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Foundations of a Connectivity Theory for Simplicial Complexes[1]

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This paper lays the foundations of a combinatorial homotopy theory, called A-theory, for simplicial complexes, which reflects their connectivity properties. A collection of bigraded groups is constructed, and methods for computation are given. A Seifert–Van Kampen type theorem and a long exact sequence of relative A-groups are derived. A related theory for graphs is constructed as well. This theory provides a general framework encompassing homotopy methods used to prove connectivity results about buildings, graphs, and matroids. © 2001 Academic Press

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1. INTRODUCTION

Simplicial complexes have been used to model connectivity properties of a variety of networks of interacting entities, such as communications, traffic, and biological networks, as well as distributed algorithms. Common representations of the static structure of such networks use graph theory. Simplicial complexes have proven very useful in studying dynamic processes in the network [4]. A first analysis tool, called Q-analysis, was developed by the British physicist R. Atkin [1, 2], who modeled social networks by simplicial complexes, with the simplices representing highly connected subnetworks. He then studied clusters of simplices any two of which were connected by a chain of simplices in which successive ones share faces of certain dimensions. Subsequently, Q-analysis has been used in a variety of social and biological settings. As part of a research project on decision networks, Kramer and Laubenbacher, following suggestions in Atkin's papers, began developing a general connectivity theory of simplicial complexes, modeled on classical higher homotopy theory of spaces [6]. In this theory, the invariants of Q-analysis play the role of the set of connected components of the space.

In this paper we develop a complete theory of connectivity for simplicial complexes, and a related theory for graphs. The theory takes the form of a bigraded family of groups

$$A_n^q(\Delta, \sigma_0), \qquad n \geq 1,\ 0 \leq q \leq \dim(\Delta),$$

for a simplicial complex Δ with distinguished base simplex σ_0 of dimension greater than or equal to q. In honor of Atkin, we call it A-theory. These groups are similar in nature to the homotopy groups of a topological space, but are quite different from the homotopy groups of Δ viewed as a space. They reflect connectivity properties of Δ in various dimensions, in the sense that the higher dimensional groups, just like the corresponding groups in topology, detect higher order structure in Atkin's connected clusters.

Based on exploratory research we are confident that A-theory is useful not only in applied problems, but also in a variety of mathematical areas, especially combinatorics, where simplicial complexes and graphs are ubiquitous. Subjects we have explored, with promising preliminary results, include order complexes of posets, in particular geometric lattices, and buildings.

The paper is organized as follows. In Section 2 we give the definition and basic properties of the group $A_1^q(\Delta, \sigma_0)$, which is the analog of the fundamental group of a topological space. In analogy to topology we prove a Seifert–Van Kampen type theorem which describes the behavior of A_1^q under decomposition of Δ into the union of subcomplexes. We also describe an algorithm to compute these groups. In Section 3, we define the higher dimensional absolute A-groups. Then in Section 4, we define the relative A-groups and derive a long exact sequence connecting the absolute A-groups of a complex and a subcomplex. Section 5 contains a related theory, denoted A_n^G, for graphs. The group $A_1^G(\Gamma)$ of a graph Γ turns out to be isomorphic to the fundamental group of the space X_Γ obtained from Γ by attaching 2-cells to all triangles and quadrilaterals. This group has been used by a number of authors to prove some connectivity results for buildings, graphs, and matroids. We mention results by Maurer [9] and Lovász [7]. Our theory provides a general setting in which their constructions arise naturally. For applications to buildings see [11, Chap. 6]. Finally, we relate the theories for graphs and simplicial complexes. Associated to Δ is its q-connectivity graph $\Gamma^q(\Delta)$, and $A_n^q(\Delta) \cong A_n^G(\Gamma^q(\Delta))$. In particular, if the graph Γ is viewed as a one-dimensional simplicial complex, then $A_n^0(\Gamma) \cong A_n^G(\Gamma^*)$, where Γ^* is the graph, whose vertices correspond to the edges of Γ, and two vertices are connected by an edge if the corresponding edges in Γ share a vertex. Sometimes Γ^* is called the *line graph* of Γ. The last section contains a brief conclusion.

2. THE COMBINATORIAL FUNDAMENTAL GROUP

2.1. *Definitions.* In this section we define a combinatorial analog for a simplicial complex of the fundamental group of a topological space and demonstrate its similarities to, as well as differences from, the fundamental group of topology. We also give an algorithm to present this group in terms of generators and relations and prove a Seifert–Van Kampen type theorem. The section concludes with some computations.

Throughout this section, let Δ be a simplicial complex of dimension d, let $0 \leq q \leq d$ be fixed, and let $\sigma_0 \in \Delta$ be a maximal simplex (with respect to inclusion) of dimension greater than or equal to q. (The maximality assumption is not strictly necessary. The whole theory can be developed without this assumption, but it avoids many technicalities.)

We first define the notion of q-connectivity and a discrete analog of a loop in Δ.

DEFINITION 2.1. (1) Two simplices σ and τ are *q-connected*, if there is a sequence of simplices

$$\sigma, \sigma_1, \sigma_2, \ldots, \sigma_n, \tau,$$

such that any two consecutive ones share a q-face, that is, they have at least $q+1$ vertices in common. Such a chain will be called a *q-chain*.

(2) The complex Δ is *q-connected*, if any two simplices in Δ of dimension greater than or equal to q are q-connected.

(3) A *q-loop in Δ based at σ_0* is a q-chain beginning and ending at σ_0. Denote a q-loop $\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_0$ by $(\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_0) = (\sigma)$. Its *length* is n. (Note that the σ_i need not be distinct.)

Note that if $\dim(\sigma_0) = q$, then $\sigma_0 = \sigma_i$ for all i, since σ_0 was assumed to be maximal. Also, if $\dim(\sigma_0) = d$, then the only q-loop based at σ_0 is the constant loop. Combinatorial q-loops of simplices can be thought of as "fattened up" ordinary loops. Two such loops are A-homotopic if they can be deformed into each other without breaking any q-dimensional connections. Following is a formal definition of A-homotopy of loops.

DEFINITION 2.2. Let $\simeq_A$ be the equivalence relation on the collection of q-loops in Δ, based at σ_0, generated by the following two conditions.

(1) The q-loop

$$(\sigma) = (\sigma_0, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0)$$

is equivalent to the q-loop

$$(\sigma') = (\sigma_0, \ldots, \sigma_i, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0).$$

That is, we can "stretch" loops by repeating a simplex without changing its equivalence class.

(2) Suppose that (σ) and (τ) have the same length. They are equivalent if there is a diagram as in Fig. 1. The diagram is to be interpreted as follows. A horizontal or vertical edge between two simplices indicates that they share a q-face. Each horizontal row in the diagram is a q-loop based at σ_0. Thus, (σ) is equivalent to (τ) $\big((\sigma) \simeq_A (\tau)\big)$ if there is a sequence of q-loops based at σ_0 connecting them. Such a diagram is said to be an A-homotopy between (σ) and (τ).

We call this equivalence relation *A-homotopy* and denote the equivalence class of a loop (σ) by $[\sigma]$. Denote the set of equivalence classes by $A_1^q(\Delta, \sigma_0)$.

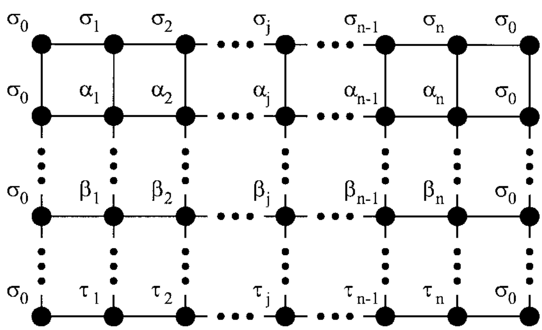

FIG. 1. Equivalent q-loops.

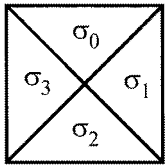

FIG. 2. A 2-dimensional complex with $A_1^1 = 1$.

The prefix "A" is intended to distinguish this notion of homotopy from the one used in algebraic topology. It is also not to be confused with the combinatorial definition of classical homotopy in simplicial complexes. It is straightforward to verify that A-homotopy defines an equivalence relation on the set of q-loops of Δ, for any q. As in the topological case, we can concatenate q-loops. If (σ) and (τ) are q-loops based at σ_0, then we denote by $(\sigma)(\tau)$ the loop obtained by first traversing (σ) followed by (τ). The proof of the following proposition is straightforward and left to the reader.

PROPOSITION 2.3. *Concatenation of q-loops defines a group structure on* $A_1^q(\Delta, \sigma_0)$. *The unit element is the equivalence class of the constant* (*or trivial*) *loop* (σ_0), *and the inverse of* $[\sigma]$ *is given by the equivalence class of the same loop traversed in the opposite direction.*

We therefore obtain a family $\{A_1^q(\Delta, \sigma_0)\}$ of groups, one for each $0 \leq q \leq \dim(\Delta)$.

EXAMPLES. (1) Let Δ_1 be the 2-dimensional complex given in Fig. 2, and let $q = 1$.

Then the loop $(\sigma) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_0)$ is A-homotopic to the constant loop (σ_0). Indeed, Fig. 3 gives an A-homotopy which contracts (σ) to the trivial loop. Thus, $A_1^1(\Delta_1, \sigma_0) = 1$, the trivial group.

(2) Let Δ_2 be the 2-dimensional complex given in Fig. 4, with $q = 1$. Then it is a straightforward exercise to show that $A_1^1(\Delta_2, \sigma_0) \cong \mathbf{Z}$, generated by the equivalence class of the loop

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0).$$

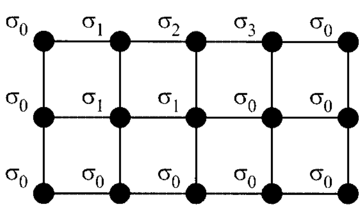

FIG. 3. Contraction of the 4-loop.

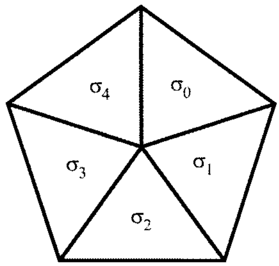

FIG. 4. A 2-dimensional complex Δ_2 with nontrivial A_1^1.

This example indicates that the A-combinatorial fundamental group is very different from the topological one, since Δ_2, viewed as a topological space, is contractible.

(3) Let Δ_3 be the complex obtained from Δ_2 by adding the simplex σ_5, as shown in Fig. 5. Then $A_1^1(\Delta_3, \sigma_0) = 1$, since the extra simplex σ_5 allows the contraction of the nontrivial loop in Δ_2. One can show this using Theorem 2.7, proven later in this section. This example shows that the A-combinatorial fundamental group is similar in nature to the topological one, in the sense that its nontrivial elements correspond to "combinatorial holes" in the complex.

(4) Let Δ_4 be the complete graph on four vertices, viewed as a 1-dimensional simplicial complex. Then $A_1^0(\Delta_4) = 1$, with any choice of base simplex, whereas the topological fundamental group $\pi_1(\Delta_4)$ is a free group on three generators.

(5) Figures 6 and 7 illustrate the concept of A-homotopy further. The former gives two homotopic loops and the latter two nonhomotopic ones. This can be seen as in Examples 1 and 2.

PROPOSITION 2.4. *If σ_0 and τ_0 are maximal simplices in Δ that are q-connected, then $A_1^q(\Delta, \sigma_0) \cong A_1^q(\Delta, \tau_0)$.*

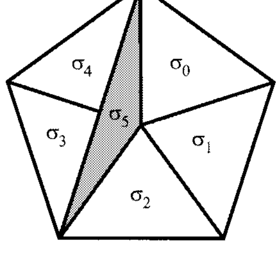

FIG. 5. Filling the combinatorial hole in Δ_2.

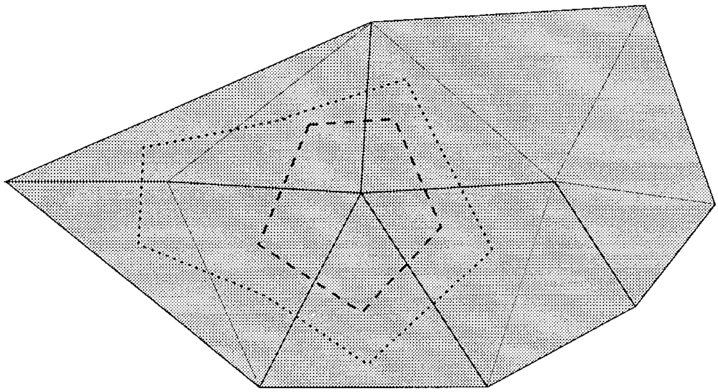

FIG. 6. Homotopic loops.

Proof. As in the topological case, an isomorphism is given by a choice of a q-chain connecting σ_0 and τ_0. ∎

We now give a description of the A-combinatorial fundamental group $A_1^q(\Delta, \sigma_0)$ in terms of generators and relations. This description provides the basis for an algorithm to compute its abelianization. Let $\Gamma = \Gamma^q(\Delta)$ be the graph with vertices corresponding to all simplices of Δ of dimension greater than or equal to q. Two vertices v and w are connected by an edge if and only if the corresponding simplices σ and τ share a q-face. Let v_0 be the distinguished vertex of Γ, corresponding to σ_0. Observe that there is a one-to-one correspondence between q-loops in Δ based at σ_0 and cycles in Γ that contain v_0. In Fig. 1, the vertices represent simplices, and the edges represent q-connection. Thus, two cycles in Γ containing v_0 correspond to A-homotopic loops in Δ if and only if they differ by 4-cycles.

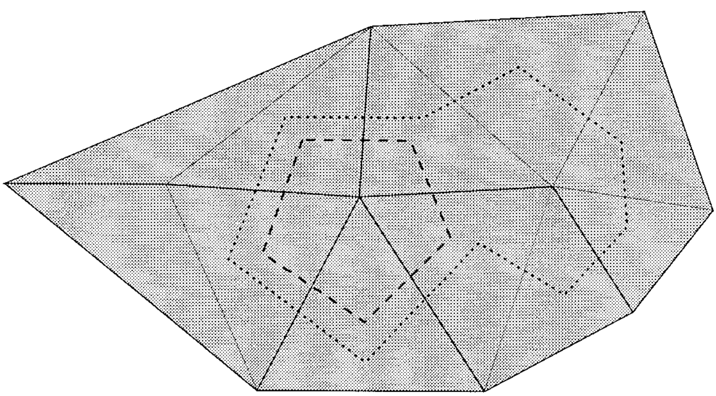

FIG. 7. Nonhomotopic loops.

One needs to be careful, however. Since simplices can be repeated in a q-loop some of these 4-cycles might in fact just be 3-cycles. Hence, two cycles in Γ correspond to A-homotopic q-loops in Δ if and only if they differ by 3- and 4-cycles.

Recall that the topological fundamental group $\pi_1(\Gamma, v_0)$ is a free group with free generators constructed in the following manner. Put a total order on the set of vertices of Γ, beginning with v_0, and choose a spanning tree T of Γ rooted at v_0. A free set of generators is in one-to-one correspondence with the edges in $\Gamma \setminus T$, subsequently referred to as "missing" edges. Namely, for each missing edge $e = uv$, let γ_e be the cycle obtained by going from v_0 to u along the unique path in T, continuing across e and back to v_0 along the unique path in T from v to v_0.

To each 3- and 4-cycle we can associate a specific element of $\pi_1(\Gamma, v_0)$ as follows. If $uvwu$ is a 3-cycle, with $u < v < w$, let g be the element of $\pi_1(\Gamma, v_0)$ represented by the cycle obtained by going from v_0 to u in T, then across uv, vw, and wu, and back to v_0 via T. We make a similar construction for 4-cycles. Let $g_1, \ldots, g_n$ be the elements constructed in this way from all 3- and 4-cycles, and let N be the normal subgroup of $\pi_1(\Gamma, v_0)$ generated by them. Let $(\tau_1), \ldots, (\tau_n)$ be the corresponding q-loops in Δ. Then each (τ_i) is homotopic to the trivial loop (σ_0).

LEMMA 2.5. *Let $uvwzu$ be a* 3- *or* 4-*cycle in* Γ, *and let p be a path in* Γ *from the base vertex v_0 to u. Then there exists an i and a cycle γ in* Γ *such that the cycle $puvwzup^{-1}$ is A-homotopic to the cycle $\gamma\tau_i\gamma^{-1}$, where τ_i induces the generator g_i of N given by the cycle $uvwzu$.*

Proof. We carry out the proof for a 4-cycle. The case of a 3-cycle is similar. For the sake of fixing notation, let wz be the edge not contained in the spanning tree, so that τ_i is given by $qwzuvwq^{-1}$, where q is the path from v_0 to w along the spanning tree. Then we obtain

$$puvwzup^{-1} \simeq_A (puvwq^{-1})(qwzuvwq^{-1})(qwvup^{-1}) = \gamma\tau_i\gamma^{-1}.$$

This completes the proof. ∎

LEMMA 2.6. *Let* $(\sigma) = (\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_0)$ *and* $(\rho) = (\sigma_0, \rho_1, \ldots, \rho_n, \sigma_0)$ *be A-homotopic q-loops in* Δ. *Then*

$$(\sigma)(\rho)^{-1} \simeq_A (\alpha_1)(\tau_{i_1})(\alpha_1)^{-1} \cdots (\alpha_r)(\tau_{i_r})(\alpha_r)^{-1},$$

for some $1 \leq i_1, \ldots, i_r \leq n$ *and some loops* (α_i).

Proof. It is clearly sufficient to show that the lemma holds if (σ) and (ρ) are two adjacent loops as represented in Fig. 8. Observe now

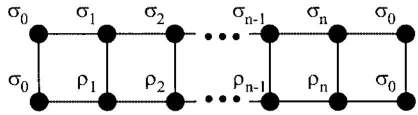

FIG. 8. Two adjacent loops.

that

$$(\sigma)(\rho)^{-1} \simeq_A (\sigma_0, \sigma_1, \rho_1, \sigma_0)(\sigma_0, \rho_1, \sigma_1, \sigma_2, \rho_2, \rho_1, \sigma_0)$$
$$(\sigma_0, \rho_1, \rho_2, \sigma_2, \sigma_3, \rho_3, \rho_2, \rho_1, \sigma_0)\cdots$$
$$(\sigma_0, \rho_1, \ldots, \rho_n, \sigma_n, \sigma_0, \rho_n, \rho_{n-1}, \ldots, \sigma_0).$$

This expression induces a homotopy equivalence of cycles in the graph Γ, based at v_0. Each of the factors on the right-hand side of this expression corresponds to a 3-or 4-cycle in Γ, connected by a path to v_0. By Lemma 2.5 each of these cycles is the conjugate of some τ_i. This completes the proof. ∎

THEOREM 2.7. $A_1^q(\Delta, \sigma_0) \cong \pi_1(\Gamma, v_0)/N$.

Proof. Define a mapping

$$\varphi : \pi_1(\Gamma, v_0) \longrightarrow A_1^q(\Delta, \sigma_0)$$

as follows. Let $[\gamma] \in \pi_1(\Gamma, v_0)$, with representative cycle γ. Define $\varphi([\gamma])$ to be the A-homotopy class of the q-loop in Δ corresponding to γ. It is clear that this map is an onto homomorphism, provided it is well-defined. The equivalence relation defining the elements of $\pi_1(\Gamma, v_0)$ is given by making all cycles of the form

$$v_0 v_1 \cdots v_{n-1} v_n v_{n-1} \cdots v_1 v_0$$

equivalent to the constant cycle at v_0. Thus it is clear that φ is well-defined. It remains to show that $\ker(\varphi) = N$.

Clearly, each generator of N is sent to a loop in Δ that is A-homotopic to the trivial loop (σ_0), which shows that N is contained in the kernel of φ. Conversely, let γ be a cycle in Γ containing v_0 such that $\varphi([\gamma]) = [\sigma] = [\sigma_0]$. Then there exists an A-homotopy from (σ) to (σ_0). Lemma 2.6 implies that (σ) is homotopic to a concatenation of (τ_i), hence $[\gamma]$ lies in N. This completes the proof. ∎

In Theorem 5.16 we will show that in order to compute A_1^q, one can replace Γ^q by the generally much smaller graph, $\Gamma^q_{\max}$, whose vertices correspond to all maximal simplices (with respect to inclusion) of Δ of dimension greater than or equal to q.

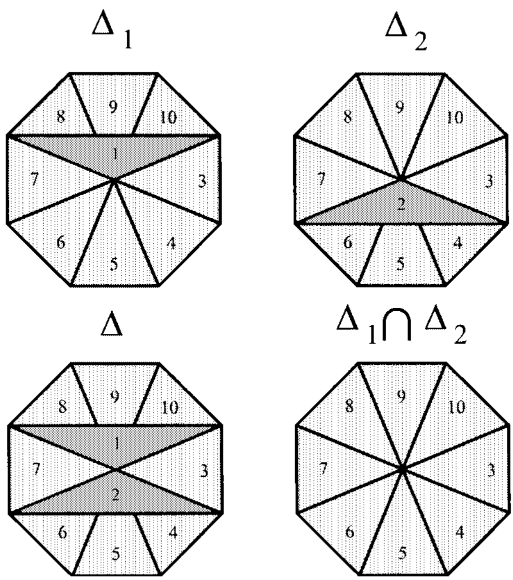

FIG. 9. $\Delta = \Delta_1 \cup \Delta_2$.

2.2. *A Seifert–Van Kampen Theorem.* It is natural to ask whether the combinatorial fundamental group of a simplicial complex satisfies a Seifert–Van Kampen type property. This amounts to asking whether one can compute A_1^q of a simplicial complex as the free product of A_1^q of appropriate subcomplexes modulo A_1^q of their intersection. While this is indeed true, an extra condition is required on the intersection of the subcomplexes, in addition to requiring q-connectivity of the complex, the subcomplexes, and the intersection. To illustrate this condition consider the following example. Let Δ be the complex given in Fig. 9.

Let $q = 1$, and let Δ_1, respectively Δ_2, be obtained by deleting simplex (2), respectively (1), from Δ. See Fig. 9. In order to compute the A_1^1-groups of the various complexes involved, we now have to consider their respective Γ^1-graphs. As mentioned above one can replace Γ^q with the much smaller graph $\Gamma^q_{\max}$ whose vertices are the maximal simplices of dimension greater than or equal to q.

From the $\Gamma^1_{\max}$-graphs in Fig. 10 we compute:

$$A_1^1(\Delta_1) \cong A_1^1(\Delta_2) \cong A_1^1(\Delta_1 \cap \Delta_2) \cong \mathbf{Z}.$$

If a Seifert–Van Kampen result would apply to this decomposition, then this would yield that $A_1^1(\Delta) \neq 1$. But it is not difficult to see from the $\Gamma^1_{\max}$-graph that $A_1^1(\Delta) = 1$, for any choice of base simplex in $\Delta_1 \cap \Delta_2$. Therefore a Seifert–Van Kampen Theorem cannot hold for this choice of subcomplexes.

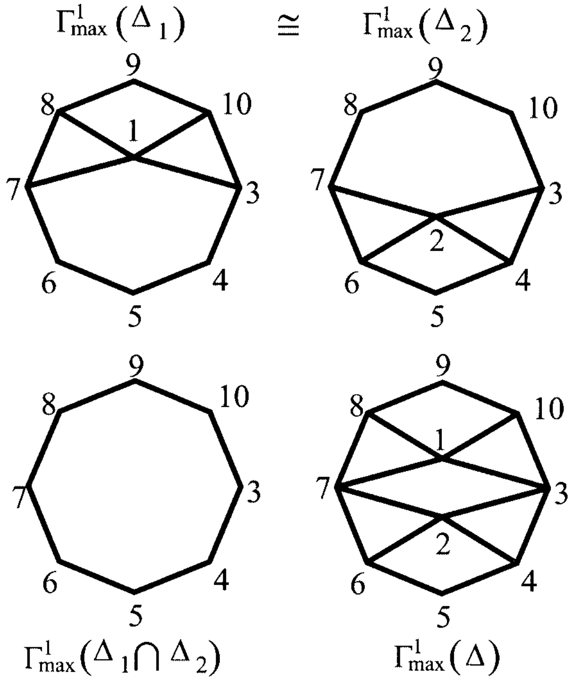

FIG. 10. The graph $\Gamma^1_{\max}(\Delta)$.

The problem lies in the fact that the 4-cycle $(1)-(3)-(2)-(7)-(1)$ in $\Gamma^1_{\max}(\Delta)$ is split up between Δ_1 and Δ_2. Its existence is the reason, however, that $A_1^1(\Delta) = 1$.

If we choose instead Δ_1 to contain the simplices 1, 2, 3, 7, 8, 9, 10, and Δ_2 to contain 1, 2, 3, 4, 5, 6, 7, then this 4-cycle lies entirely in the intersection (is in fact equal to it), and we get indeed consistent computations, with all groups being trivial. This example shows that the basic elements of the calculation of A_1, namely 3- and 4-cycles in $\Gamma^q(\Delta)$, must be preserved in the decomposition in order to have a chance of a decomposition theorem. The next result shows that this is indeed the only possible obstruction.

THEOREM 2.8. *Let Δ be a simplicial complex of dimension d, let $0 \leq q \leq d$, and let σ_0 be a maximal simplex (with respect to inclusion) of Δ of dimension greater than or equal to q. Assume that Δ is q-connected. Let Δ_i, $i = 1, 2$, be q-connected subcomplexes of Δ, such that $\Delta_1 \cup \Delta_2 = \Delta$, and $\Delta_1 \cap \Delta_2$ is q-connected and contains σ_0. Suppose further that the following condition is satisfied.*

($*$) *Let* $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ *be* (*not necessarily distinct*) *simplices in* Δ, *such that* $\dim(\sigma_i) \geq q$ *for* $i = 1, \ldots, 4$. *If these simplices form a* q-*loop*, *then they all lie in one of* Δ_1 *or* Δ_2. *See Fig.* 11.

Then there is an isomorphism

$$A_1^q(\Delta, \sigma_0) \cong A_1^q(\Delta_1, \sigma_0) * A_1^q(\Delta_2, \sigma_0)/V,$$

where V *is the normal subgroup of the free product generated by all elements of the form* $[l] * [l]^{-1}$ *for* l *a loop in* $\Delta_1 \cap \Delta_2$ *based at* σ_0.

Proof. Consider the graph $\Gamma = \Gamma^q(\Delta)$, whose vertices are all simplices of Δ of dimension greater than or equal to q, and whose edges (σ, τ) are given by pairs of simplices σ, τ that share a q-face. Choose σ_0 as base vertex and view $\Gamma = \Gamma^q(\Delta)$ as a simplicial complex. We first show that the classical Seifert–Van Kampen Theorem applies to Γ with the connectivity graphs of Δ_1 and Δ_2 as subcomplexes. Since Δ is q-connected, it follows that Γ is connected. The same holds true for the subgraphs $\Gamma_1 = \Gamma^q(\Delta_1)$ and $\Gamma_2 = \Gamma^q(\Delta_2)$, viewed as subcomplexes of Γ. Observe that

$$\Gamma_1 \cap \Gamma_2 = \Gamma^q(\Delta_1 \cap \Delta_2),$$

and, since $\Delta_1 \cap \Delta_2$ is q-connected, $\Gamma_1 \cap \Gamma_2$ is also connected. Moreover, by assumption, $\Gamma_1 \cap \Gamma_2$ contains σ_0. Finally, note that it is always true that

$$\Gamma_1 \cup \Gamma_2 \subset \Gamma^q(\Delta_1 \cup \Delta_2) = \Gamma.$$

Condition ($*$) implies that equality holds, since it forces every edge in Γ to be contained in Γ_1 or Γ_2. To see this, note that every edge in Γ between vertices corresponding to simplices σ_1 and σ_2 gives rise to the q-loop $(\sigma_1, \sigma_2, \sigma_2, \sigma_1)$, which lies in Δ_1 or Δ_2, by Condition ($*$). Figure 11

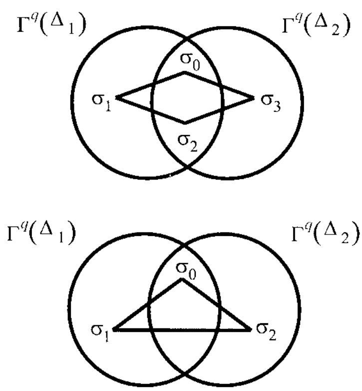

FIG. 11. Inadmissible configurations for $\Gamma^q(\Delta_1 \cup \Delta_2)$.

illustrates the inadmissible configurations in $\Gamma^q(\Delta) = \Gamma^q(\Delta_1 \cup \Delta_2)$ that Condition $(*)$ imposes on the intersection $\Gamma^q(\Delta_1) \cap \Gamma^q(\Delta_2)$.

Therefore the conditions of the Seifert–Van Kampen Theorem for the simplicial complex Γ are satisfied. We obtain an isomorphism

$$\pi_1(\Gamma, \sigma_0) \cong [\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0)/V'],$$

where V' is the normal subgroup of the free product generated by all elements of the form $[l] * [l^{-1}]$, where l is a loop in $\Gamma_1 \cap \Gamma_2$.

Recall that

$$A_1^q(\Delta, \sigma_0) = \pi_1(\Gamma, \sigma_0)/N,$$

where N is the normal subgroup generated by loops corresponding to 3- and 4-cycles in Γ. Similarly,

$$A_1^q(\Delta_i, \sigma_0) = \pi_1(\Gamma_i, \sigma_0)/N_i, \qquad i = 1, 2,$$

and

$$A_1^q(\Delta_1 \cap \Delta_2, \sigma_0) = \pi_1(\Gamma^q(\Delta_1 \cap \Delta_2), \sigma_0)/N_{12}.$$

In order to compute $\pi_1(\Gamma, \sigma_0)$ proceed as follows. First take a spanning tree of $\Gamma_1 \cap \Gamma_2$ and then extend it to spanning trees of Γ_1 and Γ_2, respectively. Then their union is a spanning tree of Γ. In this way we can view N_1, N_2, and N_{12} as subgroups of N on subsets of the generators of N.

Condition $(*)$ implies that every generator of N is a generator of N_1 or N_2, and vice versa. Therefore, N is the quotient of $N_1 * N_2$ modulo the normal subgroup V_{12} generated by all elements $[l] * [l^{-1}]$, for $[l] \in N_{12}$. Note that $V_{12} \subset V'$. Consider the group isomorphism

$$\phi : [\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0)]/V' \longrightarrow \pi_1(\Gamma, \sigma_0),$$

given by $\phi([l_1] * [l_2]) = [l_1][l_2]$, and let K be the normal subgroup of $\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0)$ generated by V' and $N_1 * N_2$. Note furthermore that N_1 and N_2, viewed as subgroups of $\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0)$, are generated by disjoint sets of generators. Therefore,

$$\begin{aligned} A_1^q(\Delta, \sigma_0) &\cong \big((\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0))/V'\big)/\phi^{-1}(N) \\ &\cong [\pi_1(\Gamma_1, \sigma_0) * \pi_1(\Gamma_2, \sigma_0)]/K \\ &\cong (\pi_1(\Gamma_1, \sigma_0)/N_1 * \pi_1(\Gamma_2, \sigma_0)/N_2)/(V'/(V' \cap N_1 * N_2)) \\ &\cong [A_1^q(\Delta_1, \sigma_0) * A_1^q(\Delta_2, \sigma_0)]/(V'/V_{12}). \end{aligned}$$

Furthermore, V'/V_{12} is isomorphic to the image of $A_1^q(\Delta_1 \cap \Delta_2, \sigma_0)$ in $A_1^q(\Delta_1, \sigma_0) * A_1^q(\Delta_2, \sigma_0)$. This completes the proof. ∎

3. HIGHER A-THEORY

In this section we define the higher A-groups of a simplicial complex in a unified manner. The combinatorial fundamental group defined in the previous section is isomorphic to the one defined here.

DEFINITION 3.1. Let Δ be a simplicial complex of dimension d. For $0 \leq q \leq d$ and $n \geq 1$, a *(q, n)-chain in* Δ is a map $f_n : \mathbf{Z}^n \to \Delta$ which satisfies the following conditions:

(1) there is an $N \geq 0$ such that if $(i_1, \ldots, i_n) \in \mathbf{Z}^n$ with $|i_k| \geqslant N$ for some $1 \leq k \leq n$, then

$$f_n(i_1, \ldots, i_n) = f_n(i_1, \ldots, i_{k-1}, \frac{|i_k|}{i_k} N, i_{k+1}, \ldots, i_n);$$

(2) for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$, we have

$$|f_n(i_1, \ldots, i_k, \ldots, i_n) \cap f_n(i_1, \ldots, i_k + 1, \ldots, i_n)| \geq q + 1$$

for all $1 \leq k \leq n$.

For a given (q, n)-chain f_n, we call the least N which satisfies (1) the *radius* of f_n.

A (q, n)-chain can be pictured as an infinite n-dimensional grid with simplices at the lattice points, which are indexed by integer coordinates. In the grid, adjacent simplices share a q-face, and there is an $N \geq 0$ such that in each coordinate of the lattice, the simplex at all lattice points with coordinate greater than N is constant equal to the one at N. Similarly, if a coordinate of the lattice point is less than $-N$, then the simplex is equal to the one at the lattice point where that coordinate is $-N$.

EXAMPLE. Let Δ be the 2-dimensional complex shown in Fig. 2, and let $q = 1$. Then the function $f_2 : \mathbf{Z}^2 \to \Delta$ given by

$$f_2(i_1, i_2) = \begin{cases} \sigma_0 & \text{for } i_1 \leq 0 \text{ or } i_1 \geq 4 \\ \sigma_{|i_1|} & \text{for } -3 \leq i_1 \leq 3 \end{cases}$$

is a $(1, 2)$-chain with radius $N = 4$.

We next define the notion of a based *(q, n)-cube*. Suppose we have a simplicial complex Δ of dimension d, and we choose a maximal simplex $\sigma_0 \in \Delta$ of dimension at least q to be our "base simplex."

DEFINITION 3.2. Let f_n be a (q, n)-chain in Δ with radius N. Then f_n is a *(q, n)-cube* based at σ_0 if for any $(i_1, \ldots, i_n) \in \mathbf{Z}^n$ such that $|i_k| \geq N$ for some k, $1 \leq k \leq n$, we have $f_n(i_1, \ldots, i_n) = \sigma_0$.

That is, a (q, n)-cube is a chain such that all simplices outside an n-dimensional cube of side length $2N$ are equal to the base simplex σ_0.

DEFINITION 3.3. Two (q, n)-cubes f_n, g_n based at σ_0 are *A-homotopic*, denoted $f_n \simeq_A g_n$, if there is a $(q, n+1)$-chain h_{n+1} with radius N such that there are $-N \leq k \leq l \leq N$ so that

(1) $h_{n+1}(i_1, \ldots, i_n, k) = f_n(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$;

(2) $h_{n+1}(i_1, \ldots, i_n, l) = g_n(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$; and

(3) for each $j \in \mathbf{Z}$ the map $h_{n+1}(i_1, \ldots, i_n, j)$ is a (q, n)-cube based at σ_0.

The A-homotopy relation is an equivalence relation on the set of (q, n)-cubes based at σ_0; we will denote the set of equivalence classes by

$$A_n^q(\Delta, \sigma_0).$$

The next lemma says that translation of (q, n)-chains leaves the equivalence class invariant. It is straightforward to prove and will be left to the reader.

LEMMA 3.4. *If $f_n : \mathbf{Z}^n \longrightarrow \Delta$ is a (q, n)-chain, then for any integers $r_1, \ldots, r_n$, the (q, n)-chain $g(x_1, \ldots, x_n) = f_n(x_1 + r_1, \ldots, x_n + r_n)$ is A-homotopic to f_n.*

PROPOSITION 3.5. *The set $A_n^q(\Delta, \sigma_0)$ forms a group under the multiplication defined as follows. Let f, g be two (q, n)-cubes in Δ based at σ_0. Let M and N be the radii of $f$and g, respectively, and define*

$$fg(i_1, \ldots, i_n) = \begin{cases} f(i_1, \ldots, i_n) & \text{for } i_1 \leq M \\ g(i_1 - (M+N), i_2, \ldots, i_n) & \text{for } i_1 > M. \end{cases}$$

This definition is illustrated in Fig. 12.

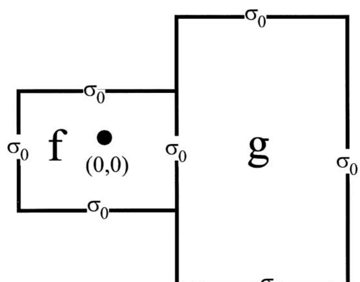

FIG. 12. Multiplication in $A_n^q(\Delta)$.

Proof. We first show that multiplication is a well-defined operation. Let f, f', g, g' be (q, n)-cubes such that $f \simeq_A f'$ and $g \simeq_A g'$. The idea for an A-homotopy between fg and $f'g'$ is simple. We know there are A-homotopies F and G from f to f' and g to g', respectively. Then we can construct an A-homotopy from fg to $f'g'$ by first using F (possibly reindexed) to deform fg into $f'g$, and then G to deform $f'g$ into $f'g'$.

To be precise, since $f \simeq_A f'$, there is a $(q, n+1)$-chain F of radius R and k, l with $-R \le k \le l \le R$, such that

(1) $F(i_1, \ldots, i_n, k) = f(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$;

(2) $F(i_1, \ldots, i_n, l) = f'(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$; and

(3) $F(i_1, \ldots, i_n, j)$ a (q, n)-cube based at σ_0 for each $j \in \mathbf{Z}$.

Similarly, there is a $(q, n+1)$-chain G of radius R' and s, t with $-R' \le s \le t \le R'$, such that

(1) $G(i_1, \ldots, i_n, s) = g(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$;

(2) $G(i_1, \ldots, i_n, t) = g'(i_1, \ldots, i_n)$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$; and

(3) $G(i_1, \ldots, i_n, j)$ is a (q, n)-cube for each $j \in \mathbf{Z}$.

By reindexing F and G, if necessary, we can assume that $s = l$. Now define

$$H : \mathbf{Z}^{n+1} \longrightarrow \Delta$$

by

$$H(i_1, \ldots, i_{n+1}) = \begin{cases} fg(i_1, \ldots, i_n) & \text{if } i_{n+1} \le k, \\ F(-, \ldots, -, i_{n+1})g(i_1, \ldots, i_n) & \text{if } k \le i_{n+1} \le l, \\ f'G(-, \ldots, -, i_{n+1})(i_1, \ldots, i_n) & \text{if } l \le i_{n+1} \le t, \\ f'g'(i_1, \ldots, i_n) & \text{if } i_{n+1} \ge t. \end{cases}$$

It is lengthy, but straightforward, to verify that H is a $(q, n+1)$-chain. This shows that the multiplication on $A_n^q(\Delta, \sigma_0)$ is well-defined.

It is easily seen that multiplication in $A_n^q(\Delta, \sigma_0)$ is associative and that the (q, n)-cube $e : \mathbf{Z}^n \to \Delta$ given by $e(i_1, \ldots, i_n) = \sigma_0$ for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$ is a multiplicative identity. For any (q, n)-cube f with radius M, its inverse is given by

$$f^{-1}(i_1, \ldots, i_n) = f(-i_1, i_2, \ldots, i_n).$$

∎

The group $A_n^q(\Delta, \sigma_0)$ is abelian for $n \ge 2$. The idea of the proof may be seen by considering the case where $n = 2$. Let f and g be $(q, 2)$-cubes of radius M and N, respectively. Picture f and g as squares with sides of length $2M$ and $2N$, respectively, with their product as shown in Fig. 12. An A-homotopy from fg to gf will then slide g up, then slide f to the right while sliding g to the left, and finally slide g down. This A-homotopy is given explicitly in the proof of the following proposition.

PROPOSITION 3.6. *For $n \geq 2$, the group $A_n^q(\Delta, \sigma_0)$ is abelian.*

Proof. Let f, g be (q, n)-cubes of radius M, N, respectively. We will show that $fg \simeq_A gf$. We can define a $(q, n+1)$-chain H that gives this A-homotopy as follows. Pick an index k, such that $2 \leqslant k \leqslant n$. For $i_{n+1} < 0$, let

$$H(i_1, \ldots, i_n, i_{n+1}) = \begin{cases} f(i_1, \ldots, i_n) & \text{for } i_1 \leq M \\ g(i_1 - (M+N), i_2, \ldots, i_k, \ldots, i_n) & \text{for } i_1 > M; \end{cases}$$

for $0 \leq i_{n+1} \leq M + N$, let

$$H(i_1, \ldots, i_n, i_{n+1}) = \begin{cases} f(i_1, \ldots, i_n) & \text{for } i_1 \leq M \\ g(i_1 - (M+N), i_2, \ldots, i_k - i_{n+1}, \ldots, i_n) & \text{for } i_1 > M; \end{cases}$$

for $M + N < i_{n+1} \leq 2(M+N)$, let

$$H(i_1, \ldots, i_n, i_{n+1}) = \begin{cases} f(i_1 + (M+N) - i_{n+1}, i_2, \ldots, i_n) & \text{for } i_k \leq M \\ g(i_1 + 2(M+N) - i_{n+1}, \ldots, i_k - (M+N), \ldots, i_n) & \text{for } i_k > M; \end{cases}$$

for $2(M+N) < i_{n+1} \leq 3(M+N)$, let

$$H(i_1, \ldots, i_n, i_{n+1}) = \begin{cases} f(i_1 - (M+N), i_2, \ldots, i_n) & \text{for } i_1 > N \\ g(i_1, \ldots, i_k - 3(M+N) + i_{n+1}, \ldots, i_n) & \text{for } i_1 \leq N; \end{cases}$$

and for $i_{n+1} > 3(M+N)$, let

$$H(i_1, \ldots, i_n, i_{n+1}) = \begin{cases} g(i_1, \ldots, i_n) & \text{for } i_1 \leq N \\ f(i_1 - (N+M), i_2 \ldots, i_n) & \text{for } i_1 > N. \end{cases}$$

Note that $H(i_1, \ldots, i_n, 0) = fg$ and $H(i_1, \ldots, i_n, 3(M+N)) = gf$. It is straightforward to check that H is a $(q, n+1)$-chain. ∎

The next proposition shows that the definition of multiplication in $A_n^q(\Delta, \sigma_0)$ is independent of the coordinate we choose to glue f and g together.

PROPOSITION 3.7. *Let f, g be two (q, n)-cubes in Δ based at σ_0. Let M and N be the radii of f and g, respectively. The multiplication of $A_n^q(\Delta, \sigma_0)$ is independent of the choice of coordinate in the n-tuple; that is, if we pick $2 \leq k \leq n$ and define*

$$f * g = \begin{cases} f(i_1, \ldots, i_n) & \textit{for } i_k \leq M \\ g(i_1, \ldots, i_k - (M+N), \ldots, i_n) & \textit{for } i_k > M, \end{cases}$$

then $fg \simeq_A f * g$.

Proof. We define a $(q, n+1)$-chain H that gives this A-homotopy as follows. Pick an index k such that $2 \leqslant k \leqslant n$. For $i_{n+1} < 0$, let

$$H(i_1, \dots, i_n, i_{n+1}) = \begin{cases} f(i_1, \dots, i_n) & \text{for } i_1 \leq M \\ g(i_1 - (M+N), i_2, \dots, i_k, \dots, i_n) & \text{for } i_1 > M; \end{cases}$$

For $0 \leq i_{n+1} \leq M + N$, let

$$H(i_1, \dots, i_n, i_{n+1}) = \begin{cases} f(i_1, \dots, i_n) & \text{for } i_1 \leq M \\ g(i_1 - (M+N), i_2, \dots, i_k - i_{n+1}, \dots, i_n) & \text{for } i_1 > M. \end{cases}$$

For $M + N < i_{n+1} \leq 2(M + N)$, let

$$H(i_1, \dots, i_n, i_{n+1}) = \begin{cases} f(i_1, \dots, i_n) & \text{for } i_k \leq M \\ g(i_1 + 2(M+N) - i_{n+1}, \dots, i_k - (M+N), \dots, i_n) & \text{for } i_k > M; \end{cases}$$

and for $i_{n+1} > 2(M + N)$, let

$$H(i_1, \dots, i_n, i_{n+1}) = \begin{cases} f(i_1, \dots, i_n) & \text{for } i_k \leq M \\ g(i_1, \dots, i_k - (M+N), \dots, i_n) & \text{for } i_k > M. \end{cases}$$

Note that $H(i_1, \dots, i_n, 0) = fg$, and $H(i_1, \dots, i_n, 2(M+N)) = f * g$. Furthermore, H is a $(q, n+1)$-chain. ∎

4. RELATIVE A-THEORY AND A LONG EXACT SEQUENCE

In this section we define the relative A-groups of a simplicial complex Δ with respect to a subcomplex Σ and derive the analog of the long exact sequence for relative homotopy theory.

Let $\sigma_0 \in \Sigma \subset \Delta$ be the base simplex. We first define groups

$$A_n^q(\Delta, \Sigma, \sigma_0), \quad n \geq 1,$$

in a way similar to the classical definition of relative homotopy groups, which we recall here. (See, e.g., [5, Chap. 4.3] for details.) Let X be a topological space, $A \subset X$ a subspace, and $x_0 \in A$ a distinguished base point. An element in the nth homotopy group $\pi_n(X, x_0)$ can be defined to be the homotopy class of a continuous map from the n-cube I_n to X which sends the boundary of the n-cube (which is homotopic to the $(n-1)$-sphere) to x_0. To define the relative group $\pi_n(X, A, x_0)$ one first chooses a distinguished $(n-1)$-face F of the n-cube I_n. Then one considers continuous maps from I_n to X, which map F into A and the rest of the boundary of I_n to x_0. The elements of $\pi_n(X, A, x_0)$ are homotopy classes of such maps (relative to these restrictions). We now imitate this construction in our setting.

DEFINITION 4.1. Let $n \geq 2$, and let $f_n : \mathbf{Z}^n \longrightarrow \Delta$ be a (q, n)-chain of radius N. Call f_n a (Σ, q, n)-*chain* (or simply Σ-chain, if the context is clear) if it satisfies the following properties.

(1) If $i_2 \geq N$ and $|i_j| < N$ for all $j \in \{1, 3, 4, \dots, n-2\}$, then all simplices $f_n(i_1, \dots, i_n)$ lie in Σ. (Note that the definition of a (q, n)-chain implies that for $i_2 > N$, the simplices along that coordinate are constant.)

(2) Otherwise, $f_n(i_1, \dots, i_n) = \sigma_0$.

This definition is illustrated for $n = 2$ in Fig. 13, center.

Define an equivalence relation on Σ-chains, in analogy to the relative homotopy from topology. We say that two Σ-chains are Σ-homotopic if they are A-homotopic via a $(\Sigma, q, n+1)$-chain.

DEFINITION 4.2. Let $A_n^q(\Delta, \Sigma, \sigma_0)$, $n \geq 2$, be the set of all (relative) A-homotopy classes of (Σ, q, n)-chains. As before, we denote the equivalence class of f_n by $[f_n]$, and by $[\sigma_0]$ the class containing the constant Σ−chain equal to σ_0.

We can define a multiplication on $A_n^q(\Delta, \Sigma, \sigma_0)$ by $[f_n][g_n] = [f_n g_n]$, where the (Σ, q, n)-chain $f_n g_n$ is defined exactly as in the absolute case described in Proposition 3.5. Following the arguments for the absolute A-groups, it is straightforward to check that this makes $A_n^q(\Delta, \Sigma, \sigma_0)$ into a group, which is abelian for all $n \geq 2$.

PROPOSITION 4.3. *The relative A-homotopy groups* $A_n^q(\Delta, \Sigma, \sigma_0)$, $n \geq 2$, *are abelian*.

The proof is similar to the one for the absolute case.

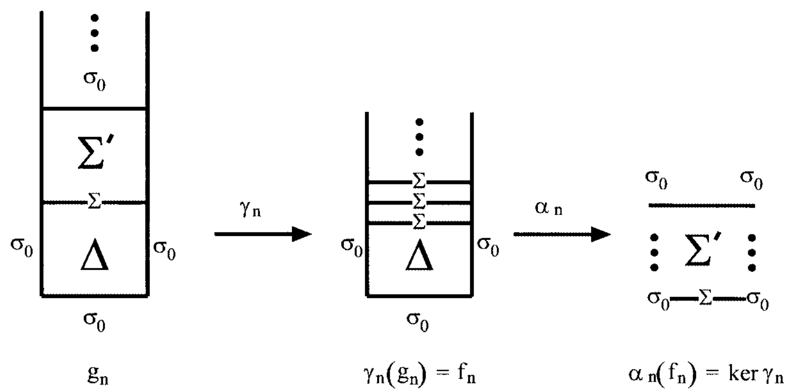

FIG. 13. $\ker(\alpha_n) \subseteq \operatorname{im}(\gamma_n)$.

THEOREM 4.4. *Let $\Sigma \subset \Delta$ be simplicial complexes of dimension $d \geq q$, with $\sigma_0 \in \Sigma$ being a distinguished maximal base simplex of dimension greater or equal than q. There is a long exact sequence for $n \geq 1$:*

$$\cdots \longrightarrow A^q_{n+1}(\Delta, \Sigma, \sigma_0) \xrightarrow{\alpha_{n+1}} A^q_n(\Sigma, \sigma_0) \xrightarrow{\beta_n} A^q_n(\Delta, \sigma_0) \xrightarrow{\gamma_n}$$
$$\longrightarrow A^q_n(\Delta, \Sigma, \sigma_0) \xrightarrow{\alpha_n} A^q_{n-1}(\Sigma, \sigma_0) \longrightarrow \cdots$$
$$\longrightarrow A^q_2(\Delta, \Sigma, \sigma_0) \xrightarrow{\alpha_2} A^q_1(\Sigma, \sigma_0) \xrightarrow{\beta_1} A^q_1(\Delta, \sigma_0).$$

Proof. The maps in the sequence are defined as follows. To describe the map α_{n+1} let $f_{n+1} : \mathbf{Z}^{n+1} \longrightarrow \Delta$ be a $(\Sigma, q, n+1)$-chain of radius N, representing an element $[f_{n+1}]$ in the domain. Then, by definition, the map

$$f_{n+1}(-, N, -, \ldots, -) : \mathbf{Z}^n \longrightarrow \Delta$$

is a (q, n)-cube in Σ based at σ_0, so it represents an element in $A^q_n(\Sigma, \sigma_0)$, which we choose as the image of $[f_{n+1}]$. The map β_n is induced by the inclusion of Σ into Δ. Likewise, the map γ_n is induced by viewing a (q, n)-cube in Δ as a (Σ, q, n)-chain. It is tedious but straightforward to verify that these maps are all well-defined and group homomorphisms. The proofs are similar to those used for the multiplication in absolute A-theory.

Exactness at $A^q_n(\Sigma, \sigma_0)$. Let $[f_n] \in A^q_n(\Sigma, \sigma_0)$ be such that $\beta_n([f_n]) = [\sigma_0]$, that is, $[f_n] \in \ker(\beta_n)$. This means that f_n is A-homotopic in Δ to the constant (q, n)-cube equal to σ_0. Such an A-homotopy between them is given by a $(q, n+1)$-chain f_{n+1} in Δ of radius N, which we can choose such that

$$f_{n+1}(i_1, \ldots, i_n, N) = f_n(i_1, \ldots, i_n)$$

for all $(i_1, \ldots, i_n) \in \mathbf{Z}^n$. Now let g_{n+1} be the $(\Sigma, q, n+1)$-chain obtained from f_{n+1} as follows

$$g_{n+1}(x_1, \ldots, x_{n+1}) = \begin{cases} f_{n+1}(x_1, x_{n+1}, x_3, \ldots, x_n, x_2) & \text{if } x_2 \leq N \\ f_{n+1}(x_1, x_{n+1}, x_3, \ldots, x_n, N) & \text{if } x_2 > N. \end{cases}$$

Then $\alpha_{n+1}([g_{n+1}]) = [f_n]$. This shows that $\ker(\beta_n) \subset \operatorname{im}(\alpha_{n+1})$. To see the other inclusion, let $[f_{n+1}]$ be a Σ-chain of radius N. Then

$$\beta_n \circ \alpha_{n+1}([f_{n+1}]) = f_{n+1}(-, N, -, \ldots, -).$$

But from the Σ-chain f_{n+1} we obtain an A-homotopy between the (q, n)-cube $f_{n+1}(-, N, -, \ldots, -)$ and the constant (q, n)-cube based at σ_0. This shows that $\beta_n \circ \alpha_{n+1}$ is the trivial homomorphism, so that the sequence is exact at $A^q_n(\Sigma, \sigma_0)$.

Exactness at $A_n^q(\Delta, \sigma_0)$. Let $\gamma_n([f_n]) = [\sigma_0]$. Let the radius of a representative (q, n)-cube f_n be N. Then we have a $(\Sigma, q, n+1)$-chain f_{n+1} which provides an A-homotopy between f_n and the constant (q, n)-cube based at σ_0. That is, $f_{n+1}(i_1, \ldots, i_n, r) = f_n(i_1, \ldots, i_n)$ and $f_{n+1}(i_1, \ldots, i_n, t) = (\sigma_0)$, for some $r \leq t$. But then $g_n = f_{n+1}(-, N, -, \ldots, -)$ is an n-cube in Σ. We next show that $[g_n] = \beta_n([g_n]) = [f_n]$ by exhibiting an explicit A-homotopy between g_n and f_n in Δ, that is, a $(q, n+1)$-chain

$$h_{n+1} : \mathbf{Z}^{n+1} \longrightarrow \Delta,$$

such that $h_{n+1}(i_1, \ldots, i_n, r) = f_n(i_1, \ldots, i_n)$ and $h_{n+1}(i_1, \ldots, i_n, 2t - r) = g_n(i_1, \ldots, i_n)$. Pictorially, the homotopy is easy to see. Observe that g_n is the lid of a box, one of whose sides is f_n. The homotopy lifts up the lid and turns it into another side of the box, thereby making the two homotopic. It is illustrated in Fig. 14. The vertical lines represent (q, n)-cubes, and the labels indicate how these get deformed. To describe h_{n+1} rigorously, for $0 \leq p \leq t - r$ let

$$h_{n+1}(i_1, \ldots, i_n, r+p) = \begin{cases} f_{n+1}(i_1, \ldots, i_n, r+p) & \text{if } i_2 \leq N \\ f_{n+1}(i_1, N, i_3, \ldots, r+p-1) & \text{if } i_2 = N+1 \\ f_{n+1}(i_1, N, i_3, \ldots, r+p-2) & \text{if } i_2 = N+2 \\ \vdots & \vdots \\ f_{n+1}(i_1, N, i_3, \ldots, r) & \text{if } i_2 = N+p \\ \sigma_0 & \text{if } i_2 > N+p. \end{cases}$$

For $1 \leq d \leq t - r$ let

$$h_{n+1}(i_1, \ldots, i_n, t+d) = h_{n+1}(i_1, \ldots, i_n + d, t).$$

Then

$$\begin{aligned} h_{n+1}(i_1, \ldots, i_n, 2t-r) &= h_{n+1}(i_1, \ldots, i_n + t - r, t) \\ &= f_{n+1}(i_1, N, i_3 \ldots, i_n) \\ &= g_n(i_1, \ldots, i_n). \end{aligned}$$

It is straightforward to see that

$$h_{n+1}(i_1, \ldots, i_n, r) = f_n(i_1, \ldots, i_n).$$

This shows that $\ker(\gamma_n) \subset \operatorname{im}(\beta_n)$.

Now let $[f_n] \in A_n^q(\Sigma, \sigma_0)$. To show that $\gamma_n \circ \beta_n([f_n]) = [\sigma_0]$ we construct an A-homotopy h_{n+1} from f_n to (σ_0) in Δ relative to Σ. Let N be the radius of f_n. Using Lemma 3.4 we can assume that $f_n(i_{1,\ldots,} i_n) = \sigma_0$ for all. Define

$$h_{n+1}(i_1, \ldots, i_{n+1}) = \begin{cases} f_n(i_1, \ldots, i_n) & \text{if } i_{n+1} \leq 0 \\ f_n(i_1, \ldots, i_n - i_{n+1}) & \text{if } i_n \leq N,\ 0 < i_{n+1} \leq N+1 \\ f_n(i_1, \ldots, N+1-i_{n+1}) & \text{if } i_n > N,\ 0 < i_{n+1} \leq N+1 \\ \sigma_0 & \text{otherwise.} \end{cases}$$

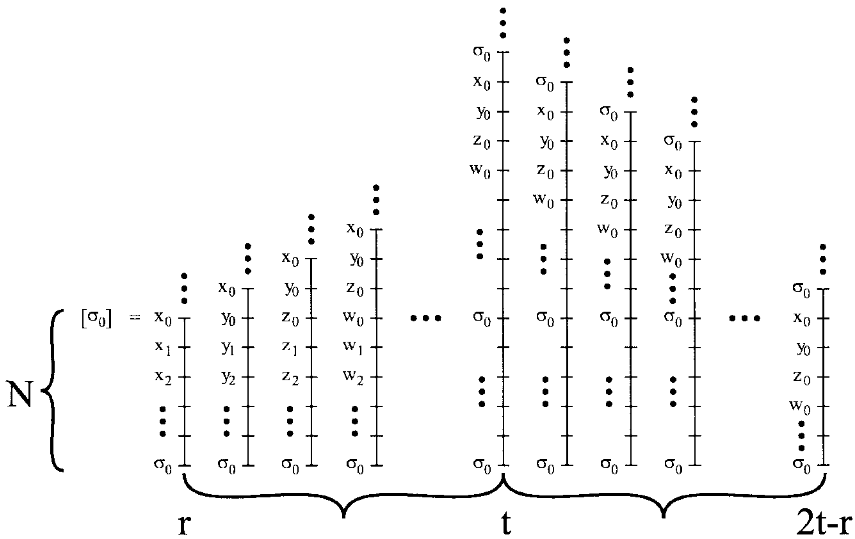

FIG. 14. $\ker(\gamma) \subseteq \operatorname{im}(\beta)$.

Then $h_{n+1}(i_1, \ldots, i_n, 0) = f_n(i_1, \ldots, i_n)$ and $h_{n+1}(i_1, \ldots, i_n, N+1) = (\sigma_0)$. See Fig. 15. Here too each vertical line represents a (q, n)-cube. This concludes the proof that $\ker(\gamma_n) = \operatorname{im}(\beta_n)$.

Exactness at $A_n^q(\Delta, \Sigma, \sigma_0)$. See Figs. 13 and 16. This completes the proof. ∎

5. A-THEORY OF GRAPHS

The computation of A_1^q of simplicial complexes via the associated connectivity graph suggests a version of A-theory for graphs, which we will

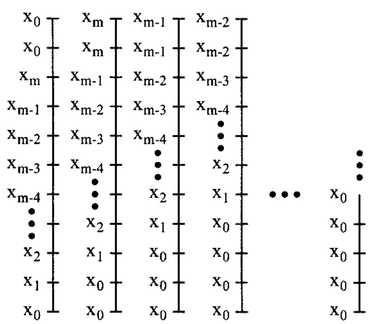

FIG. 15. $\operatorname{im}(\beta) \subseteq \ker(\gamma)$.

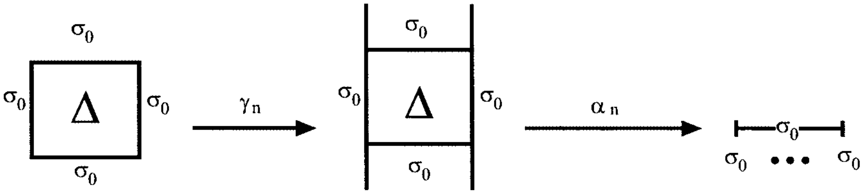

FIG. 16. $\mathrm{im}(\gamma) \subseteq \ker(\alpha)$.

develop in this section. We will also compare it to the A-theory of graphs viewed as simplicial complexes. The definition of the theory for graphs parallels closely the homotopy theory of spaces.

We first recall some standard constructions from graph theory.

DEFINITION 5.1. Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be simple graphs, that is, graphs without loops and multiple edges.

(1) The *Cartesian product* $\Gamma_1 \times \Gamma_2$ is the graph with vertex set $V_1 \times V_2$. There is an edge between (u_1, u_2) and (v_1, v_2) if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$.

(2) A *graph map* $f : \Gamma_1 \longrightarrow \Gamma_2$ is a set map $V_1 \longrightarrow V_2$ such that, if $uv \in E_1$, then either $f(u) = f(v)$ or $f(u)f(v) \in E_2$.

(3) Let $\mathbf{I}_n$ be the graph with $n+1$ vertices labeled $0, 1, \ldots, n$, and edges $(i-1)i$ for $i = 1, \ldots, n$.

(4) Let $v_1 \in \Gamma_1$, $v_2 \in \Gamma_2$ be distinguished base vertices. A *based* graph map $f : \Gamma_1 \longrightarrow \Gamma_2$ is a graph map such that $f(v_1) = v_2$.

We now define homotopy of graph maps and homotopy equivalence of graphs.

DEFINITION 5.2. (1) Let $f, g : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ be based graph maps. We call f and g *G-homotopic*, denoted by $f \simeq_G g$, if there is an integer n and a graph map

$$\phi : \Gamma_1 \times \mathbf{I}_n \longrightarrow \Gamma_2,$$

such that $\phi(-, 0) = f$, and $\phi(-, n) = g$, and such that $\phi(v_1, i) = v_2$ for all i.

(2) We call (Γ_1, v_1) and (Γ_2, v_2) *G-homotopy equivalent* if there exist based graph maps $f : \Gamma_1 \longrightarrow \Gamma_2$ and $g : \Gamma_2 \longrightarrow \Gamma_1$ such that $gf \simeq_G \mathrm{id}_{\Gamma_1}$ and $fg \simeq_G \mathrm{id}_{\Gamma_2}$. The maps f and g are called *G-homotopy inverses* of each other.

(3) If Γ_1 is a subgraph of Γ_2, with base vertex $v_1 \in \Gamma_1$, then Γ_1 is called a *G-homotopy retract* of Γ_2, if there exists a based G-homotopy inverse

of the inclusion map. This homotopy inverse will be called a *G-homotopy retraction*.

Sometimes, we will omit the prefix G.

EXAMPLES. (1) Let Γ_1 be the graph with a single vertex v, and let Γ_2 be the square with vertices $v_0, \ldots, v_3$, based at v_0. Let $f : \Gamma_1 \longrightarrow \Gamma_2$ be the graph map which sends v to v_0. Then f is a homotopy equivalence. In particular, if we view Γ_1 as a subgraph of Γ_2, then it is a homotopy retract of Γ_2, with retraction the unique map g from Γ_2 to Γ_1. To see this, observe first that $gf = \mathrm{id}_{\Gamma_1}$, trivially, hence gf and id_{Γ_1} are homotopic. To show that $fg \simeq_G \mathrm{id}_{\Gamma_2}$, let

$$\phi : \Gamma_2 \times \mathbf{I}_2 \longrightarrow \Gamma_2$$

be the graph map defined as follows. On $\Gamma_2 \times \{0\}$ the map is the identity. On $\Gamma_2 \times \{1\}$ we define ϕ by $\phi_{(v_0,1)} = \phi_{(v_3,1)} = v_0$, and $\phi_{(v_1,1)} = \phi_{(v_2,1)} = v_1$. Finally, on $\Gamma_2 \times \{2\}$ all vertices are sent to v_0. See Fig. 17, where a pair (v_i, j) is denoted by $v_{i,j}$. It is now straightforward to check that ϕ is a graph map.

(2) In a similar manner one can verify that a graph consisting of a single 3-cycle is homotopy equivalent to a graph with a single vertex.

(3) In contrast, one can verify that a 5-cycle Γ, on the other hand, is not homotopy equivalent to the trivial graph. To show this, one can verify that for any n and any map

$$\phi : \Gamma \times \mathbf{I}_n \longrightarrow \Gamma,$$

if $\phi(-, 0) = \mathrm{id}_\Gamma$, then the map $\phi(-, i)$ has to be onto Γ for all $1 \leq i \leq n$.

In complete analogy to classical topology, we now define a family of groups $\{A_n^G(\Gamma, v_0)\}$, $n \geq 1$, for a based graph (Γ, v_0).

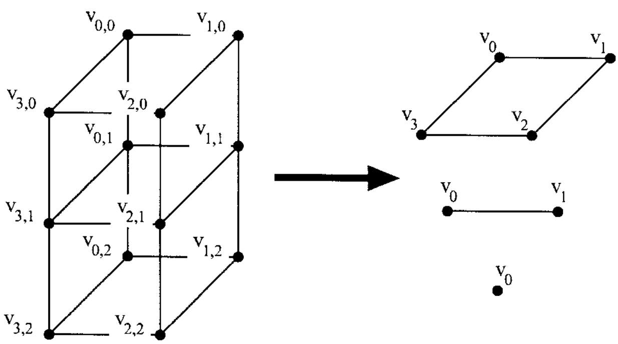

FIG. 17. Contraction of the square.

DEFINITION 5.3. (1) Let

$$\mathbf{I}_m^n = \mathbf{I}_m \times \cdots \times \mathbf{I}_m$$

be the n-fold Cartesian product of $\mathbf{I}_m$ for some m. We will call $\mathbf{I}_m^n$ an *n-cube* of *height m*. Its *distinguished base point* is $\mathbf{O} = (0, \ldots, 0)$.

(2) Define the *boundary* $\partial \mathbf{I}_m^n$ of a cube $\mathbf{I}_m^n$ of height m to be the subgraph of $\mathbf{I}_m^n$ containing all vertices with at least one coordinate equal to 0 or m.

LEMMA 5.4. *Let $f : \mathbf{I}_m^n \longrightarrow \Gamma$ be a graph map, such that $f(\partial \mathbf{I}_m^n) = v_0$. If $m \leq p$, then f can be extended to a graph map $\mathbf{I}_p^n \longrightarrow \Gamma$.*

Proof. The extension of f is defined by sending all vertices outside the smaller n-cube to the base vertex v_0.

This lemma implies that two maps from n-cubes of different height can be viewed as being defined on the larger one. Therefore, when the context is clear, we can omit the subscript m in the above notation. We will denote a map as in the previous lemma by

$$f : (\mathbf{I}^n, \partial \mathbf{I}^n) \longrightarrow (\Gamma, v_0),$$

as in classical topology.

DEFINITION 5.5. Let $A_n^G(\Gamma, v_0)$, $n \geq 1$, be the set of homotopy classes of graph maps

$$f : (\mathbf{I}^n, \partial \mathbf{I}^n) \longrightarrow (\Gamma, v_0).$$

For $n = 0$, we define $A_0^G(\Gamma, v_0)$ to be the pointed set of connected components of Γ, with distinguished element the component containing v_0. We will denote the equivalence class of a map f in $A_n^G(\Gamma, v_0)$ by $[f]$.

We can define a multiplication on the set $A_n^G(\Gamma, v_0), n \geq 1$, as follows. Given elements $[f], [g] \in A_n^G(\Gamma, v_0)$, represented by

$$f, g : (\mathbf{I}^n, \partial \mathbf{I}^n) \longrightarrow (\Gamma, v_0),$$

defined on a cube of height m, we define $[f] * [g] \in A_n^G(\Gamma, v_0)$ as the G-homotopy class of the map

$$h : (\mathbf{I}^n, \partial \mathbf{I}^n) \longrightarrow (\Gamma, v_0),$$

defined on a cube of height $2m$ as

$$h(i_1, \ldots, i_n) = \begin{cases} f(i_1, \ldots, i_n) & \text{if } i_j \leq m \text{ for all } j, \\ g(i_1 - m, \ldots, i_n) & \text{if } i_1 > m \text{ and } i_j \leq m \text{ for } j > 1, \\ v_0 & \text{otherwise.} \end{cases}$$

In light of the following proposition we call the sets $A_n^G(\Gamma, v_0), n \geq 1$ the *graph homotopy groups*, or *G-homotopy groups* of (Γ, v_0).

Proposition 5.6. *The sets* $A_n^G(\Gamma, v_0)$, $n \geq 1$ *are groups.*

Proof. The details of showing that the multiplication is well defined and satisfies the group axioms are entirely similar to the case of the multiplication for A_n^q of a simplicial complex and will be omitted.

As for A-theory of simplicial complexes, A^G also satisfies an invariance of base vertex property. The proof is similar to that of Proposition 2.4.

Proposition 5.7. *If* v *and* w *are vertices of* Γ *that are connected by a path in* Γ, *then*

$$A_n^G(\Gamma, v) \cong A_n^G(\Gamma, w),$$

for all $n \geq 1$.

The proof of the following proposition is also similar to the corresponding one for the A-groups of a simplicial complex.

Proposition 5.8. *The groups* $A_n^G(\Gamma, v_0)$ *are abelian for* $n \geq 2$.

Let $f : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ be a based graph map. Then f induces a group homomorphism

$$f_* : A_n^G(\Gamma_1, v_1) \longrightarrow A_n^G(\Gamma_2, v_2)$$

as follows. Let $[\alpha] \in A_n^G(\Gamma_1, v_1)$ be represented by a graph map

$$\alpha : (\mathbf{I}^n, \partial\mathbf{I}^n) \longrightarrow (\Gamma_1, v_1).$$

Define $f_*([\alpha])$ to be the homotopy class of the map $f \circ \alpha$. It is straightforward to verify that f_* is well-defined and a group homomorphism. The proof of the following lemma is identical to the corresponding proof for homotopic maps of topological spaces and will be omitted.

Lemma 5.9. *If* $f, g : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ *are G-homotopic based graph maps, then*

$$f_* = g_* : A_n^G(\Gamma_1, v_1) \longrightarrow A_n^G(\Gamma_2, v_2).$$

Proposition 5.10. *A based G-homotopy equivalence of based graphs induces an isomorphism of graph homotopy groups.*

Theorem 5.11. *Let* (Γ, v_0) *be a based graph, and let* N *be the normal subgroup of* $\pi_1(\Gamma, v_0)$ *defined as just before Lemma 2.5, that is, the normal subgroup generated by all 3- and 4-cycles in* Γ. *Then*

$$A_1^G(\Gamma, v_0) \cong \pi_1(\Gamma, v_0)/N.$$

The proof is similar to that of Theorem 2.7.

PROPOSITION 5.12. *Let X_Γ be the topological space obtained from Γ by attaching 2-cells to Γ, viewed as a one-dimensional cell complex, one for each generator of N. That is, we attach 2-cells along the boundary of each 3- and 4-cycle of Γ. Then*

$$A_1^G(\Gamma, v_0) \cong \pi_1(X_\Gamma, v_0).$$

Proof. The inclusion map $\Gamma \hookrightarrow X_\Gamma$ induces the quotient map $\pi_1(\Gamma) \longrightarrow \pi_1(\Gamma)/N$ on fundamental groups. ∎

This result makes it easy to generate examples of graphs with specified graph homotopy groups. For instance, the graph Γ in Fig. 18 has A_1^G equal to $\mathbf{Z}/2\mathbf{Z}$. This can be verified using the graph homotopy machinery we have developed. But it can be seen more easily by observing that the associated space X_Γ is just the projective plane, which has fundamental group equal to $\mathbf{Z}/2\mathbf{Z}$. Similarly, one can construct graphs with A_1^G equal to $\mathbf{Z}/n\mathbf{Z}$, by embedding a graph consisting of 3- and 4-cycles into the surface obtained by gluing a disc onto a circle in a way that wraps the boundary of the disc around the circle n times.

The space X_Γ associated to a graph Γ has been used in several combinatorial contexts. The first such result is a theorem of S. Maurer about matroids, which we state in terms of A^G-theory.

THEOREM 5.13 (Maurer [9]). *Let Γ be the basis graph of a matroid. Then the group $A_1^G(\Gamma, v_0)$ is trivial for any choice of base vertex v_0. Equivalently, $\pi_1(X_\Gamma, v_0) = 1$.*

Subsequently, Lovász used information about $\pi_1(X_\Gamma)$ to prove the following connectivity result in graph theory.

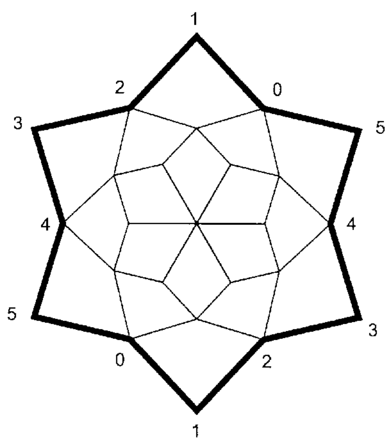

FIG. 18. A graph embedded in the projective plane.

THEOREM 5.14 (Lovász [7]). *Let Γ be a k-connected graph, $\{v_1, \dots, v_k\} \subseteq V(\Gamma)$, and $n_1, \dots, n_k$ positive integers with $n_1 + \cdots + n_k = n = |V(\Gamma)|$. Then there exists a partition $\{V_1, \dots, V_k\}$ of $V(\Gamma)$ such that*

(1) $v_i \in V_i$,
(2) $|V_i| = n_i$,
(3) V_i *spans a connected subgraph of* Γ $(i = 1, \dots, k)$.

For a survey of results of this type see [3]. Similar techniques have been used to study buildings [11, Chap. 6]. As Proposition 5.12 shows, $A_1^G(\Gamma) \cong \pi_1(X_\Gamma)$. Thus, our graph homotopy theory provides a general conceptual framework for the homotopy theory suggested by Maurer in his paper, as well as the proof technique used by Lovász and others. The next result gives an example of a graph Γ whose higher graph homotopy groups $A_n^G(\Gamma)$ differ from the higher homotopy groups of the space X_Γ.

PROPOSITION 5.15. *Let Γ be a 3-dimensional cube. Then $A_n^G(\Gamma) = 1$ for all $n \geq 1$, and $\pi_2(X_\Gamma) \equiv \mathbf{Z}$.*

Proof. Figure 19 shows a G-homotopy from the cube to the base vertex. ∎

We conclude this section by relating A^G-theory of graphs to A-theory of simplicial complexes. In particular, we prove a result that was the initial motivation for developing graph homotopy, namely to find an efficient way of computing the combinatorial fundamental group of simplicial complexes.

THEOREM 5.16. *Let Δ be a simplicial complex, with distinguished maximal simplex σ_0, $0 \leq q \leq \dim(\Delta)$. Let $\Gamma^q(\Delta)$ be the connectivity graph of Δ in dimension q, with distinguished vertex v_0 corresponding to σ_0. Let $\Gamma^q_{\max}(\Delta) \subseteq \Gamma^q(\Delta)$ be the subgraph with vertices corresponding to the maximal simplices of Δ of dimension greater than or equal to q. Then*

$$A_n^q(\Delta, \sigma_0) \cong A_n^G(\Gamma^q(\Delta), v_0) \cong A_n^G(\Gamma^q_{\max}(\Delta), v_0).$$

Proof. The first isomorphism is a lengthy but straightforward comparison of the definitions for the two theories, which is left to the reader.

To show the second isomorphism, we will show that the inclusion

$$i : \Gamma^q_{\max}(\Delta) \hookrightarrow \Gamma^q(\Delta)$$

is a homotopy equivalence by constructing a retraction

$$p : \Gamma^q(\Delta) \longrightarrow \Gamma^q_{\max}(\Delta)$$

as follows. Let $v \in \Gamma^q(\Delta)$ be a vertex corresponding to a simplex $\sigma \in \Delta$. Choose a maximal simplex τ containing σ, with corresponding vertex w in $\Gamma^q(\Delta)$. (Note that τ will in general not be unique.) If $\tau_1, \dots, \tau_r$ are

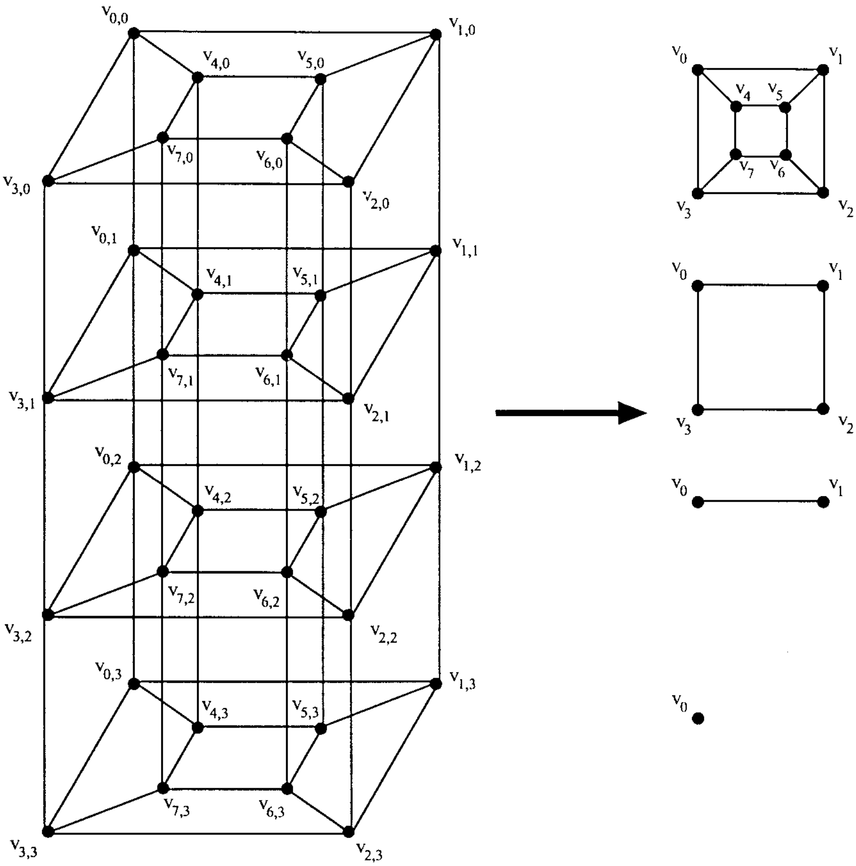

FIG. 19. Retract of the cube.

all the maximal simplices containing σ, then the corresponding vertices in $\Gamma^q(\Delta)$ form a complete subgraph on r vertices. Define $p(v) = w$. Let $\rho \in \Delta$ correspond to $z \in \Gamma^q(\Delta)$ and suppose that it shares a q-face with σ. If ρ is contained in a maximal simplex γ, then γ shares a q-face with τ, hence the corresponding vertices are connected by an edge. These observations imply that p is a well-defined graph map.

Clearly, $p \circ i = \mathrm{id}_{\Gamma^q_{\max}(\Delta)}$. To see that $i \circ p \simeq_G \mathrm{id}_{\Gamma^q(\Delta)}$, observe that the following map gives a homotopy between them. Define

$$\phi : \Gamma^q(\Delta) \times \mathbf{I}_1 \longrightarrow \Gamma^q(\Delta)$$

to be the identity on the top layer and p on the bottom layer. Then it is straightforward to check that ϕ is a graph map. ∎

Finally, we conclude this section by showing that any connected graph can occur as the connectivity graph of a simplicial complex.

THEOREM 5.17. *Given $q \geq 0$, any connected graph Γ occurs as the connectivity graph $\Gamma^q_{\max}(\Delta)$ of some simplicial complex Δ.*

Proof. Fix an integer $q \geq 0$. Let $V = \{x_1, x_2, \ldots, x_m\}$ where

$$m = (q+1)(\text{number of edges of } \Gamma) + (\text{number of vertices of } \Gamma \text{ of degree } 1).$$

Suppose the vertices of Γ are labeled $1, \ldots, n$. To each edge (ij) of Γ assign a q-simplex σ_{ij} using letters from V in such a way that they are all pairwise disjoint, that is,

$$|\sigma_{ij} \cap \sigma_{kl}| = 0,$$

if $(i, j) \neq (k, l)$. Now assign a simplex to each vertex v of Γ as follows. Let σ_v be the simplex whose vertices are all vertices contained in the q-simplices assigned to the edges emanating from v. If v has only one edge emanating from it, then add an element from V to the q-simplex σ_v, which has not been used anywhere else.

Then each σ_v is a maximal simplex in the simplicial complex Δ we obtain in this way. By definition, $\Gamma^q_{\max}(\Delta) = \Gamma$. ∎

The case $q = 0$ is a theorem of Marczewski [8]. (See also [10, Chap. 3].) To illustrate this construction, let Γ be the graph with vertices $1, 2, 3$ and edges (12), (23). Let $q = 1$. Then

$$\sigma_{12} = \{a, b\}, \qquad \sigma_{23} = \{c, d\}, \qquad \sigma_1 = \{a, b, e\},$$
$$\sigma_2 = \{a, b, c, d\}, \qquad \sigma_3 = \{c, d, f\}.$$

One can now develop relative A^G-theory of a graph with respect to a subgraph, and this theorem gives for free a long exact sequence of A^G-groups. It also gives for free a Seifert–Van Kampen theorem for A^G-theory. The details are straightforward.

6. CONCLUSION

One could say that the results in this paper fall into the category of "topological methods in combinatorics." However, while our constructions are inspired by the methods of classical algebraic topology, they are purely combinatorial. Our goal, guided by the requirements of the applications we had in mind [4], was to construct a theory for simplicial complexes that was qualitative in nature, like the homology and homotopy invariants of algebraic topology, but that reflected the combinatorial connectivity of the complex. Subsequently, we discovered that our theory, in dimension one, had already been studied in several unrelated combinatorial contexts, such as graph theory [7], matroid theory [9], and the theory of buildings [11, Chap. 6]. We now summarize our constructions and results.

Fix an integer $q \geq 0$. Let $\mathbf{SC_q}$ be the collection of simplicial complexes of dimension greater than or equal to q, with a distinguished maximal base simplex σ_0 of dimension greater than or equal to q, denoted by (Δ, σ_0). Let $\mathbf{G}$ be the collection of graphs Γ with distinguished vertex v_0, denoted by (Γ, v_0). We have defined a family of groups $A_n^q(\Delta, \sigma_0)$ for every $(\Delta, \sigma_0) \in \mathbf{SC_q}$. These are defined by forming the connectivity graph $(\Gamma^q(\Delta), v_0)$ of the complex, where v_0 corresponds to σ_0. We have also defined a collection of groups $A_n^G(\Gamma, v_0)$ for each $(\Gamma, v_0) \in \mathbf{G}$. They are related by the fact that

$$A_n^q(\Delta, \sigma_0) \cong A_n^G(\Gamma^q(\Delta), v_0),$$

which can be expressed by the diagram

$$\begin{array}{ccc} \mathrm{SC}_q & \to & \mathrm{G} \\ \downarrow & & \downarrow \\ \mathrm{Groups} & = & \mathrm{Groups} \end{array}$$

If Δ happens to be a graph, and $q = 0$, then the connectivity graph of Δ is simply its line graph Δ^*. Then, supressing the base point,

$$A_n^0(\Delta) = A_n^G(\Delta^*).$$

Finally, it is worth mentioning an alternative of developing the two theories in this paper. Instead of developing A-theory of simplicial complexes first, one could define it in terms of the A^G-theory of the connectivity graphs for the various q-values. Our choice reflects our primary interest in simplicial complexes.

Evidently, we have only begun to develop the theoretical foundations of A-theory. A lot remains to be done, in particular the development of an associated homology theory. An important next step is to explore the utility of A-theory in a variety of combinatorial contexts, such as buildings, geometric lattices, matroids, and graphs.

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