Reflexivity and Approximate Reflexivity for Bounded Boolean Algebras of Projections

DON HADWIN

Department of Mathematics, University of New Hampshire, Durham, New Hampshire 03824

AND

MEHMET ORHON

Department of Mathematics, Middle East Technical University, Ankara, Turkey

Communicated by D. Sarason

Received February 9, 1988

Let $K$ be a compact Hausdorff space. It is proven that any bounded unital representation $m$ of $C(K)$ on a Banach space $X$ has the property that the closure of $m(C(K))$ in the weak operator topology is a reflexive operator algebra. As a consequence, it is shown that if $\mathcal{B}$ is an arbitrary bounded Boolean algebra of bounded projections on a Banach space $X$, then $\text{AlgLat}(\mathcal{B})$ is the weak operator topology closure of the linear span of $\mathcal{B}$. These generalize the work of several authors. As a corollary, an alternate proof of a theorem of Bade is obtained. In addition, approximate reflexivity results are obtained for the norm closures of $m(C(K))$ and $\text{span}(\mathcal{B})$.

1. PRELIMINARIES

Suppose $\mathcal{B}$ is a complete (in the sense of Bade) Boolean algebra of projections on a Banach space $X$. The well-known result of W. G. Bade states that the uniform closure of the (linear) span of $\mathcal{B}$ is reflexive (i.e., $\overline{\text{sp} \mathcal{B}} = \text{AlgLat}(\mathcal{B})$) [3; 5, XVII.3.16]. Let $K$ be a compact Hausdorff space and let $m: C(K) \to L(X)$ be a bounded homomorphism from the continuous complex functions on $K$ into the set of (continuous linear) operators on $X$ such that $m(1) = 1$. In [11] it was shown that $m(C(K))$ is contained in the uniform closure of the span of a complete Boolean algebra of projections if and only if $m$ has weakly compact action (i.e., $\alpha \mapsto m(\alpha)x$ is a weakly compact map from $C(K)$ into $X$ for each $x$ in $X$). In this case Bade’s result
implies that the weak operator closure of $m(C(K))$ is reflexive (i.e., $\text{w-cl}(m(C(K))) = \text{AlgLat} m(C(K))$) [11, Theorem 31]. Therefore, in this sense, the theorem of Bade generalizes the von Neumann double commutant theorem (in the commutative case) for self-adjoint algebras of operators on a Hilbert space [2, 1.2.1]. In [9] an asymptotic version of the double commutant theorem was proved. The analogue in our setting would say that $m(C(K))$ is approximately reflexive, i.e., $m(C(K)) = \text{apprAlgLat} m(C(K))$.

This paper investigates the general situation in which $m$ does not have weakly compact action or when the bounded Boolean algebra $B$ is not complete. We show that

1. $\text{w-cl} m(C(K))$ is always reflexive,
2. $m(C(K))$ is always approximately reflexive,
3. the weak operator closure of $\text{sp} B$ is always reflexive,
4. the uniform closure of $\text{sp} B$ is always approximately reflexive.

We will need some preliminary results outlined in [10]. First of all, we can assume that $m$ is $1$–$1$, and, by putting an equivalent norm on $X$, we can assume that $m$ is an isometry, and that $a, b \in C(K)$, $x \in X$, and $|a| \leq |b|$ implies that $\|m(a)x\| \leq \|m(b)x\|$ [10, Lemma 2].

We also need to consider results on Arens extensions (see [1, 10]). We let $X^*$ denote the dual of a Banach space $X$. Associated with the module multiplication

1. $C(K) \times X \to X :: (a, x) \to ax = m(a)x$

we define three other bilinear maps:

2. $X \times X^* \to C(K)^* :: (x, \alpha) \to \mu_{x,\alpha} :: \mu_{x,\alpha}(a) = \alpha(ax)$,
3. $X^* \times C(K)^{**} \to X^* :: (\alpha, a) \to a\alpha :: (a\alpha)(x) = a(\mu_{x,\alpha})$,
4. $C(K)^{**} \times X^{**} \to X^{**} :: (\alpha, \beta) \to a\beta :: (a\beta)(\alpha) = \beta(a\alpha)$.

When (1) is taken as the product on $C(K)$ (i.e., $X = C(K)$), then (4) becomes the Arens product on $C(K)^{**}$, which makes $C(K)^{**}$ isomorphic to $C(S)$ with $S$ hyperstonian [1]. Furthermore, (3) defines a Banach $C(K)^{**}$-module structure on $X^*$ that gives a homomorphism $m^* : C(K)^{**} \to L(X^*)$ defined by $m^*(a)(\alpha) = ax$. The map (4) defines a Banach $C(K)^{**}$-module structure on $X^{**}$. Since the $C(K)^{**}$-module structures on $X^*$ and $X^{**}$ extend the canonical induced $C(K)$-module structures on these spaces, we call (3) and (4) the Arens extensions of the module multiplication on $X$. For information on Arens extensions the reader can consult [1].

We say that a projection $p$ in $C(K)$ is a carrier projection for a vector $x$
in \( X \) if \( \{ a \in C(K) : m(a)x = 0 \} = C(K)(1 - p) \). Carrier projections play an important role in our results. If we consider \( X \) as a subset of \( X^{**} \) considered as a \( C(K)^{**} \)-module with the Arens multiplication defined above, we define \( \langle X \rangle \) to be the norm closed \( C(K)^{**} \)-submodule generated above, we define \( \langle X \rangle \) to be the norm closed \( C(K)^{**} \)-submodule generated by \( X \), i.e., \( \langle X \rangle = \overline{\text{sp}}\{ ax : a \in C(K)^{**}, x \in X \} \). We denote by \( \sigma = \sigma(\langle X \rangle, X^{**}) \) the relative \( w^{*} \)-topology on \( \langle X \rangle \). We list some facts from [10] for easy reference.

\textbf{Lemma 1.} (1) For each \( a \) in \( C(K) \), \( m^{**}(a) \) is the adjoint in \( L(X^{**}) \) of the operator \( m(a) \) in \( L(X) \).

(2) \( m^{**} \) is \( (w^{*}, w^{*}-\text{operator}) \)-continuous.

(3) For each \( x \) in \( X^{**} \) the linear map from \( C(K)^{**} \) into \( X^{**} \) that sends \( a \) to \( ax \) is \( (w^{*}, w^{*}) \)-continuous.

(4) Each \( x \) in \( X^{**} \) has a carrier projection \( e_{x} \) in \( C(K)^{**} \).

(5) For each \( z \) in \( \langle X \rangle \), the linear map from \( C(K)^{**} \) to \( \langle X \rangle \) that sends \( a \) to \( az \) is \( (w^{*}, \sigma) \)-continuous.

(6) Each \( z \) in \( \langle X \rangle \) has a carrier projection \( e_{z} \) in \( C(K)^{**} \).

The map \( m^{**} \) need not be \( 1-1 \), but it follows from the above lemma that \( \ker(m^{**}) \) is a \( w^{*} \)-closed ideal in \( C(K)^{**} \), which implies that there is a projection \( p \) in \( C(K)^{**} \) such that \( \ker(m^{**}) = (1 - p) C(K)^{**} \). We let \( A = pC(K)^{**} \). Then \( A \) is a commutative von Neumann algebra (i.e., \( A = C(S) \) with \( S \) hyperstonian), \( A \) has a predual, and the results in the preceding lemma remain true when \( C(K)^{**} \) is replaced by \( A \). Moreover, \( m^{**} \) is \( 1-1 \) on \( A \), and if \( m \) is a contraction, then \( m^{**} \) is isometric on \( A \). Finally, suppose \( \mu \in C(K)^{*} \), i.e., \( \mu \) is a complex regular Borel measure on \( K \). The Lebesgue decomposition for measures defines a projection \( P \) from \( C(K)^{*} \) onto \( L^{1}(\mu) = \{ v \in C(K)^{*} : v \ll \mu \} \). The adjoint \( P^{*} \) defines a projection of \( C(K)^{**} \) onto \( L^{\infty}(\mu) \). In this way we can identify \( L^{\infty}(\mu) \) as a subset of \( C(K)^{**} \).

\section{Reflexivity}

Suppose \( \mathcal{S} \subset L(X) \). We let \( \mathcal{S}^{c} \) denote the commutant of \( \mathcal{S} \) in \( L(X) \), i.e., \( \mathcal{S}^{c} = \{ T \in L(X) : ST = TS \text{ for every } S \text{ in } \mathcal{S} \} \). We let \( \text{Lat} \mathcal{S} \) denote the set of all (closed linear) subspaces of \( X \) that are invariant under each \( S \) in \( \mathcal{S} \), and we let \( \text{AlgLat} \mathcal{S} \) denote the set of all operators in \( L(X) \) that leave invariant all of the subspaces in \( \text{Lat} \mathcal{S} \). We say that \( S \) is reflexive if \( \text{AlgLat} \mathcal{S} \) is the unital weakly closed algebra generated by \( \mathcal{S} \). We begin with an essentially algebraic result (part (2)).
LEMMA 2. Suppose $Y$ is a Banach space, $K$ is Stonian, $\rho : C(K) \to L(Y)$ is a contractive unital homomorphism, $T \in \rho(C(K))$. Suppose that $M$ is a linear subspace of $Y$ (not necessarily a $C(K)$-module).

(1) If $Ty = ay$ for $y$ in $Y$ and $a$ in $C(K)$, and if $y$ has a carrier projection $e_y$ in $C(K)$, then $\|ae_y\| \leq \|T\|$.

(2) If every vector in $M$ has a carrier projection in $C(K)$ and if $Ty \in C(K)y$ for every $y$ in $M$, then there is an $a$ in $C(K)$ such that $Ty = ay$ for all $y$ in $M$.

Proof. (1) Assume via contradiction that $\|ae_y\| > \|T\|$. Then there is an $\varepsilon > 0$ and a nonzero projection $e \leq e_y$ such that $|ae_y, e| \geq (\|T\| + \varepsilon)e$, since $\rho$ is a contraction. But $\|T\| \|ey\| \geq \|Tey\| = \|eTy\| = \|eae_y, y\| \geq (\|T\| + \varepsilon)\|ey\|$. This implies that $ey = 0$, and hence that $e = ee_y = 0$ ($e_y$ is a carrier projection for $y$), a contradiction.

(2) It follows from $Ty \in C(K)y$ that, for each $y$ in $M$, there is a unique $a_y$ in $C(K)$ such that $Ty = a_y y$ and $a_y e_y = a_y$ ($e_y$ denotes the carrier projection of $y$). By (1) we know that $\|a_y\| \leq \|T\|$ for every $y$ in $M$. We now wish to show, for every $y, z$ in $M$, that

$$(a_y - a_z) e_y e_z = 0. \tag{*}$$

Choose a projection $e$ in $C(K)$ that is maximal with respect to the property that $e(a_y - a_z) e_y e_z = 0$. Let $w = (1 - e)y$. Clearly, $e_w = (1 - e) e_y$ is a carrier projection for $w$. Assume via contradiction that $C(K)w \cap C(K)z \neq 0$. Then there is an $a$ in $C(K) e_w$ such that $0 \neq aw \in C(K)z$. Applying $T$ we obtain $a_y aw = a_z aw$, which implies that $a(a_y - a_z) e_y e_z = 0$. Since $a \neq 0$ and $ae = 0$, this clearly violates the maximality of $e$. Hence $C(K)w \cap C(K)z = 0$.

Hence, if $c = a_y + z$, then $(1 - e)T(y + z) = (1 - e)c(y + z) = (1 - e)a_y y + (1 - e)a_z z$, which implies that $(1 - e)(c - a_y)w = (1 - e)(a_z - c)z$. Thus $(1 - e)(c - a_y) e_y = (1 - e)(a_z - c) e_z = 0$. Multiplying by $e_y e_z$ we obtain $(1 - e)(a_y - a_z) e_y e_z = 0$, which implies (*).

It follows from the fact that $K$ is Stonian and $\|a_y\| \leq \|T\|$ that there is an $a$ in $C(K)$ such that $ae_y = a_y$ for every $y$ in $M$. Thus $Ty = ay$ for every $y$ in $M$.

LEMMA 3. Suppose $X$ is cyclic. Then $m^*(A) = m^*(A)$.

Proof. Suppose $T$ is in the commutant of $m^*(A)$, choose $x$ in $X$ so that $C(K)x$ is dense in $X$, and let $x \in X^*$. Note that $\mu_{x, ax}(b) = \mu_{x, ax}(b)$ for every $a$ in $A$ and every $x \in X$. (Proof: First check when $a \in C(K)$, then take $w^*$-limits.) Since $X$ is cyclic, it follows, for each $\beta, \gamma$ in $X^*$, that $\mu_{x, \beta} = \mu_{x, \gamma}$.

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implies $\beta = \gamma$. Since $\alpha x = 0$ if and only if $\alpha \mu_{x,\alpha} = 0$, the carrier projection $e_x$ is also the carrier projection for $\mu_{x,\alpha}$ in $A$. Since $\alpha x = 0$ implies that $aT(\alpha) = T(\alpha x) = 0$, it follows that $e_{x,T} \leq e_x$. Hence the measure $\mu_{x,T}$ is absolutely continuous with respect to the measure $\mu_{x,\alpha}$. Let $f$ be the Radon–Nikodym derivative. If $e$ is a projection in $A$ such that $ef \in A$ (i.e., $ef$ is bounded), then $\mu_{x,Tx} = e\mu_{x,Tx} = ef\mu_{x,\alpha} = \mu_{x,efx}$. The preceding lemma implies that $\|ef_{x}\| \leq \|T\|$. It follows that $a = e_x f = f \in A$. Thus $\mu_{x,Tx} = a \mu_{x,\alpha} = \mu_{x,\alpha}$, which implies that $Tz = ax$. It now follows from Lemma 2 that $T \in m^*(A)$.

**COROLLARY 4.** If $X$ is cyclic, then $m(C(K))^c = w_{\text{-cl}}(m(C(K)))$.

If $\mathcal{S} \subset L(X^*)$, we let $\text{AlgLat}^* (\mathcal{S})$ denote the set of operators that leave invariant the w*-closed invariant subspaces of $\mathcal{S}$.

**LEMMA 5.** $m^* (A) \cap \text{AlgLat}^* m^* (A) = \text{AlgLat} m^* (A) = m^* (A)$.

**Proof.** We can assume that $m^*$ is a contraction. Since $(1 - e) X^*$ and $eX^*$ are in $\text{Lat} m^* (A)$ for each projection $e$ in $A$, it follows that the elements of $\text{AlgLat} m^* (A)$ commute with the projections in $A$. But $A$ is the norm closed linear span of the projections in $A$; thus $\text{AlgLat} m^* (A) \subset m^* (A)^c$. Hence the proof of the lemma reduces to showing $m^* (A)^c \cap \text{AlgLat}^* m^* (A) \subset m^* (A)$.

Suppose $T \in m^* (A)^c \cap \text{AlgLat}^* m^* (A)$. Given an $x$ in $X$, let $X(x) = [C(K)x]^\perp$. The polar $X(x)^0$ of $X(x)$ in $X^*$ is a w*-closed $A$-submodule of $X^*$. So $T$ leaves $X(x)^0$ invariant. For any $\alpha$ in $X^*$, define $\hat{T}[\alpha] = [T\alpha]$, where $[\alpha]$ denotes $\alpha + X(x)^0$ in $X^*/X(x)^0 = X(x)^*$. Clearly $\hat{T}$ is a well-defined operator.

The Arens extension of the $C(K)$-module multiplication on $X(x)$ to $X(x)^*$ is identical with the $A$-module structure on $X(x)^*$ induced by the quotient map. To see this, let $a \in A$ and let $\{a_i\}$ be a net in $C(K)$ such that $a_i \to a(w^*)$. Let $b \in C(K)$ and $\alpha \in X^*$. One has

$$a[\alpha](bx) = \lim[a_i](a_i, bx) = \lim_\alpha (a_i, bx) = ax(bx) = [\alpha\alpha](bx).$$

It is easily checked that $\hat{T}$ is an $A$-module homomorphisms on $X(x)^*$. By Lemma 3, there is an $a_i$ in $A$ such that $\hat{T}[\alpha] = a_i[\alpha]$ for all $\alpha$ in $X^*$. Then $T(\alpha)(x) = (a_i, \alpha)(x)$ for each $x$ in $X$ and each $\alpha$ in $X^*$.

Consider the adjoint $T^*$ of $T$ on $X^{**}$. Then $T^*$ is an $A$-module homomorphism, and, for each $x$ in $X \subset X' \subset X^{**}$ and each $\alpha$ in $X^*$, we have $T^*(x)(\alpha) = (a, \alpha)(x) = \alpha(a, x) = a, x(\alpha)$. Thus $T^*(x) = a, x$ for every $x$ in $X$. It follows from Lemma 2 that there is an $a$ in $A$ such that $T^*(x) = ax$ for all $x$ in $X$. Thus $T = m^*(a)$.  


Corollary 6. $m^*(A)$ is closed in the weak operator topology on \(L(X^*)\).

Remarks. Let \(K\) be totally disconnected. The simple observation at the beginning of the proof of Lemma 5 shows that \(\text{AlgLat} \ m(C(K))\) commutes with \(m(C(K))\). That this is true for every compact Hausdorff space \(K\) follows from a theorem of Evans [7], which shows that an operator \(T\) on \(X\) commutes with \(m(C(K))\) if and only if \(m(a)x = 0\) implies \(m(a)Tx = 0\) whenever \(a \in C(K)\) and \(x \in X\). Since \(T \in \text{AlgLat}(m(C(K)))\) implies that \(Tx \in [m(C(K))x]^-\) for every \(x\), it follows that \(\text{AlgLat} \ m(C(K))\) commutes with \(m(C(K))\). Also the proof of part (1) of Lemma 2 is a variation of an argument of Evans [6, 5.1].

Theorem 7. Suppose \(m: C(K) \to L(X)\) is a bounded unital homomorphism. Then \(\text{AlgLat} \ m(C(K)) = w-cl \ m(C(K))\). Moreover, for each \(T\) in \(\text{AlgLat} m(C(K))\), there is a net \(\{a_\lambda\}\) in \(C(K)\) with \(\|a_\lambda\| \leq \|m\| \|T\|\) for each \(\lambda\), such that \(m(a_\lambda) \to T\) in the weak operator topology on \(L(X)\).

Proof. Suppose \(T \in \text{AlgLat} m(C(K))\). Then \(T\) commutes with \(m(C((K)))\) [7] (cf. the preceding remark). This implies that \(T^*\) is in \(m^*(A)^c \cap \text{AlgLat}^* m^*(A)\). It follows from Lemma 5 that there is an \(a\) in \(A\) and a net \(\{a_\lambda\}\) in \(C(K)\) with \(\|a_\lambda\| = \|a\|\) for every \(\lambda\) and such that \(a_\lambda \to a\) (w*) in \(A\). Thus \(m^*(a_\lambda) \to T^*\) in the w*-operator topology on \(L(X^*)\) (Lemma 1). This means that \(m(a_\lambda) \to T\) in the weak operator topology on \(L(X)\).

Corollary 8. There is a compact Hausdorff space \(K'\) and a bounded unital homomorphism \(\tilde{m}: C(K') \to L(X)\) such that \(\tilde{m}(C(K')) = w-cl \ m(C(K))\).

Proof. Suppose \(m^*\) is an isometry. Let \(f^*\) denote the conjugate of an \(f\) in \(A\). Let \(T \in \text{AlgLat} \ m(C(K))\) and \(\{a_\lambda\}\) is a bounded net in \(C(K)\) such that \(m(a_\lambda) \to T\) in the weak operator topology. Choose a bounded net \(\{b_\gamma\}\) in \(C(K)\), consisting of convex combinations of the \(a_\lambda\)'s so that \(m(b_\gamma) \to T\) in the strong operator topology. By Alaoglu's theorem, we can assume that there is an \(a\) in \(A\) such that \(b_\gamma \to a\) and \(\tilde{b}_\gamma \to \tilde{a}\) in the w*-topology on \(A\).

Therefore, for each \(x\) in \(X\), \(\tilde{b}_\gamma x \to \tilde{a}x\) with respect to the relative w*-topology on \(\langle X \rangle\) (Lemma 1). On the other hand, \(|b_\gamma \cdot \tilde{b}_\gamma| = |b_\gamma| |b_\gamma|\) implies \(\|b_\gamma - \tilde{b}_\gamma\| = \|b_\gamma - b_\gamma\| \to 0\). That is, there is a \(y\) in \(X\) such that \(\tilde{b}_\gamma x \to y\) in norm. Thus if we denote the embedding of \(A\) in \(L(\langle X \rangle)\) by \(m^{**}\), we have \(m^{**}(\tilde{a})(X) \subset X\). Let \(S = m^{**}(\tilde{a})\) \(X\). Then \(S \in \text{AlgLat}(m(C(K)))\) and \(S^* = m^*(\tilde{a})\). Since \((m^*)^{-1}(\{T^*: T \in \text{AlgLat} m(C(K))\})\) is a norm closed unital self-adjoint subalgebra of the commutative \(C^*\)-algebra \(A\), it is isomorphic to \(C(K')\) for some compact Hausdorff space \(K'\).
Suppose $\mathcal{B}$ is a bounded Boolean algebra of projections on $X$. Let $K$ denote the Stone representation space of $\mathcal{B}$. Then there is a bounded unital homomorphism $m : C(K) \to L(K)$ such that $m(C(K)) = \text{sp} \mathcal{B}$. The Boolean algebra $\mathcal{B}$ of projections is complete (in the sense of Bade) if $\mathcal{B}$ is complete as an abstract Boolean algebra, and if $\{e_\lambda\}$ is an increasing net with supremum $e$ in $\mathcal{B}$, then $e_\lambda x \to e x$ for each $x$ in $X$ [3, 5].

**THEOREM 9.** If $\mathcal{B}$ is a bounded Boolean algebra of projections on a Banach space $X$, then $\text{AlgLat} \mathcal{B}$ is the weak operator closure of the span of $\mathcal{B}$.

**COROLLARY 10 (Bade [3]).** If $\mathcal{B}$ is a complete Boolean algebra of projections on $X$, then $\text{AlgLat} \mathcal{B}$ is the norm closure of the span of $\mathcal{B}$.

**Proof:** Consider the induced bounded homomorphism $m : C(K) \to L(X)$, where $K$ is the Stone representation space of $\mathcal{B}$. Since $\mathcal{B}$ is complete in the sense of Bade, $K$ is hyperstonian (i.e., $C(K)$ is a dual Banach space) and $m$ is $(w^*, \text{weak-operator})$-continuous [11, Theorem 1]. Let $T \in \text{AlgLat} \mathcal{B}$. By Theorem 7 (and Alaoglu), there is a bounded net $\{a_\lambda\}$ in $C(K)$ such that $m(a_\lambda) \to T$ in the weak operator topology and converges in the $w^*$-topology to an element $a$ of $C(K)$. Then $m(a_\lambda) \to m(a)$ in the weak operator topology, so $T = m(a)$. 

The notion of $\tau$-complete bounded Boolean algebras of projections (i.e., each vector has a carrier projection) was introduced by Veksler [14] and was also considered by Rall [12]. This class strictly contains the complete Boolean algebras of projections. For this class of algebras Rall obtained the following reflexivity result [12; 13, p. 220].

**COROLLARY 11 (Rall).** Suppose $\mathcal{B}$ is a $\tau$-complete bounded Boolean algebra of projections on $X$.

1. If $X$ is cyclic, then $\mathcal{B}^* = \text{sp}(\mathcal{B})$.

2. If $\mathcal{B}$ is complete as an abstract Boolean algebra, then $\text{AlgLat} \mathcal{B} = \text{sp}(\mathcal{B})$.

**Proof.** Let $K$ be the Stone representation space of $\mathcal{B}$. It can be verified directly that $\mathcal{B}$ is $\tau$-complete on $C(K)$ and $K$ is quasi-Stonian (i.e., each sequence in $\mathcal{B}$ has a supremum in $\mathcal{B}$).

1. The crucial property of a Banach space that is cyclic with respect to a $\tau$-complete bounded Boolean algebra of projections is due to Veksler [14, Lemma 7]: Let $x_n \to x$ and $y_n \to y$ such that $e_\lambda e_n e_{\gamma_n} = 0$ for each $n$. Then $e_\lambda e_n e_{\gamma_n} = 0$.

By Corollary 8, $m(C(K))^* = \text{AlgLat} \mathcal{B} = C(K')$ for some compact
Hausdorff space $K'$. Since $K$ is a quotient of $K'$, to prove (1) it is sufficient to show that $B$ separates the points of $K'$. Let $s, t \in K'$ be distinct points. Let $\varphi \in C(K')$ be such that $\varphi(s) = 1$, $\varphi(t) = -1$, and $-1 \leq \varphi \leq 1$. There is a bounded sequence $\{a_n\}$ of real functions in $C(K)$ such that $m(a_n) \to \varphi$ in the strong operator topology in $L(X)$ (use the cyclicity of $X$). Since $|a_n^+ - \varphi^+| \leq |a_n - \varphi|$, we have $m(a_n^+) \to \varphi^+$ and $m(a_n^-) \to \varphi^-$ in the strong operator topology. Choose $e_n \in B$ so that $e_n a_n = a_n^+$ and $(1 - e_n) a_n = a_n^-$. Let $x_0$ be a cyclic vector. Then $e_n a_n x_0 \to \varphi^+ x_0$ and $(1 - e_n) a_n x_0 \to \varphi^- x_0$.

By Veksler's result, there is an $e \in B$ such that $(1 - e) \varphi^+ x_0 = 0 = e \varphi^- x_0$. Since $x_0$ is a cyclic vector, one obtains $(1 - e) \varphi = 0 = e \varphi^-$. Thus $e(s) = 1$ and $e(t) = 0$. Hence $B$ separates the points of $C(K')$, and (1) is proved.

(2) Let $T \in \operatorname{AlgLat} B$. By part (1), for each $x$ in $X$, there is an $a_x \in e_x \operatorname{sp} B$ such that $T x = a_x x$. Since $K$ is Stonian, and $T \in m(C(K))^\ast$, we may apply Lemma 2 to obtain an $a$ in $C(K)$ such that $T x = a x$ for every $x$ in $X$.

3. Asymptotic Results

We now turn to approximate reflexivity and double commutants. Suppose $\{M_j\}$ is a net of closed subspaces of $X$, and for each $j$, $\pi_j : X \to X/M_j$ is the quotient map. For an operator $T$ in $L(X)$, the net $\{M_j\}$ is asymptotically invariant if $\|\pi_j \circ (T|M_j)\| \to 0$. For a subset $\mathcal{S}$ of $L(X)$, $\operatorname{apprAlgLat}(\mathcal{S})$ is the set of all operators $T$ that leave asymptotically invariant every net of subspaces left asymptotically invariant by every element of $\mathcal{S}$. Clearly $\operatorname{apprAlgLat}(\mathcal{S})$ is a unital norm closed subalgebra of $\operatorname{AlgLat}(\mathcal{S})$. It is also clear that if $T \in \operatorname{apprAlgLat}(\mathcal{S})$, then $\|T - P_j\| \to 0$. The set $\mathcal{S}$ is called approximately reflexive if $\operatorname{apprAlgLat}(\mathcal{S})$ is the unital norm closed algebra generated by $\mathcal{S}$. We define $\mathcal{S}^{\operatorname{cc}}$ to be the set of operators $T$ such that $\|B_j T - T B_j\| \to 0$ for every bounded net $\{B_j\}$ in $L(X)$ such that $\|B_j S - S B_j\| \to 0$ for each $S$ in $\mathcal{S}$. It is clear that $\mathcal{S}^{\operatorname{cc}}$ is unital norm closed subalgebra of $\mathcal{S}^{\operatorname{cc}}$.

The following lemma contains our key asymptotic result.

**Lemma 12.** Suppose $\mathcal{A}$ is a norm closed self-adjoint unital subalgebra of $C(K)$. Then

1. $m(C(K)) \cap \operatorname{apprAlgLat}(m(\mathcal{A})) = m(\mathcal{A})$,
2. $m(C(K)) \cap \operatorname{appr}(m(\mathcal{A}))^{\operatorname{cc}} = m(\mathcal{A})$.

**Proof.** We can assume that $m$ is an isometry. Suppose $g \in C(K)$, and $g \notin \mathcal{A}$. The Bishop–Stone–Weierstrass theorem implies that there are points...
Let $s, t \in K$ such that $f(s) = f(t)$ for every $f$ in $\mathcal{A}$, but $g(s) \neq g(t)$. We can assume that $g(s) = 1$ and $g(t) = 0$.

Let $A$ be the collection of all pairs $(\mathcal{F}, \mathcal{E})$ with $\mathcal{F}$ a finite subset of $\mathcal{A}$ and $\mathcal{E} > 0$. Fix $A = (\mathcal{F}, \mathcal{E})$ in $A$. Since $m$ is $1$-$1$, we can choose nonzero functions $u, v, u_1, v_1$ in $C(K)$, with $0 \leq u, v \leq 1$, such that:

1. $uv = 0$, $uu_1 = u_1$, $vv_1 = v_1$,
2. $\|(f - f(s))u\|, \|(f - f(t))v\| < \varepsilon$ for all $f$ in $\mathcal{F} \cup \{g\}$.

If we choose unit vectors $x, y$ in $X$ with $x \in \text{range}(m(u_1))$ and $y \in \text{range}(m(v_1))$, we then have

3. $m(u)x - x, m(v)y - y$, and $m(u)y = m(v)x = 0$.

Next choose $a, a' \in X^*$ so that $\|a\| = \|a'\| = \alpha(x) = \beta(y) = 1$. Define $P_\alpha$ in $L(X)$ by $P_\alpha(z) = [\alpha(a)z + \beta(a')z].$ Then $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| \leq 2$. Moreover, since $f(s) = f(t)$ for each $f$ in $\mathcal{F}$, we have $\|P_\alpha m(f) - m(f) P_\alpha\| \leq \|P_\alpha (m(f) - f(s)) P_\alpha\| + \|(m(f) - f(s)) P_\alpha\| \leq 4 \varepsilon$. In addition, $\|P_\alpha m(g) - m(g) P_\alpha\| \geq \|(P_\alpha m(g) - m(g) P_\alpha) y\| \geq \|x\| - \|P_\alpha ((m(g) - g(t)) y)\| - \|(g(s) - m(g)) x\| \geq 1 - 3 \varepsilon$.

Thus $\{P_\alpha\}$ is a bounded net of projections, $\|P_\alpha S - SP_\alpha\| \rightarrow 0$ for every $S$ in $m(\mathcal{A})$, and $\|P_\alpha m(g) - m(g) P_\alpha\| \rightarrow 0$. Hence $m(g)$ is in neither $\text{apprAlgLat}(m(\mathcal{A}))$ nor $\text{appr}(m(\mathcal{A}))^{cc}$.  

**Corollary 13.** If $X$ is cyclic, then $\text{appr}(m(C(K)))^{cc} = m(C(K))$.

**Proof.** It follows from Corollary 4 and Corollary 8 that $\text{appr}(m(C(K)))^{cc} \subset m(C(K))^{cc} = \text{w-cl}(m(C(K))) = C(K')$ for some compact Hausdorff space $K'$. We now apply the preceding lemma with $\mathcal{A} = C(K)$ and $K'$ playing the role of $K$.

The method of the preceding proof also yields the following result.

**Theorem 14.** Suppose $m : C(K) \rightarrow L(X)$ is a bounded unital homomorphism. Then

1. $\text{apprAlgLat}(m(C(K))) = m(C(K))$, and
2. $\text{AlgLat}(m(C(K))) \cap \text{appr}(m(C(K)))^{cc} = m(C(K))$.

**Corollary 15.** If $\mathcal{B}$ is a bounded Boolean algebra of projections on $X$, then $\text{apprAlgLat}(\mathcal{B}) = \overline{\text{Sp}}(\mathcal{B})$.

### 4. Questions and Comments

1. It was shown by T. A. Gillespie [8] that if $\mathcal{B}$ is a complete Boolean algebra of projections on $X$, and if $\varphi$ is a weak-operator continuous linear
functional on $L(X)$ and $\varepsilon > 0$, then there are an $x$ in $X$ and an $\alpha$ in $X^*$ such that $\varphi(T) = \alpha(Tx)$ for every $T$ in $\overline{\text{sp}} \mathcal{B}$ and $\|x\| \alpha\| \leq (1 + \varepsilon)\|\varphi\|$. As a consequence, Gillespie [8] proved that every weak-operator closed unital subalgebra of $\overline{\text{sp}} \mathcal{B}$ is reflexive. If $\mathcal{B}$ is a complete Boolean algebra of projections on $X$, then must every unital norm closed subalgebra of $\overline{\text{sp}} \mathcal{B}$ be approximately reflexive?

2. The proof of Corollary 13 works whenever $m(C(K))^{\text{cc}} = \text{w-cl}(m(C(K)))$. However, Dieudonné [4] has shown that this is not always the case. In spite of this fact, we conjecture that $m(C(K)) = \text{appr}(m(C(K)))^{\text{cc}}$ always holds. (It holds for Dieudonné's example.) It is sufficient to show that $\text{appr}(m(C(K)))^{\text{cc}} \subseteq \text{w-cl}(m(C(K)))$.

3. There are several questions of interest related to Lemma 5. When is $m^*(A)$ (or its commutant) closed in $L(X^*)$ with respect to the $w^*$-operator topology? When do the operators in $\text{AlgLat}^* m^*(A)$ commute with $m^*(A)$? The answer to these questions is affirmative, if every operator on $X^*$ that commutes with $m^*(C(K))$ also commutes with $m^*(A)$. A sufficient condition is that $m^*(C(K))$ be weak-operator dense in $m^*(A)$. Another sufficient (but not necessary) condition is that the projections in $m^*(A)$ be a complete (in the sense of Bade) Boolean algebra of projections on $X^*$ [11, Theorem 1].

ACKNOWLEDGMENTS

We gratefully acknowledge support from the National Science Foundation (first author) and Tübitak (second author) while this research was undertaken. The first author also thanks the University of New Hampshire for a grant that provided leave time to do research at Middle East Technical University. We also thank Varan for providing a suitable research environment.

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