Multivalued generalizations of probabilistic contractions

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Abstract

We generalize some well-known fixed point theorems for probabilistic contractions to multivalued contractions. The contraction maps are considered in generalized Menger spaces.

Keywords: Menger space; Probabilistic contraction; Multivalued mapping

In this work we deal with multivalued contractions in generalized Menger spaces. The outline of the paper is as follows. Section 1 recalls some notions and results concerning probabilistic metric spaces and probabilistic contractions. Section 2 is devoted to the main results of this paper: a general fixed point principle for multivalued contractions in generalized Menger spaces (Theorem 2.2) and other fixed point theorems, including Theorems 2.6 and 2.11 for two types of contractions introduced here (Definitions 2.3 and 2.10). These can be seen as generalizations of some classical results.
1. Preliminaries

The notions and the results concerning probabilistic metric spaces used in this paper are classical ones and follow the books [6] and [16]. We recall some of them.

Definition 1.1 [16]. A mapping \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a triangular norm (shortly \( t \)-norm) if it satisfies the following conditions:

\[
\begin{align*}
(\text{N1}) & \quad T(a, b) = T(b, a), \forall a, b \in I; \\
(\text{N2}) & \quad (a \leq c, b \leq d) \Rightarrow T(a, b) \leq T(c, d); \\
(\text{N3}) & \quad T(a, 1) = a, \forall a \in I; \\
(\text{N4}) & \quad T(a, T(b, c)) = T(T(a, b), c), \forall a, b, c \in I.
\end{align*}
\]

Among the important examples of \( t \)-norms we mention the \( t \)-norms \( T_L \), \( T_P \) and Min, defined by \( T_L(a, b) = \max(a + b - 1, 0) \) (Lukasiewicz \( t \)-norm), \( T_P(a, b) = a \cdot b \) and \( \text{Min}(a, b) = \min(a, b) \).

Definition 1.2 [3, 4]. We say that the \( t \)-norm \( T \) is of Hadžić type if the family \( \{ T^n \}_{n \in N} \) of its iterates defined, for each \( x \) in \([0, 1]\), by

\[ T^0(x) = 1 \quad \text{and} \quad T^{n+1}(x) = T(T^n(x), x), \quad \forall n \geq 0, \]

is equicontinuous at \( x = 1 \), that is

\[ \forall \varepsilon \in (0, 1) \exists \delta \in (0, 1) \text{ s.t. } x > 1 - \delta \quad \Rightarrow \quad T^n(x) > 1 - \varepsilon, \quad \forall n \geq 1. \]

Definition 1.3 [6]. If \( T \) is a \( t \)-norm and \( (x_n)_{n \geq 1} \) is a sequence of numbers in \([0, 1]\), one defines recurrently \( T^1_{i=1} x_i \) by \( T^1_{i=1} x_i = x_1 \) and \( T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n), \forall n \geq 2, T^\infty_{i=1} x_i \) is defined as \( \lim_{n \to \infty} T^\infty_{i=1} x_i \).

If \( q \in (0, 1) \) is given, we say that the \( t \)-norm \( T \) is \( q \)-convergent if \( \lim_{n \to \infty} T^n_{i=1} (1 - q^i) = 1 \). We remark that if \( T \) is \( q \)-convergent then

\[ \forall \lambda \in (0, 1) \exists s = s(\lambda) \in N: \quad T^n_{i=1} (1 - q^i) > 1 - \lambda, \quad \forall n \in N. \]

Also note that if the \( t \)-norm \( T \) is \( q \)-convergent then \( \sup_{a \leq 1} T(a, a) = 1 \).

Examples 1.4 [6]. Each \( t \)-norm of Hadžić type is \( q \)-convergent for every \( q \in (0, 1) \).

Among other examples of \( t \)-norms which are \( q \)-convergent for each \( q \in (0, 1) \) we mention:

- Lukasiewicz \( t \)-norm \( T_L \);
- Sugeno–Weber family, defined by
  \[ T^\text{SW}_\lambda = \max \left( 0, \frac{x + y - 1 + \lambda xy}{1 + \lambda} \right), \lambda \in (-1, \infty); \]
- Dombi family, defined by
  \[ T^D_\lambda = \left( 1 + \left( \frac{1 - x}{x} \right)^{\lambda} + \left( \frac{1 - y}{y} \right)^\lambda \right)^{1/\lambda} - 1, \lambda \in (0, \infty); \]
Aczel–Alsina family, defined by
\[ T^{AA}_\lambda = e^{-[(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}]}, \quad \lambda \in (0, \infty). \]

**Definition 1.5** [16]. The class of generalized distance distribution functions (denoted by \( \Delta_+ \)) is the class of functions \( F : [0, \infty) \to [0, 1] \) with the properties:

(a) \( F(0) = 0 \);
(b) \( F \) is increasing;
(c) \( F \) is left continuous on \( (0, \infty) \).

\( D_+ \) is the subset of \( \Delta_+ \) containing functions \( F \) that also satisfy the condition \( \lim _{x \to \infty} F(x) = 1 \).

A special element of \( D_+ \) is the function \( \varepsilon_0 \), defined by
\[
\varepsilon_0(t) = \begin{cases} 
0, & \text{if } t = 0, \\
1, & \text{if } t > 0.
\end{cases}
\]

If \( X \) is a nonempty set, a mapping \( F : X \times X \to \Delta_+ \) is called a probabilistic distance on \( X \) and \( F(x, y) \) will be denoted by \( F_{xy} \).

**Definition 1.6** [9,16]. If \( X \) is a nonempty set, \( F \) is a probabilistic distance on \( X \) and \( T \) is a \( t \)-norm, the triple \( (X, F, T) \) is called a generalized Menger space if the following axioms are satisfied:

\[ (PM0): \quad F_{xy} = \varepsilon_0 \quad \text{iff} \quad x = y, \]
\[ (PM1): \quad F_{xy} = F_{yx}, \quad \forall x, y \in X, \]
\[ (PM2M): \quad F_{xy}(t+s) \geq T(F_{xz}(t), F_{zy}(s)), \quad \forall x, y, z \in X, \forall t, s > 0. \]

A Menger space is a generalized Menger space with \( F(X \times X) \subset D_+ \).

**Definition 1.7** [16]. Let \( (X, F, T) \) be a generalized Menger space. It is known [7,16,17] that if \( \sup _{a < 1} T(a, a) = 1 \) then the family \( \{U_\varepsilon\}_{\varepsilon > 0} \), where
\[ U_\varepsilon = \{(x, y) \in X \times X : F_{xy}(\varepsilon) > 1 - \varepsilon\}, \]
is a base for a metrizable uniformity on \( X \), called the \( F \)-uniformity.

The \( F \)-uniformity naturally determines a metrizable topology on \( X \), called [15] the strong topology or the \( F \)-topology: a subset \( O \) of \( X \) is \( F \)-open if for every \( p \in O \) there exists \( t > 0 \) such that \( N_p(t) = \{q \in X \mid F_{pq}(t) > 1 - t\} \subset O \).

In the following all topological notions refer to the \( F \)-topology.

Probabilistic contractions were first defined and studied by V.M. Sehgal in his Ph.D. thesis, partially published in [19]. Next we will point out some results from the probabilistic metric space fixed point theory.
Definition 1.8 [19]. Let $S$ be a nonempty set and $F$ be a probabilistic distance on $S$. A mapping $f : S \to S$ is called a probabilistic contraction (or $B$-contraction) if there exists $k \in (0, 1)$ such that

$$F_{f(p)f(q)}(kt) \geq F_{pq}(t), \quad \forall p, q \in S, \forall t > 0.$$ 

Theorem 1.9 [14]. Every $B$-contraction in a complete Menger space $(S, F, T)$ with $T$ continuous has a (unique) fixed point iff $T$ is of Hadžić type.

Definition 1.10 [7]. Let $S$ be a nonempty set and $F$ be a probabilistic distance on $S$. A mapping $f : S \to S$ is called a $C$-contraction if there exists $k \in (0, 1)$ such that, for all $p, q \in S$,

$$(H): \quad t > 0, \quad F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$ 

Unlike Sehgal’s contractions, any contraction of Hicks type has a fixed point even in complete Menger spaces $(S, F, T)$ with $\sup_{a < 1} T(a, a) = 1$ (the weakest condition which ensures the existence of the $F$-uniformity [12]).

Theorem 1.11 [14]. Let $(S, F, T)$ be a complete Menger space such that $\sup_{a < 1} T(a, a) = 1$. Then every $C$-contraction $f$ on $S$ has a unique fixed point which is the limit of the sequence $(f^n(p))_{n \in \mathbb{N}}$ for every $p \in S$.

Multivalued generalizations of probabilistic contractions were considered in [6–8]. For more details concerning these classical types of probabilistic contractions we refer the reader to [1,2,5,6,18,20,21].

Finally, we mention two larger classes of probabilistic contractions, defined as follows.

Definition 1.12 [10]. Let $S$ be a nonempty set and $F$ be a probabilistic distance on $S$. If $(b_n)_{n \in \mathbb{N}}$ is a sequence in $(0, 1)$ such that $b_n \not\to 1$, we say that the mapping $f : S \to S$ is a $(b_n)$-probabilistic contraction if there exists $k \in (0, 1)$ such that $\forall t > 0, \forall p, q \in S$,

$$F_{pq}(t) > b_n \quad \Rightarrow \quad F_{f(p)f(q)}(kt) > b_n.$$ 

Definition 1.13 [11]. Let $S$ be a nonempty set and $F$ be a probabilistic distance on $S$. A mapping $f : S \to S$ is called a weak-Hicks contraction (shortly $w$-$H$ contraction) if there exists $k \in (0, 1)$ such that, for all $p, q \in S$, the following implication holds:

$$(w\text{-}H): \quad t \in (0, 1), \quad F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$ 

2. Main results

In the following $2^S$ denotes the class of all nonempty subsets of the set $S$ and $C(S)$ is the class of all nonempty closed (in the $F$-topology) subsets of $S$.

Definition 2.1. Let $F$ be a probabilistic distance on $X$ and $M \in 2^S$. A mapping $A : X \to 2^X$ is called continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F_{xy}(\delta) > 1 - \delta \Rightarrow \forall p \in A(x) \exists q \in A(y): F_{pq}(\varepsilon) > 1 - \varepsilon$. 
Theorem 2.2. Let \((X,F,T)\) be a complete Menger space with \(\sup_{a<1} T(a,a) = 1\) and \(A:X \rightarrow C(X)\) be a continuous mapping. If there exist a sequence \((t_n)_{n \in N} \subset (0,\infty)\) with \(\sum_{i=1}^{\infty} t_n < \infty\) and a sequence \((x_n)_{n \in N} \subset X\) with the properties:

\[
x_{n+1} \in A(x_n) \quad \text{for all } n \quad \text{and} \quad \lim_{n \rightarrow \infty} T_{i=1}^{\infty} f_{n+i-1} = 1,
\]

where \(f_n := F_{x_n,x_{n+1}}(t_n)\), then \(A\) has a fixed point.

Proof. We will prove that \((x_n)\) is a Cauchy sequence, that is

\[
\forall \varepsilon > 0 \exists n_0 = n_0(\varepsilon) \in N: F_{x_n,x_{n+m}}(\varepsilon) > 1 - \varepsilon, \quad \forall n \geq n_0, \forall m \in N.
\]

Let \(\varepsilon > 0\) be given. Then

\[
\lim_{n \rightarrow \infty} T_{i=1}^{\infty} f_{n+i-1} = 1 \quad \Rightarrow \quad \exists n_1 = n_1(\varepsilon) \in N: T_{i=1}^{m} f_{n+i-1} > 1 - \varepsilon, \quad \forall n \geq n_1, \forall m \in N.
\]

Since the series \(\sum_{i=1}^{\infty} t_i\) is convergent, there exists \(n_2 (= n_2(\varepsilon))\) such that

\[
\sum_{i=n_2}^{\infty} t_i < \varepsilon.
\]

Let \(n_0 = \max\{n_1, n_2\}\). Then, for all \(n \geq n_0\) and \(m \in N\) we have

\[
F_{x_n,x_{n+m}}(\varepsilon) \geq F_{x_n,x_{n+m}} \left( \sum_{i=n}^{n+m-1} t_i \right) \geq T_{i=1}^{m} F_{x_{n+i-1},x_{n+i}}(t_{n+i-1}) = T_{i=1}^{m} f_{n+i-1} > 1 - \varepsilon,
\]

as desired. By the completeness of \(X\) it follows that \((x_n)\) converges to some \(x \in X\).

We will prove that \(x \in A(x)\). Since \(A(x) = \overline{A(x)}\) we have to prove that for every \(\varepsilon > 0\) there exists \(y = y(\varepsilon) \in A(x)\) such that \(F_{x,y}(\varepsilon) > 1 - \varepsilon\).

Let \(\varepsilon > 0\) be given. Since \(\sup_{a<1} T(a,a) = 1\) and \((S,F,T)\) is a Menger space, we can find the positive numbers \(\delta\) and \(\delta_1\) such that \(F_{pq}(\delta) > 1 - \delta\), \(F_{pq}(\delta_1) > 1 - \delta_1\) and \(F_{pq}(1) > 1 - \delta\) and \(F_{pq}(\delta_1) > 1 - \delta_1\) and \(F_{pq}(\delta) > 1 - \delta\).

From \(x_n \rightarrow x\) it follows that there exists \(n_1 = (n_1(\varepsilon)) \in N\) such that

\[
F_{x_n,x_{n+1}}(\delta_1) > 1 - \delta_1, \quad \forall n \geq n_1.
\]

Also, there exists \(n_2 = n_2(\varepsilon) \in N\) such that

\[
F_{x_{n_2},x_{n_2+1}}(\delta_1) > 1 - \delta_1, \quad \forall n \geq n_2.
\]

From the continuity of \(A\) it follows that there exists \(\delta_2 > 0\) such that \(F_{x,y}(\delta) > 1 - \delta_2 \Rightarrow \forall p \in A(x) \exists q \in A(y): F_{pq}(\delta) > 1 - \delta\).

Let \(n_3 \in N\) be such that \(F_{x_n,x_{n+1}}(\delta_2) > 1 - \delta_2\) for all \(n \geq n_3\) and \(s = \max\{n_1, n_2, n_3\}\).

Then there exists \(y \in A(x)\) such that \(F_{x,y}(\delta_1) > 1 - \delta\). We also have \(F_{x_{n_3+1},x_{n_3+2}}(\delta) > 1 - \delta, \quad \text{therefore} \quad F_{x,y}(\varepsilon) > 1 - \varepsilon\). The theorem is proved. \(\square\)

In the following we present some applications of this theorem.
2.1. A multivalued generalization of B-contractions

Definition 2.3. Let \((b_n)_{n \in \mathbb{N}}\) be a sequence in \((0, 1)\) such that \(b_n \not\to 1\) and \(F\) be a probabilistic distance on \(S\). A mapping \(A : S \to 2^S\) is called a multivalued \((b_n)\)-contraction if for every \(n \in \mathbb{N}\) there exists \(k_n \in (0, 1)\) such that for all \(p, q \in S\) and \(t > 0\) the following relation holds:

\[(b_n) - \bar{B}: \quad F_{pq}(t) > b_n \Rightarrow \forall u \in A(p) \exists v \in A(q): \quad F_{uv}(k_nt) > b_n.\]

Lemma 2.4. Any multivalued \((b_n)\)-contraction is continuous.

Proof. Let \(\varepsilon > 0\) be given. We can find \(m \in \mathbb{N}\) such that \(b_m > 1 - \varepsilon\). Then, for \(\delta := \min\{\varepsilon/k_m, 1 - b_m\}\) we have \(F_{xy}(\delta) > 1 - \delta \Rightarrow F_{xy}(\varepsilon/k_m) > b_m \Rightarrow \forall p \in A(x) \exists q \in A(y): F_{pq}(\varepsilon) > b_m > 1 - \varepsilon.\)

Lemma 2.5. Let \(A\) be a multivalued \((b_n)\)-contraction such that \(F_{pq} \in D_+\) for some \(p \in S, q \in A(p)\). Then there exists a sequence \((p_n)_{n \in \mathbb{N}}\) of elements of \(S\) with the properties

\[p_n \in A(p_{n-1}) \quad \text{for all } n \geq 1\]

and

\[\forall s \in \mathbb{N} \exists \varepsilon = \varepsilon(s) > 0: \quad F_{p_{n-1}p_n}(k_n^{n-1}\varepsilon) > b_s \quad \text{for all } n \geq 1.\]

Proof. Let \(s \in \mathbb{N}\) be given. We define \(p_0 = p\) and \(p_1 = q\).

Since \(F_{pq} \in D_+\) it follows that there exists \(\varepsilon > 0\) such that \(F_{p_0p_1}(\varepsilon) > b_s\).

Using \((b_n) - \bar{B}\) we can find \(p_2 \in A(p_1)\) such that \(F_{p_1p_2}(k_2\varepsilon) > b_s\) and, by induction, \(p_n \in A(p_{n-1})\) such that \(F_{p_{n-1}p_n}(k_n^{n-1}\varepsilon) > b_s\) for all \(n \geq 1.\)

Theorem 2.6. Let \(A : S \to C(S)\) be a multivalued \((b_n)\)-contraction in a complete generalized Menger space \((S, F, T)\) under the \(t\)-norm \(T\) of Hadžić type. If there exist \(p \in S\) and \(q \in A(p)\) such that \(F_{pq} \in D_+\), then \(A\) has a fixed point.

Proof. Let \(s \in \mathbb{N}\) be such that \(\lim_{n \to \infty} T_{i=n}^\infty T_{i=n}^\infty b_{s+i} = 1\) (such a number does exist, for \(T\) is of Hadžić type). From Lemma 2.5 it follows that there exist \(\varepsilon > 0\) and a sequence \((p_n)_{n \in \mathbb{N}}\) of elements of \(S\) with the properties

\[p_n \in A(p_{n-1}) \quad \text{for all } n \geq 1\]

and

\[\forall s \in \mathbb{N} \exists \varepsilon = \varepsilon(s) > 0: \quad F_{p_{n-1}p_n}(k_n^{n-1}\varepsilon) > b_s \quad \text{for all } n \geq 1.\]

By choosing \(t_n = k_n^{n-1}\varepsilon\), we have \(\sum_{i=1}^{\infty} t_n < \infty\) and \(\lim_{n \to \infty} T_{i=1}^\infty f_{n+i-1} = \lim_{n \to \infty} T_{i=1}^\infty b_{s+i} = 1\), so we can apply Theorem 2.2 to find a fixed point of \(A\).

The theorem is proved.
Corollary 2.7 [6, Theorem 3.49]. Let \((S, F, T)\) be complete Menger space under the \(t\)-norm \(T\) of Hadžić type and \(f : S \rightarrow S\) be a mapping with the property that for every \(n \in N\) there exists \(k_n \in (0, 1)\) such that for all \(p, q \in S\) and \(t > 0\),

\[ F_{pq}(t) > b_n \quad \Rightarrow \quad F_{f(p)f(q)}(k_nt) > b_n. \]

Then \(f\) has a unique fixed point.

**Proof.** The existence of a fixed point follows from Theorem 2.6 (as we have already mentioned, if \(T\) is a \(t\)-norm of Hadžić type then \(\sup_{a<1} T(a, a) = 1\)).

For the uniqueness, let us suppose that \(p\) and \(q\) are fixed points of \(f\). If \(\varepsilon > 0\) and \(\delta \in (0, 1)\) are given, since \(F_{pq} \in D_+\) we can choose \(n\) such that \(b_n > 1 - \delta\) and \(\varepsilon_1 > 0\) such that \(F_{pq}(\varepsilon_1) > b_n\).

Then for every \(s \in N\) we have \(F_{pq}(k^s_n \varepsilon_1) > b_n\), therefore (choosing \(s\) such that \(k^s_n \varepsilon_1 < \varepsilon\)) \(F_{pq}(\varepsilon) > 1 - \delta\). So \(F_{pq} = \varepsilon_0\), from where we deduce that \(p = q\). \(\Box\)

The following result follows from Theorem 2.6. It can be found also in [23] (see [6, Remark 4.7]).

**Corollary 2.8.** Let \(k\) be a fixed number in \((0, 1)\), \((S, F, T)\) be a complete generalized Menger space under the \(t\)-norm \(T\) of Hadžić type and \(A : S \rightarrow C(S)\) be a mapping with the property that for every \(p, q \in S\) and every \(u \in A(p)\) there exists \(v \in A(q)\) such that

\[ F_{uv}(kt) \geq F_{pq}(t) \quad \text{for every} \quad t > 0. \]

If there exist \(p \in S\) and \(q \in A(p)\) such that \(F_{pq} \in D_+\), then \(A\) has a fixed point.

2.2. Multivalued generalizations of \(C\)-contractions

The following theorem refers to the class of \((\Psi, C)\)-contractions, introduced in [13].

**Theorem 2.9** [13,22]. Let \((S, F, T)\) be a complete Menger space with \(\sup_{a<1} T(a, a) = 1\) and \(f : S \rightarrow C(S)\) be a mapping with the property

\[ x, y \in S, \ t > 0, \ F_{xy}(t) > 1 - t \quad \Rightarrow \quad \forall p \in f(x) \ \exists q \in f(y): \ F_{pq}(\Psi(t)) > 1 - \Psi(t), \]

where \(\Psi : [0, \infty) \rightarrow [0, \infty)\) is such that \(\sum_{n=1}^{\infty} \Psi^n(s) < \infty\) for some \(s > 1\). If

\[ \lim_{n \rightarrow \infty} T_{\infty}^{n=1} \left(1 - \Psi^{n+i-1}(s)\right) = 1 \]

then \(A\) has a fixed point.

**Proof.** It is easy to see that \(f\) is continuous.

Let \(s > 1\) be such that the series \(\sum_{n=1}^{\infty} \Psi^n(s)\) converges and \(p\) be an arbitrary element of \(S\). Next, let \(p_0 = p\) and \(p_1\) be in \(f(p_0)\). Since \(F_{pq}(s) > 1 - s\) for all \(p, q \in S\), we have \(F_{p_0p_1}(s) > 1 - s\).
Using the contraction relation we can find \( p_2 \in A(p_1) \) such that \( F_{p_1 p_2}(\varphi(t)) > 1 - \varphi(t) \), and, by induction, \( p_n \in A(p_{n-1}) \) and \( F_{p_{n-1} p_n}(\varphi^{n-1}(s)) > 1 - \varphi^{n-1}(s) \) for all \( n \geq 1 \).

Defining \( x_n = \varphi^n(s) \), we have \( f_j = F_{p_j p_{j+1}}(t_j) \geq 1 - \varphi^j(s), \forall j \), so \( \lim_{n \to \infty} T_{i=1}^\infty f_{n+i} \geq \lim_{n \to \infty} T_{i=1}^\infty (1 - \varphi^{n+i-1}(s)) = 1 \). \( \Box \)

**Definition 2.10.** Let \( F \) be a probabilistic distance on \( S \) and \( k \in (0, 1) \). A mapping \( A : S \to 2^S \) is called a multivalued weak \( k-C \) contraction if for all \( x, y \in S \) and \( u \in (0, 1) \),

\( (k-C): \quad F_{xy}(u) > 1 - u \quad \Rightarrow \quad \forall p \in A(x) \exists q \in A(y): \quad F_{pq}(ku) > 1 - ku. \)

**Theorem 2.11.** Let \( A : S \to C(S) \) be a multivalued weak \( k-C \) contraction on a complete generalized Menger space \( (S, F, T) \). If \( T \) is \( k \)-convergent and \( F_{pq}(1) > 0 \) for some \( p, q \in S \) and \( q \in A(p) \) then there exists \( x \in S \) such that \( x \in A(x) \).

**Proof.** Again, we will define the sequences \( (p_n) \) and \( (t_n) \) as in Theorem 2.2.

We put \( p_0 = p \) and \( p_1 = q \). Since \( F_{pq}(1) > 0 \), it follows that there exists \( \delta \in (0, 1) \) such that \( F_{p_0 p_1}(\delta) > 1 - \delta \). Indeed, if we suppose that \( F_{p_0 p_1}(\delta) \leq 1 - \delta \) for all \( \delta \in (0, 1) \), then by the left continuity of \( F_{p_0 p_1} \) we deduce that \( F_{p_0 p_1}(1) \leq 0 \), which is a contradiction. Next, using \( (k-C) \) we can find \( p_2 \in A(p_1) \) such that \( F_{p_1 p_2}(k\delta) > 1 - k\delta \).

It follows \( F_{p_1 p_2}(k) > 1 - k \) and, by induction, we obtain \( p_n \) such that \( p_n \in A(p_{n-1}) \) and \( F_{p_n, 1} p_n(k^{n-1}) > 1 - k^n \) for all \( n \geq 2 \).

Defining \( t_n = k^n \), we have \( \lim_{n \to \infty} T_{i=1}^\infty f_{n+i} \geq \lim_{n \to \infty} T_{i=1}^\infty (1 - k^n + 1) = 1 \).

It remains to prove that \( A \) is continuous. Let \( \varepsilon > 0 \) be given and \( \delta \in (0, 1) \) be such that \( k\delta < \varepsilon \). If \( F_{pq}(\delta) > 1 - \delta \) then, due to \( (k-C) \), for each \( x \in A(p) \) we can find \( y \in A(q) \) such that \( F_{xy}(k\delta) > 1 - k\delta \), from where we obtain that \( F_{xy}(\varepsilon) > 1 - \varepsilon \).

The theorem is proved. \( \Box \)

**Corollary 2.12.** Let \( (S, F, T) \) be a complete generalized Menger space under a \( k \)-convergent \( t \)-norm \( T \) and \( f : S \to S \) be a mapping with the property that, for all \( p, q \in S \) and \( t \in (0, 1) \),

\[ F_{pq}(t) > 1 - t \quad \Rightarrow \quad F_{f(p)f(q)}(kt) > 1 - kt. \]

If there exist \( p \in S \) such that \( F_{pf(p)}(1) > 0 \) then \( A \) has a fixed point.

We end with another application of Theorem 2.11 to operator equations (for details see [13, Section 5]).

Let \( (M, d) \) be a complete separable metric space, \( (\Omega, \mathcal{K}, P) \) a probability space with \( P \) continuous and \( S \) be the space of all equivalence classes of measurable mappings \( X : \Omega \to M \) (an element of \( S \) is denoted by \( X(\omega) \) if \( \{X(\omega)\} \in X \)).

Then \( (S, F, T) \), where \( F_{XY}(u) := P\{\omega \in \Omega : d(X(\omega), Y(\omega)) < u\} \) is a Menger space. For the sake of simplicity, \( P\{\omega \in \Omega : d(X(\omega), Y(\omega)) < u\} \) will be written as \( P[d(X, Y) < u] \).
Let \( f : \Omega \times M \to M \) be a random operator, which means that for every measurable mapping \( X : \Omega \to M \), the mapping \( \omega \to f(\omega, X(\omega)) \) is measurable, and \( \hat{f} : S \to S \), \( \hat{f}(X)(\omega) := f(\omega, X(\omega)) \), for every \( \omega \in \Omega \) and \( X \in S \).

If there exists \( \hat{X} \in S \) such that \( P\{\omega \in \Omega, \ d(X(\omega), f(\omega, X(\omega))) < 1\} > 0 \) and, for every \( \hat{X}, \hat{Y} \in S \) and \( u \in (0, 1) \),
\[
P\{d(X, Y) < u\} > 1 - u \implies P\{d(\hat{f}(X), \hat{f}(Y)) < ku\} > 1 - ku
\]
\( (k \in (0, 1) \) is given), then there exists a measurable mapping \( X : \Omega \to M \) such that \( X(\omega) = f(\omega, X(\omega)) \) for every \( \omega \in \Omega \) and \( P(\Omega_0) = 1 \).

The proof is similar to that from [6, Theorem 5.30].

References