Existence of Transformed Rational Complex Chebyshev Approximations, II*

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This paper is a continuation of the author’s existence theory [4]. The problem is the same except that a different convention for defining approximations is also considered and it is assumed that \( |a(t)| \to \infty \) as \( |t| \to \infty \). Readers should consult [4] for the problem statement, notation and preliminaries.

We need a convention for defining approximations \( F(A, \cdot) \) where the denominator \( Q(A, \cdot) \) vanishes. We will adapt one due to Goldstein [2, 3, 85ff].

Convention. Let \( Q(A, x) = 0 \). If \( P(A, x) \neq 0 \), \( F(A, x) = \sigma(\infty) \). If \( P(A, x) = 0 \), \( F(A, x) = f(x) \).

While this convention can be applied to all \( X \), it is most appropriate for finite \( X \), as no other convention seems practical. In particular, even if we could apply Boehm’s convention [4], we would not necessarily get existence—in Example 1, below, an approximation zero on \( \{ \frac{1}{2}, 1 \} \) is identically zero with Boehm’s convention. If \( X \) is infinite, Goldstein’s may not be the most satisfactory convention, as the convention may give \( F(A, \cdot) \) discontinuities which need not exist with other conventions.

**Theorem.** Let \( P \) be a non-empty closed subset of \( \hat{P} \). With Goldstein’s convention, there exists a best parameter from \( P \) to all \( f \in C(X) \).

**Proof.** The proof is exactly the same as the proof for the corresponding result in [4] up to the case \( Q(A, x) \neq 0 \).

If \( Q(A, x) = 0 \) we have two possibilities. First, we might have \( P(A, x) \neq 0 \) in which case \( P(A^k, x)/Q(A^k, x) \to \infty \), \( F(A^k, x) \to \infty \), and

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\[ |f(x) - F(A^k, x)| \to \infty, \] which contradicts the choice of \( A^k \). There only remains the possibility that \( P(A, x) = 0 \), in which case

\[ |f(x) - F(A, x)| = 0 \leq \rho(f'). \]

Three classes of closed sets of coefficients are given in [4]. A fourth follows:

Let \( Y = \{y_1, ..., y_s\} \) be a finite subset of \( X \). Let

\[ P_t = \{ A : A \in \hat{P}, F(A, y_i) = f(y_i), \ i = 1, ..., s \}. \]

\( P_t \) is closed. Assume \( \{A^k\} \subseteq P_t, \{A^k\} \to A \). Take \( y \in Y \). If \( Q(A, y) \neq 0 \), \( F(A^k, y) = f(y) \to F(A, y) \). If \( Q(A, y) = 0 \) and \( P(A, y) \neq 0 \), \( |F(A^k, y)| \to \infty \), which is a contradiction. If \( Q(A, y) = P(A, y) = 0 \), \( F(A, y) = f(y) \) by convention. It should be noted that \( P_t \) may not be closed with Boehm's convention [4].

**Admissible Approximation**

Dolganov [1] defines a rational function \( R(A, \cdot) \) to be *admissible* if \( \text{Re}(Q(A, \cdot)) > 0 \). Even in approximation by ratios of power polynomials (ordinary rational functions), a best approximation need not exist if we do not permit denominators to have zeros.

**Example 1.** Let \( X = \{0, \frac{1}{2}, 1\} \) and \( F(A, x) = a_0/(a_0 + a_1 x) \). Let \( f(0) = 1 \) and \( f(\frac{1}{2}) = f(1) = 0 \). As \( 1/(1 + kx) \to f \) uniformly on \( X \), \( \rho(f) = 0 \). But if \( Q(A, \cdot) \) cannot have zeros, the only approximant vanishing on \( \{\frac{1}{2}, 1\} \) is identically zero.

Examples of non-existence on sets with no isolated points are given in [4].

**Real Approximation**

Consider the case in which all basis functions are real, all coefficients are real, \( \sigma \) is a continuous mapping of the real line into the extended real line, and \( f \) is real. This is the case of real Chebyshev approximation by transformed rational functions. A special case is where \( \sigma(x) = x \), which has already been studied by Goldstein. The existence theorem obtained earlier in this paper applies. Of particular interest is the case where \( P \) is the subset of \( \hat{P} \) with \( Q(A, \cdot) \geq 0 \). This is \( P_r \) with \( K \) being the non-negative real line.
Approximation with Unbounded Bases

In this and following sections we drop the requirement that \( X \) be compact but still require that basis functions be continuous on \( X \). A problem of this type with all basis functions bounded has already been considered by Boehm [5]. We consider the case in which basis functions may be unbounded on \( X \). A classical case where this happens is when \( X \) is an unbounded subset of the real line or complex plane, for example, \([0, \infty)\) or \((-\infty, \infty)\), and we approximate by ordinary rational functions

\[
R_m^n = \{ p/q : q \in H_n, q \in H_m, q \neq 0 \},
\]

where \( H_l \) is the set of power polynomials of degree \( l \). We claim that the existence theorem of the previous paper [4] and of this paper holds for the case of unbounded basis functions providing \( f \) is bounded on \( X \) (in addition to being continuous).

If all approximants are unbounded, \( \rho(f) = \infty \) and we have existence trivially. If at least one approximant is bounded, \( \rho(f) < \infty \) and we follow the existence proof of [4] down to an inequality involving \( |R(A, x)| \) on the left-hand side. We replace that inequality by the following discussion. By the normalization \( \sum_{k=1}^{m} |a_{n+k}| = 1 \),

\[
|Q(A, x)| \leq \sum_{k=1}^{m} |\psi_k(x)|.
\]

If the right-hand side of (2) is >0, we can write the inequality

\[
|R(A, x)| = |P(A, x)|/|Q(A, x)| \geq |P(A, x)| \sum_{k=1}^{m} |\psi_k(x)|.
\]

If the right-hand side of (2) is zero, we have

\[
|R(A, x)| < \infty \rightarrow P(A, x) = 0.
\]

Let \( Y \) be any \( n \)-point subset on which \( \{\phi_1, \ldots, \phi_n\} \) is linearly independent. Let there exist \( M \) such that

\[
\max \{|R(A^k, x)| : x \in Y\} < M;
\]

then (4) implies by (3), (3') that

\[
\max \{|P(A^k, x)| : x \in Y\}
\]

is bounded. This implies that the numerator coefficients of the sequence \( \{A^k\} \) are bounded and we use the rest of the proof of [4] and its modification for Goldstein's convention earlier in this paper.
ORDINARY RATIONAL APPROXIMATIONS

Consider approximation by $R^m_n$ (defined by (1)) on $X$, a subset of the complex plane or real line. First, let us assume $X$ has no isolated points and use Boehm’s convention. The existence theorem of [4] applies and we have existence of a best approximation $p/q$. By considering multiplicities of zeros of polynomials, it can be seen by standard arguments that $p/q$ can be replaced by $p_0/q_0$, $p_0$ and $q_0$ relatively prime and $q_0$ having no zeros on $\bar{X}$, the closure of $X$.

Remark. If we approximate with constraints, the above may not be true. For example, we might want denominators $\geq 0$. In the case $X = \{x : |x| \geq 1\}$, $x/x^2$ has a positive denominator on $X$ but removing common factors gives $1/x$.

In the case of real approximation and $X$ an interval, $q_0$ is of one sign on $\bar{X}$; hence (by changing the sign of $p_0$ and $q_0$ if necessary) we can assume $q_0 > 0$ and there is a best approximation which is admissible (that is, its denominator is $> 0$). The same results hold if we transform $R^m_n$ by $\sigma$. Second, let us apply Goldstein’s convention. The existence theorem of this paper applies and we get existence of a best approximation $p/q$. However, we cannot always cancel out common factors (see Example 1). Existence remains if we transform $R^m_n$ by $\sigma$.

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