra

Let V be a simple VOA of CFT-type and $e \in V$ a rational conformal vector with central charge $1/2$. Set $T_e(h) := \{v \in V \mid L^e(0)v = h \cdot v\}$ for $h = 0, 1/2, 1/16$. $T_e(h)$ describes the space of highest weight vectors for $\mathrm{Vir}(e)$ and it is canonically isomorphic to $\mathrm{Hom}_{\mathrm{Vir}(e)}(L(1/2, h), V)$ for $h = 0, 1/2, 1/16$. Therefore, $V_e(h) \simeq L(1/2, h) \otimes T_e(h)$ and we have a decomposition as follows:

$$V = L(1/2, 0) \otimes T_e(0) \oplus L(1/2, 1/2) \otimes T_e(1/2) \oplus L(1/2, 1/16) \otimes T_e(1/16).$$

One can verify that a decomposition $\omega = e + (\omega - e)$ is orthogonal by using [FZ, Theorem 5.1]. Recall the commutant subalgebra $\mathrm{Com}_V(\mathrm{Vir}(e)) := \mathrm{Ker}_V\, L^e(-1)$ defined in [FZ]. It is easy to see that $T_e(0) = \mathrm{Ker}_V\, L^e(-1)$. So $(T_e(0), \omega - e)$ forms a sub VOA of V whose action on V is commutative with that of $\mathrm{Vir}(e)$ on V. In particular, $T_e(h)$, $h = 0, 1/2, 1/16$, are $T_e(0)$-modules. By the quantum Galois theory [DM], $T_e(0)$ is a simple subalgebra and $T_e(1/2)$ is an irreducible $T_e(0)$-module if $V_e(1/2) \neq 0$.

The commutant subalgebra $T_e(0)$ affords an extension to a superalgebra by its module $T_e(1/2)$ if $V_e(1/2) \neq 0$.

Theorem 2.2 [Hö1,Y2].

(1) *Suppose that* $V_e(1/2) \neq 0$. *There exists a simple SVOA structure on* $T_e(0) \oplus T_e(1/2)$ *such that the even part of a tensor product of SVOAs* $\{L(1/2, 0) \oplus L(1/2, 1/2)\} \otimes \{T_e(0) \oplus T_e(1/2)\}$ *is isomorphic to* $V_e(0) \oplus V_e(1/2)$ *as a VOA.*
(2) *Suppose that* $V_e(1/2) \neq 0$ *and* $V_e(1/16) \neq 0$. *Then* $T_e(1/16)$ *carries a structure of an irreducible* $\mathbb{Z}_2$*-twisted* $T_e(0) \oplus T_e(1/2)$*-module. Moreover,* $V_e(1/16)$ *is isomorphic to a tensor product of an irreducible* $\mathbb{Z}_2$*-twisted* $L(1/2, 0) \oplus L(1/2, 1/2)$*-module* $L(1/2, 1/16)$ *and an irreducible* $\mathbb{Z}_2$*-twisted* $T_e(0) \oplus T_e(1/2)$*-module* $T_e(1/16)$.

Proof. (1) First, we introduce a vertex operator map on $T_e(0) \oplus T_e(1/2)$. Let $a \in T_e(0)$ and $x \in L(1/2, 0)$. By [ADL, Theorem 2.10], there are $T_e(0)$-intertwining operators $I^i(\cdot, z)$ of type $T_e(0) \times T_e(i/2) \to T_e(i/2)$, $i = 0, 1$, such that $Y_V(x \otimes a, z)|_{V_e(i/2)} = Y_{\mathfrak{M}}(x, z) \otimes I^i(a, z)$, where $Y_{\mathfrak{M}}(\cdot, z)$ is the vertex operator map on the SVOA $L(1/2, 0) \oplus L(1/2, 1/2)$ constructed in Section 2.1. Similarly, for $u \in T_e(1/2)$ and $y \in L(1/2, 1/2)$, there are $T_e(0)$-intertwining operators $J^0(\cdot, z)$ and $J^1(\cdot, z)$ of types $T_e(1/2) \times T_e(0) \to T_e(1/2)$ and $T_e(1/2) \times T_e(1/2) \to T_e(0)$, respectively, such that $Y_V(y \otimes u, z)|_{V_e(0)} = Y_{\mathfrak{M}}(y, z) \otimes J^0(u, z)$ and $Y_V(y \otimes u, z)|_{V_e(1/2)} = Y_{\mathfrak{M}}(y, z) \otimes J^1(u, z)$ again by [ADL, Theorem 2.10].

We define the vertex operator map $\widehat{Y}(\cdot,z)$ on $T_e(0)\oplus T_e(1/2)$ as follows: for $a,b\in T_e(0)$ and $u,v\in T_e(1/2)$,

$$\widehat{Y}(a,z)b := I^0(a,z)b, \qquad \widehat{Y}(a,z)u := I^1(a,z)u,$$
$$\widehat{Y}(u,z)a := J^0(u,z)a, \qquad \widehat{Y}(u,z)v := J^1(u,z)v.$$

We claim that the quadruple $(T_e(0)\oplus T_e(1/2),\widehat{Y}(\cdot,z),\mathbb{1}_{T_e(0)},\omega-e)$ forms an SVOA, where $\mathbb{1}_V=|0\rangle\otimes\mathbb{1}_{T_e(0)}$. It is clear that $\widehat{Y}(\mathbb{1}_{T_e(0)},z)=\mathrm{id}_{T_e(0)\oplus T_e(1/2)}$ as the substructure $(T_e(0),I^0(\cdot,z),\mathbb{1}_{T_e(0)},\omega-e)$ is exactly $\mathrm{Com}_V(\mathrm{Vir}(e))$. The $L(-1)$-derivation property for $\widehat{Y}(\cdot,z)$ is also clear as $\widehat{Y}(\cdot,z)$ is made of $T_e(0)$-intertwining operators. By considering $Y_V(\psi_{-1/2}|0\rangle\otimes u,z)(|0\rangle\otimes a)$, we obtain a skew-symmetric property $J^0(u,z)a=e^{z(L(-1)-L^e(-1))}I^1(a,-z)u$ as both $Y_V(\cdot,z)$ and $Y_{\mathfrak{M}}(\cdot,z)$ satisfy the skew-symmetry. Therefore, for any $w\in T_e(0)\oplus T_e(1/2)$, the following creation property holds:

$$\widehat{Y}(w,z)\mathbb{1}_{T_e(0)}\in w+T_e(0)\oplus T_e(1/2)[\![z]\!]z.$$

Hence, in order to prove that the quadruple is an SVOA, it suffices to show that $\widehat{Y}(\cdot,z)$ satisfies the locality (cf. [Li1]):

$$\begin{aligned}&(z_1-z_2)^{N(w^1,w^2)}\widehat{Y}\big(w^1,z_1\big)\widehat{Y}\big(w^2,z_2\big)\\&\quad=(-1)^{\varepsilon(w^1,w^2)}(-z_2+z_1)^{N(w^1,w^2)}\widehat{Y}\big(w^2,z_2\big)\widehat{Y}\big(w^1,z_1\big),\end{aligned}\tag{2.1}$$

where w^1, w^2 are $\mathbb{Z}_2$-homogeneous elements in $T_e(0)\oplus T_e(1/2)$, ε is the standard parity function and $N(w^1,w^2)$ is a sufficiently large integer. Since $\widehat{Y}(\cdot,z)$ is made of $T_e(0)$-intertwining operators, we only need to show the locality (2.1) in the case of $w^1,w^2\in T_e(1/2)$. Let $u,v\in T_e(1/2)$ be arbitrary and N a positive integer such that

$$\begin{aligned}&(z_1-z_2)^N\big[Y_V(\psi_{-1/2}|0\rangle\otimes u,z_1),Y_V(\psi_{-1/2}|0\rangle\otimes v,z_2)\big]=0\\&\quad\text{on } V_e(0)\oplus V_e(1/2).\end{aligned}\tag{2.2}$$

The equality (2.1) is equivalent to the following two equalities:

$$(z_1-z_2)^N J^1(u,z_1)J^0(v,z_2)a=-(z_1-z_2)^N J^1(v,z_2)J^0(u,z_1)a,\tag{2.3}$$
$$(z_1-z_2)^N J^0(u,z_1)J^1(v,z_2)w=-(z_1-z_2)^N J^0(v,z_2)J^1(u,z_1)w,\tag{2.4}$$

where $a\in T_e(0)$ and $w\in T_e(1/2)$ are arbitrary. For simplicity, we set

$$A_0=(z_1-z_2)^N J^1(u,z_1)J^0(v,z_2)a,\qquad B_0=(z_1-z_2)^N J^1(v,z_2)J^0(u,z_1)a,$$
$$A_1=(z_1-z_2)^N J^0(u,z_1)J^1(v,z_2)w,\qquad B_1=(z_1-z_2)^N J^0(v,z_2)J^1(u,z_1)w.$$

We should prove both $A_0=-B_0$ and $A_1=-B_1$. By (2.2), we have

$$(z_1-z_2)^N\big[Y_V(\psi_{-1/2}|0\rangle\otimes u,z_1),Y_V(\psi_{-1/2}|0\rangle\otimes v,z_2)\big]\cdot(|0\rangle\otimes a)=0.$$

In terms of $\psi(z)$, the equality above becomes

$$\psi(z_1)\psi(z_2)|0\rangle \otimes A_0 = \psi(z_2)\psi(z_1)|0\rangle \otimes B_0. \tag{2.5}$$

By a direct computation, we obtain

$$\psi(z_1)\psi(z_2)|0\rangle = |0\rangle \cdot (z_1 - z_2)^{-1} + \sum_{m>n\geqslant 0} \psi_{-m-1/2}\psi_{-n-1/2}|0\rangle \cdot \bigl(z_1^m z_2^n - z_1^n z_2^m\bigr).$$

So by multiplying $z_1 - z_2$ both sides of (2.5) and comparing the coefficient of $|0\rangle$, we obtain $A_0 = -B_0$. Therefore, (2.3) holds. By (2.2), we have

$$(z_1 - z_2)^N \bigl[Y_V(\psi_{-1/2}|0\rangle \otimes u, z_1), Y_V(\psi_{-1/2}|0\rangle \otimes v, z_2)\bigr] \cdot (\psi_{-1/2}|0\rangle \otimes w) = 0.$$

Rewriting the equality above in terms of $\psi(z)$, we get

$$\psi(z_1)\psi(z_2)\psi_{-1/2}|0\rangle \otimes A_1 = \psi(z_2)\psi(z_1)\psi_{-1/2}|0\rangle \otimes B_1. \tag{2.6}$$

By a direct computation, we have

$$\begin{aligned}\psi(z_1)\psi(z_2)\psi_{-1/2}|0\rangle = {} & \psi_{-1/2}|0\rangle \cdot \bigl\{(z_1 - z_2)^{-1} + (z_1 - z_2)/z_1 z_2\bigr\} \\ & + \sum_{m>0} \psi_{-m-3/2}|0\rangle \cdot \bigl(z_1^{m+1} z_2^{-1} - z_1^{-1} z_2^{m+1}\bigr) \\ & + \sum_{m\geqslant n\geqslant 0} \psi_{-m-5/2}\psi_{-n-3/2}\psi_{-1/2}|0\rangle \cdot \bigl(z_1^{m+2} z_2^{n+1} - z_1^{n+1} z_2^{m+2}\bigr).\end{aligned}$$

Multiplying $z_1 - z_2$ both sides of (2.6) and comparing the coefficient of $\psi_{-1/2}|0\rangle$ in (2.6), we obtain $(z_1^2 - z_1 z_2 + z_2^2)(A_1 + B_1) = 0$. Then multiplying $z_1 + z_2$, we get $(z_1^3 + z_2^3) \times (A_1 + B_1) = 0$. On the other hand, by comparing the coefficient of $\psi_{-5/2}|0\rangle$ in (2.6), we obtain

$$\bigl(z_1^2 z_2^{-1} - z_1^{-1} z_2^2\bigr)(A_1 + B_1) = 0,$$

or equivalently $(z_1^3 - z_2^3)(A_1 + B_1) = 0$. Combining this with $(z_1^3 + z_2^3)(A_1 + B_1) = 0$, we obtain $A_1 = -B_1$ and (2.4) also holds. Hence, $\widehat{Y}(\cdot, z)$ satisfies the locality and thus $(T_e(0) \oplus T_e(1/2), \widehat{Y}(\cdot, z), \mathbb{1}_{T_e(0)}, \omega - e)$ forms an SVOA.

By the construction of the vertex operator map $\widehat{Y}(\cdot, z)$, the remaining part of (1) of Theorem 2.2 is obvious except for the simplicity of $T_e(0) \oplus T_e(1/2)$, which is almost trivial. For, as V is simple, none of $I^i(\cdot, z)$, $J^j(\cdot, z)$, $i, j = 0, 1$, is zero map by [DL]. Then $T_e(0) \oplus T_e(1/2)$ is also simple since $T_e(0)$ is a simple VOA and $T_e(1/2)$ is an irreducible $T_e(0)$-module.

(2) Recall that the vertex operator map $Y_{\mathfrak{N}^+}(\cdot, z)$ on $\mathfrak{N}^+$ we constructed in Section 2.1 is an $L(1/2, 0)$-intertwining operator of type

$$\bigl(L(1/2, 0) \oplus L(1/2, 1/2)\bigr) \times L(1/2, 1/16) \to L(1/2, 1/16).$$

We make use of $Y_{\mathfrak{N}^+}(\cdot,z)$ to factorize $Y_V(\cdot,z)|_{V_e(1/16)}$. Let $a \in T_e(0)$, $u \in T_e(1/2)$, $x \in L(1/2,0)$ and $y \in L(1/2,1/2)$. By [ADL, Theorem 2.10], there are $T_e(0)$-intertwining operators $X^i(\cdot,z)$ of types $T_e(i/2) \times T_e(1/16) \to T_e(1/16)$, $i = 0,1$, such that $Y_V(x \otimes a,z)|_{V_e(1/16)} = Y_{\mathfrak{N}^+}(x,z) \otimes X^0(a,z)$ and $Y_V(y \otimes u,z)|_{V_e(1/16)} = Y_{\mathfrak{N}^+}(y,z) \otimes X^1(u,z)$, as $V_e(1/16) \simeq \mathfrak{N}^+ \otimes T_e(1/16)$ as $\mathrm{Vir}(e) \otimes T_e(0)$-modules. We define a $\mathbb{Z}_2$-twisted vertex operator map $X(\cdot,z)$ of $T_e(0) \oplus T_e(1/2)$ on $T_e(1/16)$ as follows:

$$X(a,z) := X^0(a,z) \quad \text{for } a \in T_e(0), \qquad X(u,z) := X^1(u,z) \quad \text{for } u \in T_e(1/16).$$

Then we prove $(T_e(1/16), X(\cdot,z))$ is an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. As $X(\cdot,z)$ is made of $T_e(0)$-intertwining operators, we only need to prove the $\mathbb{Z}_2$-twisted Jacobi identity for $X(\cdot,z)$, which is equivalent to the following commutativity and associativity for $u, v \in T_e(1/2)$ and $w \in T_e(1/16)$ (cf. [Li2]):

$$(z_1 - z_2)^{N_1}\big[X(u,z_1), X(v,z_2)\big]_+ = 0, \tag{2.7}$$

$$(z_0 + z_2)^{N_2+1/2} X(u, z_0 + z_2) X(v,z_2) w = (z_2 + z_0)^{N_2+1/2} X\big(\widehat{Y}(u,z_0)v, z_2\big) w, \tag{2.8}$$

where N_1 and N_2 are sufficiently large integers. We can take $N > 0$ which is independent of w such that

$$(z_1 - z_2)^N \big[Y_V(\psi_{-1/2}|0\rangle \otimes u, z_1), Y_V(\psi_{-1/2}|0\rangle, z_2)\big] \cdot \big(v_{1/16}^+ \otimes w\big) = 0,$$

where $v_{1/16}^+ = (\phi_0 + \sqrt{2})|\frac{1}{16}\rangle \in \mathfrak{N}^+$. Since $Y_{\mathfrak{N}^+}(\psi_{-1/2}|0\rangle, z) = \phi(z)$, we can rewrite the above as follows:

$$\begin{aligned}
&\phi(z_1)\phi(z_2)v_{1/16}^+ \otimes (z_1 - z_2)^N X(u,z_1)X(v,z_2)w \\
&\quad = \phi(z_2)\phi(z_1)v_{1/16}^+ \otimes (z_1 - z_2)^N X(v,z_2)X(u,z_1)w.
\end{aligned} \tag{2.9}$$

For simplicity, we set

$$A_2 = (z_1 - z_2)^N X(u,z_1)X(v,z_2)w, \qquad B_2 = (z_1 - z_2)^N X(v,z_2)X(u,z_1)w.$$

By a direct computation, one has the following:

$$\begin{aligned}
z_1^{1/2} z_2^{1/2} \phi(z_1)\phi(z_2)v_{1/16}^+ &= v_{1/16}^+ \cdot p(z_1,z_2) + \sum_{m>0} \phi_{-m} v_{1/16}^+ \cdot \frac{1}{\sqrt{2}} q_m(z_1,z_2) \\
&\quad + \sum_{m>n>0} \phi_{-m}\phi_{-n} v_{1/16}^+ \cdot r_{m,n}(z_1,z_2),
\end{aligned} \tag{2.10}$$

where we have set

$$p(z_1,z_2) := -\frac{1}{2} + \sum_{i=0}^{\infty} \left(\frac{z_2}{z_1}\right)^i, \qquad q_m(z_1,z_2) := z_1^m - z_2^m,$$
$$r_{m,n}(z_1,z_2) := z_1^m z_2^n - z_1^n z_2^m.$$

It is easy to see

$$(z_1 - z_2)p(z_1, z_2) = (z_1 + z_2)/2 = (z_2 - z_1)p(z_2, z_1),$$
$$q_m(z_2, z_1) = -q_m(z_1, z_2) \quad \text{and} \quad r_{m,n}(z_2, z_1) = -r_{m,n}(z_1, z_2). \tag{2.11}$$

By (2.10), the left-hand side of (2.9) can be expressed as follows:

$$v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} p(z_1, z_2) A_2 + \sum_{m>0} \phi_{-m} v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} \frac{1}{\sqrt{2}} q_m(z_1, z_2) A_2$$
$$+ \sum_{m>n>0} \phi_{-m}\phi_{-n} v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} r_{m,n}(z_1, z_2) A_2.$$

Similarly, the right-hand side of (2.9) becomes:

$$v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} p(z_2, z_1) B_2 + \sum_{m>0} \phi_{-m} v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} \frac{1}{\sqrt{2}} q_m(z_2, z_1) B_2$$
$$+ \sum_{m>n>0} \phi_{-m}\phi_{-n} v_{1/16}^{+} \otimes z_1^{-1/2} z_2^{-1/2} r_{m,n}(z_2, z_1) B_2.$$

Thus, we get the following relations:

$$p(z_1, z_2) A_2 = p(z_2, z_1) B_2, \tag{2.12}$$
$$q_m(z_1, z_2) A_2 = q_m(z_2, z_1) B_2, \tag{2.13}$$
$$r_{m,n}(z_1, z_2) A_2 = r_{m,n}(z_2, z_1) B_2. \tag{2.14}$$

Multiplying $(z_1 - z_2)$ to (2.12) and using (2.11), we obtain $\frac{1}{2}(z_1 + z_2)(A + B) = 0$. And by (2.13), we have $(z_1^m - z_2^m)(A + B) = 0$ for any $m > 0$. Combining them, we obtain $A + B = 0$ and so (2.7) follows.

Next, we prove the associativity (2.8). As

$$\phi(z) v_{1/16}^{+} = \frac{1}{\sqrt{2}} v_{1/16}^{+} z^{-1/2} + \sum_{n>0} \phi_{-n} v_{1/16}^{+} z^{n-1/2},$$

we see that $z^{1/2}\phi(z) v_{1/16}^{+} \in L(1/2, 1/16)[[z]]$. Therefore, by [Li2], we have the following associativity on $\mathfrak{N}^{+}$:

$$(z_0 + z_2)^{1/2} \phi(z_0 + z_2) \phi(z_2) v_{1/16}^{+} = (z_2 + z_0)^{1/2} Y_{\mathfrak{N}^{+}}\big(\psi(z_0)\psi_{-1/2}|0\rangle, z_2\big) v_{1/16}^{+}. \tag{2.15}$$

Let k be an integer such that

$$z^k Y_V(\psi_{-1/2}|0\rangle \otimes u, z) v_{1/16}^{+} \otimes w \in V_e(1/16)[[z]].$$

On $V_e(1/16)$, we have the following associativity by [Li1]:

$$\begin{aligned}&(z_0+z_2)^{k+1}Y_V(\psi_{-1/2}|0\rangle\otimes u,z_0+z_2)Y_V(\psi_{-1/2}|0\rangle\otimes v,z_2)v^+_{1/16}\otimes w\\&\quad=(z_2+z_0)^{k+1}Y_V\big(Y_V(\psi_{-1/2}|0\rangle\otimes u,z_0)\psi_{-1/2}|0\rangle\otimes v,z_2\big)v^+_{1/16}\otimes w.\end{aligned}$$

In terms of $\phi(z)$ and $X(\cdot,z)$, we can rewrite the above as follows:

$$\begin{aligned}&(z_0+z_2)^{1/2}\phi(z_0+z_2)\phi(z_2)v^+_{1/16}\otimes(z_0+z_2)^{k+1/2}X(u,z_0+z_2)X(v,z_2)w\\&\quad=(z_2+z_0)^{1/2}Y_{\mathfrak{N}^+}\big(\psi(z_0)\psi_{-1/2}|0\rangle,z_2\big)v^+_{1/16}\otimes(z_2+z_0)^{k+1/2}X\big(\widehat{Y}(u,z_0)v,z_2\big)w.\end{aligned}$$

Using (2.15), we get

$$(z_0+z_2)^{1/2}\phi(z_0+z_2)\phi(z_2)v^+_{1/16}\otimes C=0, \tag{2.16}$$

where we have set

$$C:=(z_0+z_2)^{k+1/2}X(u,z_0+z_2)X(v,z_2)w-(z_2+z_0)^{k+1/2}X\big(\widehat{Y}(u,z_0)v,z_2\big)w.$$

By (2.10), we find that the coefficient of $\phi_{-1}v^+_{1/16}$ in $(z_0+z_2)^{1/2}\phi(z_0+z_2)\phi(z_2)v^+_{1/16}$ is just a monomial $z_0z_2^{-1/2}/\sqrt{2}$. Therefore, Eq. (2.16) leads to the associativity relation $C=0$, or the equality (2.8). Hence, $(T_e(1/16),X(\cdot,z))$ is a $\mathbb{Z}_2$-twisted $T_e(0)\oplus T_e(1/2)$-module. The remaining part of the assertion is clear except for the irreducibility, which is easy to show. If $T_e(1/16)$ contains a non-trivial $\mathbb{Z}_2$-twisted $T_e(0)\oplus T_e(1/2)$-submodule, say P, then $L(1/2,1/16)\otimes P$ forms a non-trivial $V_e(0)\oplus V_e(1/2)$-submodule of $V_e(1/16)\simeq L(1/2,1/16)\otimes T_e(1/16)$. This yields a contradiction as $V_e(1/16)$ is an irreducible $V_e(0)\oplus V_e(1/2)$-module by [DM]. This completes the proof of Theorem 2.2. □

Remark 2.3. As we mentioned, there are exactly two inequivalent $\mathbb{Z}_2$-twisted irreducible $L(1/2,0)\oplus L(1/2,1/2)$-module structures on $L(1/2,1/16)$ (cf. [LLY]). In the statement (2) of the theorem above, we have to choose one of them and the irreducible $\mathbb{Z}_2$-twisted $T_e(0)\oplus T_e(1/2)$-module structure on $T_e(1/16)$ may depend on this choice.

2.4. Automorphisms of commutant superalgebra

In the rest of this section we will work over the following setup:

Hypothesis 1.

(1) V is a holomorphic VOA of CFT-type.
(2) e is a rational conformal vector of V with central charge $1/2$.
(3) $V_e(h)\neq 0$ for $h=0,1/2,1/16$.
(4) $V_e(0)$ and $T_e(0)$ are rational C_2-cofinite VOAs of CFT-type.
(5) $V_e(1/16)$ is a simple current $V^{\langle\tau_e\rangle}=V_e(0)\oplus V_e(1/2)$-module.
(6) $T_e(1/2)$ is a simple current $T_e(0)$-module.

We define the one point stabilizer by $C_{\mathrm{Aut}(V)}(e) := \{\rho \in \mathrm{Aut}(V) \mid \rho(e) = e\}$. Clearly $C_{\mathrm{Aut}(V)}(e)$ forms a subgroup of $\mathrm{Aut}(V)$. Since $\tau_{\rho(e)} = \rho\tau_e\rho^{-1}$ for any $\rho \in \mathrm{Aut}(V)$, we have $C_{\mathrm{Aut}(V)}(e) \leqslant C_{\mathrm{Aut}(V)}(\tau_e)$, where $C_{\mathrm{Aut}(V)}(\tau_e)$ denotes the centralizer of an involution $\tau_e \in \mathrm{Aut}(V)$.

Lemma 2.4. *There are group homomorphisms* $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ *and* $\psi_2 : C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$ *such that* $\mathrm{Ker}(\psi_1) = \langle\tau_e\rangle$ *and* $\mathrm{Ker}(\psi_2) = \langle\sigma_e\rangle$.

Proof. Let $\rho \in C_{\mathrm{Aut}(V)}(e)$. Then ρ preserves the space of highest weight vectors $T_e(h)$ for $h = 0, 1/2, 1/16$ so that ρ definitely acts on $T_e(h)$. Therefore, we have group homomorphisms $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ and $\psi_2 : C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$. Assume that $\psi_1(\rho) = \mathrm{id}_{V^{\langle\tau_e\rangle}}$ for $\rho \in C_{\mathrm{Aut}(V)}(e)$. Since ρ commutes with τ_e, ρ acts on $V_e(1/16)$ and commutes with the action of $V^{\langle\tau_e\rangle} = V_e(0) \oplus V_e(1/2)$ on its module $V_e(1/16)$. Therefore, ρ on $V_e(1/16)$ is a scalar by Schur's lemma and hence $\rho \in \langle\tau_e\rangle \leqslant C_{\mathrm{Aut}(V)}(\tau_e)$. Similarly, one can verify that $\mathrm{Ker}(\psi_2) = \langle\sigma_e\rangle$. $\square$

The following result will be used frequently (cf. [Y2, Theorem 9.1.7]).

Theorem 2.5. *Let $V = V^0 \oplus V^1$ be a simple SVOA such that the even part V^0 is a rational C_2-cofinite VOA of CFT type and the odd part V^1 is a simple current V^0-module. Then V is both rational and $\mathbb{Z}_2$-rational. Let W be an irreducible V^0-module.*

(1) *If $V^1 \boxtimes_{V^0} W \not\simeq W$ as V^0-modules, then W is uniquely lifted to either an irreducible untwisted V-module or an irreducible $\mathbb{Z}_2$-twisted V-module given by $W \oplus (V^1 \boxtimes_{V^0} W)$.*
(2) *If $V^1 \boxtimes_{V^0} W \simeq W$ as V^0-modules, then there are exactly two inequivalent irreducible $\mathbb{Z}_2$-twisted V-module structures on W and these two modules are mutually $\mathbb{Z}_2$-conjugate.*

Lemma 2.6. *Under Hypothesis* 1, *every irreducible $T_e(0)$-module is contained in an untwisted irreducible $V^{\langle\tau_e\rangle}$-module as a submodule.*

Proof. Let X be an irreducible $T_e(0)$-module. By Theorem 2.5, X is contained in an irreducible $T_e(0) \oplus T_e(1/2)$-module or an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Let $\widetilde{X}$ be such a $T_e(0) \oplus T_e(1/2)$-module. If $\widetilde{X}$ is an untwisted representation, then a tensor product $\{L(1/2, 0) \oplus L(1/2, 1/2)\} \otimes \widetilde{X}$ has a structure of an untwisted $V^{\langle\tau_e\rangle}$-module and contains X as a submodule. If $\widetilde{X}$ is a $\mathbb{Z}_2$-twisted representation, then a tensor product $L(1/2, 1/16) \otimes \widetilde{X}$ has a structure of an untwisted $V^{\langle\tau_e\rangle}$-module and contains X as a submodule. $\square$

Theorem 2.7. *Under Hypothesis* 1, *$V^{\langle\tau_e\rangle}$ has exactly four inequivalent irreducible modules, $V^{\langle\tau_e\rangle}$, $V_e(1/16)$, $W^0 := L(1/2, 0) \otimes T_e(1/2) \oplus L(1/2, 1/2) \otimes T_e(0)$ and*

$$W^1 := V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} W^0.$$

Their fusion rules are as follows:

$$V_e(1/16) \times V_e(1/16) = V^{\langle \tau_e \rangle}, \qquad V_e(1/16) \times W^0 = W^1, \qquad V_e(1/16) \times W^1 = W^0,$$
$$W^0 \times W^0 = V^{\langle \tau_e \rangle}, \qquad W^0 \times W^1 = V_e(1/16), \qquad W^1 \times W^1 = V^{\langle \tau_e \rangle}.$$

Therefore, the fusion algebra for $V^{\langle \tau_e \rangle}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since $V = V^{\langle \tau_e \rangle} \oplus V_e(1/16)$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle \tau_e \rangle}$, every irreducible $V^{\langle \tau_e \rangle}$-module is lifted to either an irreducible V-module or an irreducible τ_e-twisted V-module by [Y1, Theorem 3.3]. Moreover, the τ_e-twisted V-module is unique up to isomorphism by [DLM, Theorem 10.3]. Since both $L(1/2,0) \oplus L(1/2,1/2)$ and $T_e(0) \oplus T_e(1/2)$ are simple SVOAs, the space $W^0 = L(1/2,1/2) \otimes T_e(0) \oplus L(1/2,0) \otimes T_e(1/2)$ has a unique structure of an irreducible $V^{\langle \tau_e \rangle}$-module. As the top weight of W^0 is half-integral, the induced module

$$W = W^0 \oplus W^1, \qquad W^1 = V_e(1/16) \boxtimes_{V^{\langle \tau_e \rangle}} W^0,$$

becomes an irreducible τ_e-twisted V-module again by [Y1, Theorem 3.3]. It is clear from $V_e(1/16) \boxtimes_{V^0} W^1 = W^0$ that W^1 and $V_e(1/16)$ are inequivalent $V^{\langle \tau_e \rangle}$-modules. Therefore, $V^{\langle \tau_e \rangle}$ has exactly four irreducible modules as in the assertion. We remark that only $V^{\langle \tau_e \rangle}$, $V_e(1/16)$ and W^1 have integral top weights.

Consider fusion rules for $V^{\langle \tau_e \rangle}$-modules. By [SY, Lemma 3.12], we have the fusion rule $W^0 \times W^0 = V^{\langle \tau_e \rangle}$. Then it follows from the forthcoming Lemma 3.5 that W^0 is a simple current $V^{\langle \tau_e \rangle}$-module. Since $V_e(1/16)$ is also a simple current $V^{\langle \tau_e \rangle}$-module, so is $W^1 = V_e(1/16) \boxtimes_{V^{\langle \tau_e \rangle}} W^0$. By looking at the τ_e-twisted V-module structure on $W^0 \oplus W^1$, we easily find the following fusion rules:

$$V_e(1/16) \times V_e(1/16) = V^{\langle \tau_e \rangle}, \qquad V_e(1/16) \times W^0 = W^1, \qquad V_e(1/16) \times W^1 = W^0.$$

Since V is holomorphic, V is self-dual. Hence $V^{\langle \tau_e \rangle}$ and $V_e(1/16)$ are self-dual $V^{\langle \tau_e \rangle}$-modules. Then by considering top weights we see that all irreducible $V^{\langle \tau_e \rangle}$-modules are self-dual. Then by the S_3-symmetry of fusion rules (cf. [FHL]), we have the desired fusion rules. □

By the fusion rules for $L(1/2,0)$-modules, we note that W^1 as a $\mathrm{Vir}(e)$-module is a direct sum of copies of $L(1/2,1/16)$. Set the space of highest weight vectors of W^1 by $Q_e(1/16) := \{v \in W^1 \mid L^e(0)v = (1/16) \cdot v\}$. Then as a $\mathrm{Vir}(e) \otimes T_e(0)$-module, $W^1 \simeq L(1/2,1/16) \otimes Q_e(1/16)$. By Theorem 2.5, the space $Q_e(1/16)$ naturally carries an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module structure, which may depend on a choice of irreducible $\mathbb{Z}_2$-twisted $L(1/2,0) \oplus L(1/2,1/2)$-module structures on $L(1/2,1/16)$.

Proposition 2.8. *If the $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then its $\mathbb{Z}_2$-conjugate is isomorphic to $Q_e(1/16)$ as a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. In this case there are exactly three inequivalent irreducible $T_e(0)$-modules, $T_e(0)$, $T_e(1/2)$ and $T_e(1/16)$. Conversely, if $T_e(1/16)$ as a $T_e(0)$-module is reducible, then so is $Q_e(1/16)$ and in this case there are exactly six inequivalent irreducible $T_e(0)$-modules.*

Proof. Assume that $T_e(1/16)$ is irreducible as a $T_e(0)$-module. In this case there are exactly two inequivalent irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module structures on $T_e(1/16)$ by Theorem 2.5. Therefore, an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module structure on $T_e(1/16)$ given in Theorem 2.2 and its $\mathbb{Z}_2$-conjugate are inequivalent. This implies that there are exactly two inequivalent irreducible untwisted $V^{\langle \tau_e \rangle}$-module structures on $L(1/2, 1/16) \otimes T_e(1/16)$. Thus by the classification in Theorem 2.7, $V_e(1/16)$ and W^1 are isomorphic as $L(1/2, 0) \otimes T_e(0)$-modules. By Lemma 2.6, every irreducible $T_e(0)$-module appears in an irreducible $V^{\langle \tau_e \rangle}$-module as a submodule. Thus $T_e(0)$ has exactly three inequivalent irreducible modules as in the assertion.

Conversely, if $T_e(1/16)$ as a $T_e(0)$-module is reducible, then it is a direct sum of two inequivalent irreducible $T_e(0)$-module by Theorem 2.5. In this case we note that $V_e(1/16)$ is a σ_e-stable $V^{\langle \tau_e \rangle}$-module, that is, the σ_e-conjugate $V_e(1/16)^{\sigma_e}$ of $V_e(1/16)$ is isomorphic to $V_e(1/16)$ itself as a $V^{\langle \tau_e \rangle}$-module. We note that $Q_e(1/16)$ is also a reducible $T_e(0)$-module. For, if $Q_e(1/16)$ is irreducible, then $T_e(1/16)$ and $Q_e(1/16)$ are in the relation of $\mathbb{Z}_2$-conjugate, and hence $T_e(1/16)$ is also irreducible, a contradiction. Thus $Q_e(1/16)$ is a direct sum of two inequivalent irreducible $T_e(0)$-submodule. If $T_e(1/16)$ and $Q_e(1/16)$ contain isomorphic irreducible $T_e(0)$-submodules, then $T_e(1/16)$ and $Q_e(1/16)$ are isomorphic irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-modules by Theorem 2.5. This implies that $V_e(1/16)$ is isomorphic to W^1 as a $V^{\langle \tau_e \rangle}$-module, which is a contradiction. Now the assertion follows from Lemma 2.6. □

Corollary 2.9. *If $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then $V^{\langle \tau_e \rangle} \oplus W^1$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle \tau_e \rangle}$ which is isomorphic to $V = V^{\langle \tau_e \rangle} \oplus V_e(1/16)$.*

Proof. If $T_e(1/16)$ is an irreducible $T_e(0)$-module, then by the previous proposition the $\mathbb{Z}_2$-conjugate of $T_e(1/16)$ is isomorphic to $Q_e(1/16)$ as $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-modules. Hence the σ_e-conjugate $V_e(0) \oplus V_e(1/2)$-module of $V_e(1/16) = L(1/2, 1/16) \otimes T_e(1/16)$ is isomorphic to $W^1 = L(1/2, 1/16) \otimes Q_e(1/16)$ and so $\sigma_e \in \mathrm{Aut}(V^{\langle \tau_e \rangle})$ induces a VOA isomorphism between two extensions $V^{\langle \tau_e \rangle} \oplus V_e(1/16)$ and $V^{\langle \tau_e \rangle} \oplus W^1$ of $V^{\langle \tau_e \rangle}$. □

Remark 2.10. The corollary above implies that the τ_e-twisted orbifold construction applied to V yields V itself again.

Theorem 2.11. *Under Hypothesis* 1,

(1) *ψ_2 is surjective, that is, $C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e) \simeq \langle \sigma_e \rangle . \mathrm{Aut}(T_e(0))$.*
(2) $\mathrm{Aut}(T_e(0) \oplus T_e(1/2)) \simeq 2.(C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e)/\langle \sigma_e \rangle)$, *where* 2 *denotes the canonical $\mathbb{Z}_2$-symmetry on the SVOA $T_e(0) \oplus T_e(1/2)$.*
(3) $|C_{(\mathrm{Aut}(V^{\langle \tau_e \rangle}))}(e) : C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle| \leqslant 2$.
(4) *If $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$ is simple or has an odd order, then extensions in* (1) *and* (2) *split. That is, $C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e) \simeq \langle \sigma_e \rangle \times C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$ and* $\mathrm{Aut}(T_e(0) \oplus T_e(1/2)) \simeq 2 \times \mathrm{Aut}(T_e(0))$.

Proof. We have an injection from $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle$ to $\mathrm{Aut}(T_e(0))$ by Lemma 2.4. We will show that every element in $\mathrm{Aut}(T_e(0))$ has its preimage in $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. By Proposition 2.8, every irreducible $T_e(0)$-module is contained in one of $T_e(0)$, $T_e(1/2)$, $T_e(1/16)$ or $Q_e(1/16)$ as a submodule. In particular, we find that $T_e(0)$ is the only irreducible $T_e(0)$-module whose top weight is integral and $T_e(1/2)$ is the only irreducible $T_e(0)$-module whose top weight is in $1/2+\mathbb{N}$. Let $\rho \in \mathrm{Aut}(T_e(0))$. Then by considering top weights we can immediately see that $T_e(0)^\rho \simeq T_e(0)$ and $T_e(1/2)^\rho \simeq T_e(1/2)$. Then by [Sh, Theorem 2.1] we have a lifting $\tilde{\rho} \in \mathrm{Aut}(T_e(0)\oplus T_e(1/2))$ such that $\tilde{\rho}T_e(0) = T_e(0)$, $\tilde{\rho}T_e(1/2) = T_e(1/2)$ and $\tilde{\rho}|_{T_e(0)} = \rho$. Since $\tilde{\rho}$ is uniquely determined up to the canonical $\mathbb{Z}_2$-symmetry on $T_e(0)\oplus T_e(1/2)$, we have $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) \simeq 2.\mathrm{Aut}(T_e(0))$. Now we define $\tilde{\tilde{\rho}} \in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ by

$$\tilde{\tilde{\rho}}|_{L(1/2,h)\otimes T_e(h)} = \mathrm{id}_{L(1/2,h)} \otimes \tilde{\rho}, \quad h = 0, 1/2.$$

Then by this lifting $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ contains a subgroup isomorphic to $2.\mathrm{Aut}(T_e(0))$. Moreover, the canonical $\mathbb{Z}_2$-symmetry on the SVOA $T_e(0)\oplus T_e(1/2)$ is naturally extended to $\sigma_e \in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. Clearly $\psi_2(\tilde{\tilde{\rho}}) = \rho$ and hence ψ_2 is surjective. Therefore, we have the desired isomorphisms $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle.\mathrm{Aut}(T_e(0))$ and $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) \simeq 2.(C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle)$. This completes the proofs of (1) and (2).

Consider (3). By Theorem 2.7, there are exactly three irreducible $V^{\langle\tau_e\rangle}$-modules whose top weights are integral, namely, $V^{\langle\tau_e\rangle}$, $V_e(1/16)$ and W^1. Thus $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ acts on the 2-point set $\{V_e(1/16), W^1\}$ as a permutation and so there is a subgroup H of $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ with index at most 2 such that $V_e(1/16)^\pi \simeq V_e(1/16)$ as a $V^{\langle\tau_e\rangle}$-module for all $\pi \in H$. Then by [Sh, Theorem 2.1] there is a lifting $\tilde{\pi} \in C_{\mathrm{Aut}(V)}(e)$ of π such that $\psi_1(\tilde{\pi}) = \pi$ for each $\pi \in H$. Thus $|C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle| \leqslant 2$ and (3) holds.

Consider (4). Suppose $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ is either simple or odd. By (3), $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ contains a subgroup isomorphic to $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ with index at most 2. Since $\langle\sigma_e\rangle$ is a normal subgroup of $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ of order 2, the index $|C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle|$ must be 2 by the assumption and hence we obtain the desired isomorphism $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$. In this case, it is easy to see that the extension $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) = 2.\mathrm{Aut}(T_e(0))$ splits. □

Corollary 2.12. *If $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ is simple, then $V_e(1/16)$ is an irreducible $V_e(0)$-module and $T_e(1/16)$ is an irreducible $T_e(0)$-module. Therefore, $V = V^{\langle\tau_e\rangle}\oplus V_e(1/16)$ and $V^{\langle\tau_e\rangle}\oplus W^1$ are equivalent extensions of $V^{\langle\tau_e\rangle}$.*

Proof. Let H be the subgroup of $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ which fixes $V_e(1/16)$ in the action on the 2-point set $\{V_e(1/16), W^1\}$. It is shown in the proof of (3) of Theorem 2.11 that we have inclusions

$$H \leqslant C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle \lneqq C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) = \langle\sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle.$$

Therefore, $\sigma_e \notin H$ and hence the σ_e permutes $V_e(1/16)$ and W^1. Then $V_e(1/16)$ is an irreducible $V_e(0)$-module by Proposition 2.8 and hence $T_e(1/16)$ as a $T_e(0)$-module is irreducible. The rest of the assertion follows from Corollary 2.9. □

3. 2A-framed VOA

In this section we consider VOAs with unitary Virasoro frames. For convention, we introduce the following notion:

Definition 3.1. A simple vertex operator algebra (V,ω) is called *2A-framed* if there is an orthogonal decomposition $\omega = e^1 + \cdots + e^n$ such that each e^i generates a sub VOA isomorphic to $L(1/2,0)$. The decomposition $\omega = e^1 + \cdots + e^n$ is called a *2A-frame* of V.

Remark 3.2. Any 2A-framed VOA is rational and C_2-cofinite (cf. [DGH,Z]).

3.1. Structure codes

For a 2A-framed VOA, we can associate two linear binary codes in the following way (cf. [M2,DGH]). Let (V,ω) be a 2A-framed VOA with a 2A-frame $\omega = e^1 + \cdots + e^n$. Set $F := \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$. Then $F \simeq L(1/2,0)^{\otimes n}$ and V is a direct sum of irreducible F-submodules $L(1/2,h_1) \otimes \cdots \otimes L(1/2,h_n)$, $h_i \in \{0,1/2,1/16\}$. Assign to an irreducible F-module $\bigotimes_{i=1}^n L(1/2,h_i)$ its $1/16$-word $(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}_2^n$ by $\alpha_i = 1$ if and only if $h_i = 1/16$. For each $\alpha \in \mathbb{Z}_2^n$, denote by V^α the sum of all irreducible F-submodules whose $1/16$-words are equal to α and define a linear code $S \subset \mathbb{Z}_2^n$ by $S = \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}$. Then we have the $1/16$-*word decomposition* $V = \bigoplus_{\alpha\in S} V^\alpha$. By the fusion rules for $L(1/2,0)$-modules, we have an S-graded structure $V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$. Namely, the dual group S^* of an abelian 2-group S acts on V, and we find that this automorphism group coincides with the elementary abelian 2-group generated by Miyamoto involutions $\{\tau_{e^i} \mid 1 \leqslant i \leqslant n\}$. Therefore, all V^α, $\alpha \in S$, are inequivalent irreducible $V^{S^*} = V^0$-modules by [DM].

Since there is no $L(1/2,1/16)$-component in V^0, the fixed point subalgebra $V^{S^*} = V^0$ is of the following form:

$$V^0 = \bigoplus_{h_i \in \{0,1/2\}} m_{h_1,\ldots,h_n} L(1/2,h_1) \otimes \cdots \otimes L(1/2,h_n),$$

where $m_{h_1,\ldots,h_n}$ denotes the multiplicity. On V^0 we can define σ-type Miyamoto involutions σ_{e^i} for $i = 1,\ldots,n$. Denote by I the elementary abelian 2-subgroup of $\mathrm{Aut}(V^0)$ generated by $\{\sigma_{e^i} \mid 1 \leqslant i \leqslant n\}$. Then we have $(V^0)^I = F$ and each $m_{h_1,\ldots,h_n} L(1/2,h_1) \otimes \cdots \otimes L(1/2,h_n)$ is an irreducible F-submodule by [DM]. Thus $m_{h_1,\ldots,h_n} \in \{0,1\}$ and we obtain an even linear code $D := \{(2h_1,\ldots,2h_n) \in \mathbb{Z}_2^n \mid m_{h_1,\ldots,h_n} \neq 0\}$ such that

$$V^0 = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in D} L(1/2,\alpha_1/2) \otimes \cdots \otimes L(1/2,\alpha_n/2). \tag{3.1}$$

The VOA V^0 is a D-graded simple current extension of F and is refereed to as a *code VOA associated to code D*. We call a pair (D,S) the *structure codes* of a 2A-framed VOA V. Since powers of z in an $L(1/2,0)$-intertwining operator of type $L(1/2,1/2) \times L(1/2,1/2) \to L(1/2,1/16)$ are half-integral, structure codes satisfy $D \subset S^\perp$.

3.2. Construction of 2A-framed VOA

In this subsection we recall Miyamoto's construction of 2A-framed VOAs in [M3]. Here we assume the following:

Hypothesis 2.

(1) (D, S) is a pair of even linear even codes of $\mathbb{Z}_2^n$ such that
 (1-i) $D \subset S^\perp$,
 (1-ii) for each $\alpha \in S$, there is a subcode $E^\alpha \subset D$ such that E^α is a direct sum of the $[8, 4, 4]$ Hamming code H_8 and $\mathrm{Supp}(E^\alpha) = \mathrm{Supp}(\alpha)$, where $\mathrm{Supp}(A)$ denotes $\bigcup_{\beta \in A} \mathrm{Supp}(\beta)$ for a subset A of $\mathbb{Z}_2^n$.
(2) V^0 is the code VOA associated to the code D.
(3) $\{V^\alpha \mid \alpha \in S\}$ is a set of irreducible V^0-modules such that
 (3-i) the $1/16$-word of V^α is equal to α for all $\alpha \in S$,
 (3-ii) all V^α, $\alpha \in S$, have integral top weights,
 (3-iii) the fusion product $V^\alpha \boxtimes_{V^0} V^\beta$ contains at least one $V^{\alpha+\beta}$. That is, there is a non-trivial V^0-intertwining operator of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ for any $\alpha, \beta \in S$.

Theorem 3.3 [M3,Y2].

(1) *Under the condition* (1) *of Hypothesis* 2*, all* V^α*,* $\alpha \in D$*, are simple current* V^0*-modules.*
(2) *Under Hypothesis* 2*, the space* $V = \bigoplus_{\alpha \in S} V^\alpha$ *carries a unique structure of a simple VOA as an* S*-graded simple current extension of* V^0*.*

Remark 3.4. In [M3], Miyamoto assumed stronger conditions than those in Hypothesis 2. In particular, he assumed that the structure codes (D, S) are of length $8k$ for some positive integer k. A refinement in [Y2] enables us to construct 2A-framed VOAs with structure codes of any length as long as Hypothesis 2 is satisfied.

3.3. Superalgebras associated to 2A-framed VOA

Let V be a 2A-framed VOA with structure codes (D, S). We assume that the pair (D, S) satisfies the condition (1-ii) of Hypothesis 2 and $D = S^\perp$. Then V is holomorphic by [M4,DGH]. Let $\omega = e^1 + \cdots + e^n$ be the 2A-frame of V. We consider the commutant subalgebra of $\mathrm{Vir}(e^1)$. For simplicity, we set $e = e^1$. Assume that $\{1\} \cap \mathrm{Supp}(S) \neq \emptyset$. Then by the condition (1-ii) of Hypothesis 2, we have $V_e(1/2) \neq 0$. Let $V = \bigoplus_{\alpha \in S} V^\alpha$ be the $1/16$-word decomposition according to the structure codes (D, S). Set $S^0 = \{\alpha \in S \mid \{1\} \cap \mathrm{Supp}(\alpha) = \emptyset\}$ and $S^1 = \{\alpha \in S \mid \{1\} \cap \mathrm{Supp}(\alpha) = \{1\}\}$. Then $S = S^0 \sqcup S^1$ (disjoint union) and we have a $\mathbb{Z}_2$-grading $V = (\bigoplus_{\alpha \in S^0} V^\alpha) \oplus (\bigoplus_{\beta \in S^1} V^\beta)$ such that $V_e(0) \oplus V_e(1/2) = \bigoplus_{\alpha \in S^0} V^\alpha$ and $V_e(1/16) = \bigoplus_{\beta \in S^1} V^\beta$. We shall prove that $V_e(1/16)$ is a simple current $V^{\langle \tau_e \rangle}$-module. We quote the following simple lemma from [Y2]:

Lemma 3.5 [Y2]. *Let V be a simple rational C_2-cofinite VOA of CFT-type. If two V-modules M^1 and M^2 satisfy $M^1 \times M^2 = V$ in the fusion algebra, then both M^1 and M^2 are simple current V-modules. In particular, if V is self-dual, then the set of all the simple current V-modules form a finite abelian group in the fusion algebra.*

Lemma 3.6. *$V_e(1/16)$ is a simple current $V^{\langle\tau_e\rangle}$-module.*

Proof. By Lemma 3.5, it suffices to show that $V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} V_e(1/16) = V^{\langle\tau_e\rangle}$. Let M be an irreducible $V^{\langle\tau_e\rangle}$-submodule of $V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} V_e(1/16)$. Since $V^\alpha \boxtimes_{V^0} V^\alpha = V^0$ for any $\alpha \in S$ by (1) of Theorem 3.3, M contains V^0 as a V^0-submodule. Thus M contains a non-zero vacuum-like vector and hence M is isomorphic to $V^{\langle\tau_e\rangle}$ as a $V^{\langle\tau_e\rangle}$-module by [Li3]. Therefore, we have $V_e(1/16) \times V_e(1/16) = nV^{\langle\tau_e\rangle}$ for some $n \in \mathbb{N}$. As V is holomorphic, both $V^{\langle\tau_e\rangle}$ and $V_e(1/16)$ are self-dual $V^{\langle\tau_e\rangle}$-modules. Now by using the S_3-symmetry of fusion rules, we obtain the desired fusion rule $V_e(1/16) \times V_e(1/16) = V^{\langle\tau_e\rangle}$ from the canonical fusion rule $V^{\langle\tau_e\rangle} \times V_e(1/16) = V_e(1/16)$. □

Write $V_e(h) = L(1/2, h) \otimes T_e(h)$ for $h = 0, 1/2, 1/16$ as we did before. By Theorem 2.2, $T_e(0) \oplus T_e(1/2)$ forms a simple SVOA. The Virasoro vector of $T_e(0)$ is given by $\omega - e^1 = e^2 + \cdots + e^n$ and so $T_e(0)$ is a 2A-framed VOA. We compute the structure codes of $T_e(0)$. Define $\phi_\varepsilon : \mathbb{Z}_2^{n-1} \hookrightarrow \mathbb{Z}_2^n$ by $\mathbb{Z}_2^{n-1} \ni \alpha \mapsto (\varepsilon, \alpha) \in \mathbb{Z}_2^n$ for $\varepsilon = 0, 1$, and set

$$D^\varepsilon := \left\{\alpha \in \mathbb{Z}_2^{n-1} \;\middle|\; \phi_\varepsilon(\alpha) \in D\right\}, \quad \varepsilon = 0, 1, \qquad S^{0,0} := \left\{\beta \in \mathbb{Z}_2^{n-1} \;\middle|\; \phi_0(\beta) \in S^0\right\}.$$

Proposition 3.7.

(1) *The structure codes of $T_e(0)$ with respect to the 2A-frame $e^2 + \cdots + e^n$ are $(D^0, S^{0,0})$.*
(2) *$T_e(1/2)$ has the $1/16$-word decomposition $T_e(1/2) = \bigoplus_{\alpha \in S^{0,0}} T_e(1/2)^\alpha$.*

Proof. For $\alpha \in S^0$, define $V^{\alpha,\varepsilon}$ to be the sum of all irreducible $\bigotimes_{i=1}^n \mathrm{Vir}(e^i)$-submodules of V^α whose $\mathrm{Vir}(e^1)$-components are isomorphic to $L(1/2, \varepsilon/2)$ for $\varepsilon = 0, 1$. By (1-ii) of Hypothesis 2, $V^{\alpha,\varepsilon} \neq 0$ for all $\alpha \in S^0$ and $\varepsilon = 0, 1$. Therefore, $V^\alpha = V^{\alpha,0} \oplus V^{\alpha,1}$ and we obtain the 1/16-word decompositions $V_e(0) = \bigoplus_{\alpha \in S^0} V^{\alpha,0}$ and $V_e(1/2) = \bigoplus_{\alpha \in S^0} V^{\alpha,1}$. Since $D = \phi_0(D^0) \sqcup \phi_1(D^1)$, $V^{0,0}$ is isomorphic to $\mathrm{Vir}(e^1) \otimes U_{D^0}$, where U_{D^0} denotes the code VOA associated to the even code D^0. Thus $T_e(0)$ has the 1/16-word decomposition $T_e(0) = \bigoplus_{\alpha \in S^{0,0}} T_e(0)^\alpha$ such that $T_e(0)^0 \simeq U_{D^0}$. Hence the structure codes of $T_e(0)$ are $(D^0, S^{0,0})$. The proof of (2) is similar. □

The following is easy to see:

Lemma 3.8. *If the structure codes (D, S) satisfy the condition* (1) *of Hypothesis* 2*, then so do $(D^0, S^{0,0})$.*

Thus $T_e(0) = \bigoplus_{\alpha \in S^{0,0}} T_e(0)^\alpha$ is an $S^{0,0}$-graded simple current extension of $T_e(0)^0$ by (1) of Theorem 3.3. In addition, by using (2) of Theorem 3.3, we can reconstruct $T_e(0)$ without reference to V.

Proposition 3.9. *$T_e(1/2)$ is a simple current $T_e(0)$-module.*

Proof. It suffices to show that $T_e(1/2) \boxtimes_{T_e(0)} T_e(1/2) = T_e(0)$ by Lemma 3.5. Let M be an irreducible $T_e(0)$-submodule of $T_e(1/2) \boxtimes_{T_e(0)} T_e(1/2)$. Since $T_e(1/2)$ has a 1/16-word decomposition $T_e(1/2) = \bigoplus_{\alpha \in S^{0,0}} T_e(1/2)^\alpha$ by Proposition 3.7, $T_e(1/2)^0$ as a $\bigotimes_{i=2}^n \mathrm{Vir}(e^i)$-module is isomorphic to

$$\bigoplus_{\beta=(\beta_2,\dots,\beta_n)\in D^1} L(1/2, \beta_2/2) \otimes \cdots \otimes L(1/2, \beta_n/2).$$

Therefore, by the fusion rules of $L(1/2, 0)$, M contains $L(1/2, 0)^{\otimes n-1}$ as a $\bigotimes_{i=2}^n \mathrm{Vir}(e^i)$-submodule. So M contains a non-trivial vacuum-like vector and hence M is isomorphic to $T_e(0)$ as a $T_e(0)$-module by [Li3]. Therefore, there exists an $n \in \mathbb{N}$ such that $T_e(1/2) \times T_e(1/2) = nT_e(0)$. Since V is holomorphic, both $T_e(0)$ and $T_e(1/2)$ are self-dual $T_e(0)$-modules. So by the S_3-symmetry of fusion rules, we obtain the desired fusion rule $T_e(1/2) \times T_e(1/2) = T_e(0)$ from the canonical fusion rule $T_e(0) \times T_e(1/2) = T_e(1/2)$. □

To summarize, we obtain:

Proposition 3.10. *Let V be a 2A-framed VOA with a 2A-frame $\omega = e^1 + \cdots + e^n$ and its associated structure codes (D, S). Suppose that the pair (D, S) satisfies the condition* (1-ii) *of Hypothesis* 2*, $D = S^\perp$ and $V_{e^1}(1/16) \neq 0$. Then V and e^1 satisfy Hypothesis* 1.

4. The baby-monster SVOA

Let $(V^\natural, \omega^\natural)$ be the moonshine VOA constructed in [FLM]. The full automorphism group of $V^\natural$ is the Monster $\mathbb{M}$, the largest sporadic finite simple group. We apply our results to $V^\natural$ and study the baby-monster SVOA. As shown in [DMZ], $V^\natural$ has a 2A-frame $\omega^\natural = e^1 + \cdots + e^{48}$, and one of its structure codes are determined in [DGH,M4].

Theorem 4.1 [DGH,M4]. *The moonshine VOA $V^\natural$ has a 2A-frame such that its associated structure codes $(D^\natural, S^\natural)$ are as follows*:

$$S^\natural := \mathrm{Span}_{\mathbb{Z}_2}\{(\alpha,\alpha,\alpha), (1^{16}0^{32}), (0^{32}1^{16}) \in \mathbb{Z}_2^{48} \mid \alpha \in \mathrm{RM}(1,4)\}, \quad D^\natural := (S^\natural)^\perp,$$

where $\mathrm{RM}(1,4)$ *is a Reed–Müller code defined as follows*:

$$\mathrm{RM}(1,4) := \mathrm{Span}_{\mathbb{Z}_2}\{(1^{16}), (1^8 0^8), (1^4 0^4 1^4 0^4), (\{1100\}^4), (\{10\}^8)\} < \mathbb{Z}_2^{16}.$$

Lemma 4.2. *For any conformal vector e of $V^\natural$ with central charge $1/2$, $V^\natural$ and e satisfy Hypothesis* 1.

Proof. It is shown in [C] and [M1] that all the conformal vectors with central charge $1/2$ are conjugate under the Monster $\mathbb{M} = \mathrm{Aut}(V^\natural)$. Thus we may assume that $e = e^1$. Since

$\{1\} \subset \mathrm{Supp}(S^\natural)$, $V_{e^1}(1/16) \neq 0$. It is easy to verify that the structure codes $(D^\natural, S^\natural)$ satisfy (1-ii) of Hypothesis 2. Therefore, $V^\natural$ and e^1 satisfy Hypothesis 1 by Proposition 3.10. □

Now set $e = e^1$ and consider the commutant subalgebra $T_e^\natural(0)$ of $\mathrm{Vir}(e)$ in $V^\natural$. By the lemma above, we have the following decomposition:

$$V^\natural = L(1/2, 0) \otimes T_e^\natural(0) \oplus L(1/2, 1/2) \otimes T_e^\natural(1/2) \oplus L(1/2, 1/16) \otimes T_e^\natural(1/16)$$

with $T_e^\natural(h) \neq 0$ for $h = 0, 1/2, 1/16$. By Theorem 2.2, we know that $T_e^\natural(0) \oplus T_e^\natural(1/2)$ forms a simple SVOA and $T_e^\natural(1/16)$ is an irreducible $\mathbb{Z}_2$-twisted $T_e^\natural(0) \oplus T_e^\natural(1/2)$-module. Moreover, the algebraic structures on $T_e^\natural(0) \oplus T_e^\natural(1/2)$ and $T_e^\natural(1/16)$ are independent of choice of a conformal vector $e = e^1 \in V^\natural$ because all the conformal vectors with central charge $1/2$ are conjugate under $\mathbb{M} = \mathrm{Aut}(V^\natural)$.

Lemma 4.3. *$C_{\mathrm{Aut}(V^\natural)}(e)/\langle \tau_e \rangle$ is the baby-monster sporadic finite simple group $\mathbb{B}$.*

Proof. It is shown in [C] and [M1] that the map $e \mapsto \tau_e$ defines a one-to-one correspondence between conformal vectors in $V^\natural$ with central charge $1/2$ and involutions of 2A-conjugacy class of $\mathbb{M}$. Therefore, $C_{\mathrm{Aut}(V^\natural)}(e) = C_{\mathrm{Aut}(V^\natural)}(\tau_e)$. We know that $C_{\mathbb{M}}(\tau_e)$ is isomorphic to a 2-fold central extension $\langle \tau_e \rangle \cdot \mathbb{B}$ of the baby-monster simple group $\mathbb{B}$ (cf. [ATLAS]). So the assertion holds. □

By the lemma above, the commutant subalgebra $T_e^\natural(0)$ affords an action of $\mathbb{B}$. We set $VB^0 := T_e^\natural(0)$, $VB^1 := T_e(1/2)$ and $VB := T_e^\natural(0) \oplus T_e^\natural(1/2)$ and we call VB the *baby-monster vertex operator superalgebra*. We also set $VB_T := T_e^\natural(1/16)$ for convention. Now we state our main result which gives a new proof of [Hö2].

Theorem 4.4.

(1) $\mathrm{Aut}(VB^0) \simeq \mathbb{B}$ *and* $\mathrm{Aut}(VB) \simeq 2 \times \mathbb{B}$.
(2) *VB_T as a VB^0-module is irreducible. Thus, there are exactly three irreducible VB^0-modules, VB^0, VB^1 and VB_T.*
(3) *The fusion rules for irreducible VB^0-modules are as follows*:

$$VB^1 \times VB^1 = VB^0, \qquad VB^1 \times VB_T = VB_T, \qquad VB_T \times VB_T = VB^0 + VB^1.$$

Proof. (1) follows from Theorem 2.11 and Lemma 4.3. By Corollary 2.12, VB_T as a VB^0-module is irreducible. Then (2) follows from Proposition 2.8. Consider (3). We only have to show the fusion rule $VB_T \times VB_T = VB^0 + VB^1$. By considering the $1/16$-word decomposition of VB_T, we have $VB_T \times VB_T = n_0 VB^0 + n_1 VB^1$ for some $n_0, n_1 \in \mathbb{N}$. Since top weights of VB^0, VB^1 and VB_T are distinct, every irreducible VB^0-module is self-dual. Then by the S_3-symmetry of fusion rules we obtain the desired fusion rule. □

The classification of irreducible VB^0-modules has interesting corollaries.

Corollary 4.5. *The irreducible 2A-twisted $V^\natural$-module as an $L(1/2,0)\otimes VB^0$-module has a shape*

$$L(1/2,1/2)\otimes VB^0\oplus L(1/2,0)\otimes VB^1\oplus L(1/2,1/16)\otimes VB_T.$$

Proof. Follows from Theorems 4.4, 2.7 and Proposition 2.8. □

Remark 4.6. A straightforward construction of the 2A-twisted and 2B-twisted $V^\natural$-modules is already obtained by Lam [L].

Corollary 4.7. *For any conformal vector $e\in V^\natural$ with central charge $1/2$, there is no automorphism ρ on $V^\natural$ such that $\rho(V_e^\natural(h))=V_e^\natural(h)$ for $h=0,1/2$ and $\rho|_{(V^\natural)^{\langle\tau_e\rangle}}=\sigma_e$.*

Proof. Suppose such an automorphism ρ exists. We remark that ρ also preserves the space $V_e^\natural(1/16)$ as $\rho\in C_{\mathrm{Aut}(V^\natural)}(e)$. We view $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module by a restriction of the vertex operator map $Y_{V^\natural}(\cdot,z)$ on $V^\natural$. Consider the σ_e-conjugate $(V^\natural)^{\langle\tau_e\rangle}$-module $V_e^\natural(1/16)^{\sigma_e}$. By Theorem 4.4 and Proposition 2.8, $V_e^\natural(1/16)^{\sigma_e}$ is not isomorphic to $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module. On the other hand, we can take a canonical linear isomorphism $\varphi: V_e^\natural(1/16)\to V_e^\natural(1/16)^{\sigma_e}$ such that $Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi v=\varphi Y_{V^\natural}(\sigma_e a,z)v$ for any $a\in(V^\natural)^{\langle\tau_e\rangle}$ and $v\in V_e^\natural(1/16)$ by definition of the conjugate module. Then we have

$$Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi\rho v=\varphi Y_{V^\natural}(\sigma_e a,z)\rho v=\varphi Y_{V^\natural}(\rho a,z)\rho v=\varphi\rho Y_{V^\natural}(a,z)v$$

for any $a\in(V^\natural)^{\langle\tau_e\rangle}$ and $v\in V_e^\natural(1/16)$. Thus $\varphi\rho$ defines a $(V^\natural)^{\langle\tau_e\rangle}$-isomorphism between $V_e^\natural(1/16)$ and $V_e^\natural(1/16)^{\sigma_e}$, which is a contradiction. □

Corollary 4.8. *The 2A-orbifold construction applied to the moonshine VOA $V^\natural$ yields $V^\natural$ itself again.*

Proof. Follows from Theorem 4.4 and Corollary 2.12. □

Remark 4.9. The statement in the corollary above was conjectured by Tuite [Tu]. In [Tu], Tuite has shown that any $\mathbb{Z}_p$-orbifold construction of $V^\natural$ yields either the moonshine VOA $V^\natural$ or the Leech lattice VOA V_Λ under the uniqueness conjecture of the moonshine VOA which states that $V^\natural$ constructed by Frenkel et al. [FLM] is the unique holomorphic VOA with central charge 24 whose weight one subspace is trivial.

Finally, we end this paper by presenting the modular transformations of characters of VB^0-modules. Here the character means the conformal character, not the q-dimension, of modules. Recall the characters of $L(1/2,0)$-modules. By an explicit construction of $L(1/2,0)$-modules in Section 2.1 (cf. [FFR]), one can easily prove the following:

$$\mathrm{ch}_{L(1/2,0)}(\tau)=\frac{1}{2}q^{-1/48}\left\{\prod_{n=0}^{\infty}\left(1+q^{n+1/2}\right)+\prod_{n=0}^{\infty}\left(1-q^{n+1/2}\right)\right\},$$

$$\mathrm{ch}_{L(1/2,1/2)}(\tau)=\frac{1}{2}q^{-1/48}\left\{\prod_{n=0}^{\infty}\left(1+q^{n+1/2}\right)-\prod_{n=0}^{\infty}\left(1-q^{n+1/2}\right)\right\},$$

$$\mathrm{ch}_{L(1/2,1/16)}(\tau)=q^{-1/24}\prod_{n=1}^{\infty}\left(1+q^{n}\right).$$

The following modular transformations are well known:

$$\mathrm{ch}_{L(1/2,0)}(-1/\tau)=\frac{1}{2}\,\mathrm{ch}_{L(1/2,0)}(\tau)+\frac{1}{2}\,\mathrm{ch}_{L(1/2,1/2)}(\tau)+\frac{1}{\sqrt{2}}\,\mathrm{ch}_{L(1/2,1/16)}(\tau),$$

$$\mathrm{ch}_{L(1/2,1/2)}(-1/\tau)=\frac{1}{2}\,\mathrm{ch}_{L(1/2,0)}(\tau)+\frac{1}{2}\,\mathrm{ch}_{L(1/2,1/2)}(\tau)-\frac{1}{\sqrt{2}}\,\mathrm{ch}_{L(1/2,1/16)}(\tau),$$

$$\mathrm{ch}_{L(1/2,1/16)}(-1/\tau)=\frac{1}{\sqrt{2}}\,\mathrm{ch}_{L(1/2,0)}(\tau)-\frac{1}{\sqrt{2}}\,\mathrm{ch}_{L(1/2,1/2)}(\tau).$$

Set $j(\tau):=J(\tau)-744$, where $J(\tau)$ is the famous $\mathrm{SL}_2(\mathbb{Z})$-invariant. Since $\mathrm{ch}_{V^\natural}(\tau)=j(\tau)$ and

$$\mathrm{ch}_{V^\natural}(\tau)=\mathrm{ch}_{L(1/2,0)}(\tau)\,\mathrm{ch}_{VB^0}(\tau)+\mathrm{ch}_{L(1/2,1/2)}(\tau)\,\mathrm{ch}_{VB^1}(\tau)+\mathrm{ch}_{L(1/2,1/16)}(\tau)\,\mathrm{ch}_{VB_T}(\tau),$$

we can write down the characters of irreducible VB^0-modules by using those of $V^\natural$ and $L(1/2,0)$-modules. This computation is already done in [Ma] by using Matsuo–Norton trace formula. The results are written as a rational expression involving the functions $j(\tau)$, $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h=0,1/2,1/16$, their first and second derivatives and the Eisenstein series $E_2(\tau)$ and $E_4(\tau)$, see [Ma].

By Zhu's theorem [Z], the linear space spanned by $\{\mathrm{ch}_{VB^0}(\tau),\mathrm{ch}_{VB^1}(\tau),\mathrm{ch}_{VB_T}(\tau)\}$ affords an $\mathrm{SL}_2(\mathbb{Z})$-action. Using the modular transformations for $j(\tau)$ and $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h=0,1/2,1/16$, we can show the following modular transformations:

$$\mathrm{ch}_{VB^0}(-1/\tau)=\frac{1}{2}\,\mathrm{ch}_{VB^0}(\tau)+\frac{1}{2}\,\mathrm{ch}_{VB^1}(\tau)+\frac{1}{\sqrt{2}}\,\mathrm{ch}_{VB_T}(\tau),$$

$$\mathrm{ch}_{VB^1}(-1/\tau)=\frac{1}{2}\,\mathrm{ch}_{VB^0}(\tau)+\frac{1}{2}\,\mathrm{ch}_{VB^1}(\tau)-\frac{1}{\sqrt{2}}\,\mathrm{ch}_{VB_T}(\tau),$$

$$\mathrm{ch}_{VB_T}(-1/\tau)=\frac{1}{\sqrt{2}}\,\mathrm{ch}_{VB^0}(\tau)-\frac{1}{\sqrt{2}}\,\mathrm{ch}_{VB^1}(\tau).$$

Namely, we have exactly the same modular transformation laws for the Ising model $L(1/2,0)$. As in Theorem 4.4, we also note that the fusion algebra for VB^0 is also canonically isomorphic to that of $L(1/2,0)$. Therefore, we may say that $L(1/2,0)$ and VB^0 form a dual-pair inside the moonshine VOA $V^\natural$.

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