Isomorphisms of Certain CSL Algebras

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We show that an isomorphism between two reflexive operator algebras on
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tinuous and induces an isomorphism between the lattices. For such algebras with
completely distributive subspace lattices, CDC algebras, the “quasi-spatially”
implemented isomorphisms are shown to be exactly those that preserve the rank of
all finite-rank operators. We present some examples of nonspatial isomorphisms
and discuss some sufficient conditions on the lattices that ensure that all
isomorphisms be spatially implemented. The relationship between isomorphisms
and derivations of CSL algebras is also investigated.

1. PRELIMINARIES

A map from one Banach algebra into another has both an algebraic
color (e.g., is it a homomorphism?) and a topological character (is it
continuous?). If the algebras consist of operators acting on some space,
then the map may have a spatial character as well (is it spatially implemen-
ted?).

In this paper we show that an isomorphism between two reflexive
operator algebras on Hilbert space with commutative subspace lattices is
automatically continuous and induces an isomorphism between the lattices.

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For such algebras with completely distributive subspace lattices, CDC algebras, the “quasi-spatially” implemented isomorphisms are shown to be exactly those that preserve the rank of all finite-rank operators. Finally we present some examples of nonspatial isomorphisms and discuss some sufficient conditions on the lattices that ensure that all isomorphisms be spatially implemented.

The relationship between the algebraic, topological, and spatial properties of maps has been the subject of much study. For example, if \( \mathcal{A} \) is a semi-simple Banach algebra, then any homomorphism onto \( \mathcal{A} \) is automatically continuous, as shown by Johnson in [10]. Kadison and Ringrose [11] have proved that an automorphism \( \rho \) of a C*-algebra must be spatially implemented in case \( \| \rho - I \| < 2 \) (\( I \) being the identity isomorphism). If \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are strictly dense subalgebras of the algebras of all bounded operators on Banach spaces \( X_1 \) and \( X_2 \), respectively, and if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) contain finite-rank operators, then Rickart [18] has shown that any algebraic isomorphism from \( \mathcal{A}_1 \) to \( \mathcal{A}_2 \) is not only continuous, but spatially implemented. Ringrose obtained the same conclusion for isomorphisms between nest algebras [19], and Lambrou [13] has an even stronger result in case the algebras in question are reflexive with lattices that are completely atomic and complemented. A recurrent theme in the above work is the exploitation of rank-one operators.

Laurie and Longstaff [15] have recently obtained a result concerning density of rank-one operators in certain reflexive algebras of operators on Hilbert space, namely, those algebras whose lattice is commutative and completely distributive; we call these CDC algebras. Specifically, their result is that the collection of finite sums of rank-one operators in a CDC algebra is strongly dense in that algebra. We use this result to investigate isomorphisms of CDC algebras.

This work is a direct generalization of Ringrose's results in [19] and to some extent is modeled upon Lambrou's paper on homomorphisms of algebras with Boolean lattices [14]. We would like to thank A. Hopenwasser, C. Laurie, and T. Trent for many enlightening conversations.

Let \( \mathcal{H} \) be a complex separable Hilbert space. A subspace lattice \( \mathcal{L} \) is a strongly closed lattice of orthogonal projections on \( \mathcal{H} \), containing 0 and 1. If \( \mathcal{L} \) is a subspace lattice, \( \text{Alg } \mathcal{L} \) denotes the algebra of all bounded operators on \( \mathcal{H} \) that leave invariant every projection in \( \mathcal{L} \). \( \text{Alg } \mathcal{L} \) is a weakly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \), the algebra of all bounded operators on \( \mathcal{H} \). Dually, if \( \mathcal{A} \) is a subalgebra of \( \mathcal{B}(\mathcal{H}) \), then \( \text{Lat } \mathcal{A} \) is the lattice of all projections invariant for each operator in \( \mathcal{A} \). An algebra \( \mathcal{A} \) is reflexive if \( \mathcal{A} = \text{Alg Lat } \mathcal{A} \) and a lattice \( \mathcal{L} \) is reflexive if \( \mathcal{L} = \text{Lat Alg } \mathcal{L} \). A lattice \( \mathcal{L} \) is a commutative subspace lattice, or CSL, if each pair of projections in \( \mathcal{L} \) commute; \( \text{Alg } \mathcal{L} \) is then called a CSL algebra. All lattices in this paper will be commutative. Since every CSL is reflexive [1], reflexivity of lattices will
not be a problem. A totally ordered (and thus commutative) subspace lattice is called a nest, and the associated reflexive algebra is a nest algebra.

Let \( \mathcal{L} \) be a lattice and let \( E \) be a projection in \( \mathcal{L} \). We define

\[
E = \bigvee \{ F : F \in \mathcal{L}, F \supseteq E \},
\]

\[
E^* = \bigwedge \{ F : F \in \mathcal{L}, F \subseteq E \}.
\]

A lattice \( \mathcal{L} \) is called completely distributive if \( E^*_E = E \) for every \( E \) in \( \mathcal{L} \).

There is a standard lattice-theoretic definition of complete distributivity which Longstaff has shown equivalent to this one \cite{16}. If \( \mathcal{L} \) is completely distributive and commutative, we will call Alg \( \mathcal{L} \) a CDC algebra.

If \( x \) and \( y \) are vectors in \( \mathcal{H} \), we use the notation \( x \otimes y \) for the rank-one operator defined by

\[
(x \otimes y)f = (f, x)y.
\]

The following lemma, due to Longstaff \cite{B}, will get repeated use.

**Lemma 1.1.** The rank-one operator \( x \otimes y \) belongs to Alg \( \mathcal{L} \) if and only if there is a projection \( E \) in \( \mathcal{L} \) such that \( y \in E \) and \( x \in E^* \). (The notation \( E^* \) means \( (E^*)^\perp \).)

Let \( \mathcal{H}_x \) be the linear span of the rank-one operators in Alg \( \mathcal{L} \). Longstaff \cite{17} has shown that if \( \mathcal{H}_x \) is ultraweakly dense in Alg \( \mathcal{L} \), then \( \mathcal{L} \) is completely distributive (even if \( \mathcal{L} \) is noncommutative). Recently, Laurie and Longstaff \cite{15} have proved the converse for commutative \( \mathcal{L} \). This result provides a striking example of the duality between a subspace lattice property and a topological property of the corresponding invariant algebra.

**Theorem 1.2 (Laurie, Longstaff).** Let \( \mathcal{L} \) be a commutative subspace lattice. Then \( \mathcal{L} \) is completely distributive if and only if \( \mathcal{H}_x \) is ultraweakly dense in Alg \( \mathcal{L} \).

By an isomorphism \( \rho : \text{Alg} \mathcal{L}_1 \to \text{Alg} \mathcal{L}_2 \) we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. No assumption is made about the continuity of \( \rho \) in any topology. If \( T \) is a bounded invertible operator and \( \mathcal{L}_1 \) is a subspace lattice, then the collection

\[
\mathcal{L}_2 = \{ \text{ran}(T^{-1}ET) : E \in \mathcal{L} \}
\]

also forms a subspace lattice, and the map \( \rho(A) = TAT^{-1} \) is an isomorphism of Alg \( \mathcal{L}_1 \) onto Alg \( \mathcal{L}_2 \). In this case we say that the isomorphism \( \rho \) is spatially implemented, or simply spatial. A slightly weaker condition is that \( \rho \) be quasi-spatial; in this case we drop the assumption...
that $T$ be bounded but we require that $T$ be one-to-one with dense domain $\mathcal{D}$, that $\mathcal{D}$ be an invariant linear manifold for $\text{Alg} \mathcal{L}$, and that

$$\rho(A) T f = T A f$$

for all $A$ in $\text{Alg} \mathcal{L}$, and $f \in \mathcal{D}$. Any quasi-spatial isomorphism is automatically continuous in the norm topology [13].

We end this section with a lemma that will be used repeatedly.

**Lemma 1.3.** Let $\mathcal{L}$ be commutative and completely distributive. Then

$$\sqrt{\{ E : E \in \mathcal{L} \text{ and } E \neq I \}} = I$$

and

$$\sqrt{\{ E : E \in \mathcal{L} \text{ and } E \neq 0 \}} = I.$$

**Proof.** Let $E_0 = \sqrt{\{ E : E \in \mathcal{L} \text{ and } E \neq I \}}$. By definition, $E_0^\ast = \wedge \{ E : E \leq E_0 \}$. But if $E \leq E_0$ then $E - I$. Thus $E_0^\ast = I$ and since $\mathcal{L}$ is completely distributive, $E_0 = E_0^\ast = I$.

Now let $E_1 = \wedge \{ E : E \in \mathcal{L} \text{ and } E \neq 0 \}$. Evidently, $E_1 = \wedge \{ E : E \leq 0 \} = 0^\ast = 0$ by complete distributivity. Thus

$$\sqrt{\{ E : E \in \mathcal{L}, E \neq 0 \}} = (\wedge \{ E : E \in \mathcal{L}, E \neq 0 \})^\ast = I.$$

2. **A Reduction Formula**

Before proceeding, we present a result which greatly simplifies any discussion of isomorphisms of CSL algebras. Specifically, an isomorphism of two algebras is the composition of a spatially implemented isomorphism and an automorphism of a special kind. The result, due essentially to Ringrose [19, Theorems 4.1 and 4.2], was originally formulated for nest algebras and its proof is included here for the reader's convenience. Alan Hopenwasser noticed the distilled version presented here. We first note that any CSL algebra $\text{Alg} \mathcal{L}$ contains a maximal abelian self-adjoint subalgebra (masa) which contains $\mathcal{L}$.

**Theorem 2.1.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and suppose that

$$\phi: \text{Alg} \mathcal{L}_1 \to \text{Alg} \mathcal{L}_2$$

is an algebraic isomorphism. Let $\mathcal{M}$ be a masa contained in $\text{Alg} \mathcal{L}_1$. Then there exists a bounded invertible operator $Y: \mathcal{H}_1 \to \mathcal{H}_2$ and an automorphism $\rho: \text{Alg} \mathcal{L}_1 \to \text{Alg} \mathcal{L}_1$ such that
(i) $\rho(M) = M$ for all $M \in \mathcal{M}$

(ii) $\phi(A) = Y\rho(A)Y^{-1}$ for all $A \in \text{Alg } \mathcal{L}_1$.

**Proof.** First, observe that $\mathcal{M}$ is maximally abelian as a subalgebra of $\text{Alg } \mathcal{L}_1$. Consequently, $\phi(\mathcal{M})$ is a maximally abelian subalgebra of $\text{Alg } \mathcal{L}_2$, and is therefore norm-closed (in fact, weakly closed). If we let $\phi_{\mathcal{M}}$ be the restriction of $\phi$ to $\mathcal{M}$ then $\phi_{\mathcal{M}}$ is an algebraic isomorphism from the masa $\mathcal{M}$ onto the Banach algebra $\phi(\mathcal{M})$. Letting $X$ be the maximal ideal space of $\mathcal{M}$, we have that $\mathcal{M}$ is isomorphic to $C(X)$. Define a new norm $\| \cdot \|_0$ on $\mathcal{M}$ by $\|M\|_0 = \|\phi(M)\|$; we have $\|MN\|_0 = \|\phi(MN)\| \leq \|\phi(M)\| \|\phi(N)\| = \|M\|_0 \|N\|_0$. By a theorem of Kaplansky [12], the operator norm is minimal among all submultiplicative norms on $C(X)$, and so $\|\phi_{\mathcal{M}}(M)\| \geq \|M\|$. It now follows from the closed graph theorem that both $\phi_{\mathcal{M}}$ and $\phi_{\mathcal{M}}^{-1}$ are bounded.

Let $\Omega$ be the class of all closed and open subsets of $X$. To each $G$ in $\Omega$ there corresponds a projection in $\mathcal{M}$, say $E_G$, given by multiplication by the characteristic function of $G$. Let $F_G = \phi(E_G)$. The function $G \mapsto F_G$ is a bounded, finitely-additive idempotent-valued measure on $\Omega$, such that $F_0 = 0$, $F_X = 1$ and $F_{G_1 \wedge G_2} = F_{G_1}F_{G_2}$. Moreover, $\|F_G\| = \|\phi_{\mathcal{M}}(E_G)\| \leq \|\phi_{\mathcal{M}}\| \|E_G\| = \|\phi_{\mathcal{M}}\|$. By a theorem of Dixmier [4] there exist a Hilbert space $\mathcal{H}_0$ and an invertible operator $P : \mathcal{H}_2 \to \mathcal{H}_0$ such that for all $G$ in $\Omega$, the operator $K_G = PF_GP^{-1}$ is a projection on $\mathcal{H}_0$.

Now let $\chi(A) = P\phi(A)P^{-1}$ for each $A$ in $\text{Alg } \mathcal{L}_1$. It is easy to see that the range of $\chi$ is the algebra $\text{Alg}\{\chi(E) : E \in \mathcal{L}_1\}$ of operators on $\mathcal{H}_0$, and that $\chi$ is an isomorphism. Each projection in $\mathcal{M}$ has the form $E_G$ for some $G$. Thus $\chi(E_G) = K_G$ is self-adjoint. Since linear combinations of projections are dense in $\mathcal{M}$, we have $\chi(M)^* = \chi(M^*)$ for each $M$ in $\mathcal{M}$.

Let $\chi_{\mathcal{M}}$ be the restriction of $\chi$ to $\mathcal{M}$. Then $\chi_{\mathcal{M}}$ is a bounded *-isomorphism of $\mathcal{M}$ onto the Banach *-algebra $\chi(\mathcal{M})$. We assert that $\chi(\mathcal{M})$ is actually a masa. Suppose that $B \in \chi(\mathcal{M})$. Then, in particular, $B$ commutes with $\chi(E)$ for every $E$ in $\mathcal{L}_1$, so that $B \in \text{Alg}\{\chi(E) : E \in \mathcal{L}_1\}$. Thus $\chi^{-1}(B)$ makes sense and $\chi^{-1}(B) \in \mathcal{M}$. Since $\mathcal{M}' = \mathcal{M}$ it follows that $B \in \chi(\mathcal{M})$.

Thus $\chi_{\mathcal{M}}$ is a *-isomorphism of masas, and consequently [4] can be implemented by a unitary operator $U : \mathcal{H}_0 \to \mathcal{H}_1$, where $\chi_{\mathcal{M}}(M) = U^*MU$. Finally, define the isomorphism $\rho : \text{Alg } \mathcal{L}_1 \to \text{Alg } \mathcal{L}_1$ by

$$\rho(A) = U\chi(A)U^*.$$ 

Then $\rho$ is an automorphism of $\text{Alg } \mathcal{L}_1$. If $M \in \mathcal{M}$, 

$$\rho(M) = U\chi(M)U^* = U\chi_{\mathcal{M}}(M)U^* = U(U^*MU)U^* = M.$$
Moreover,
\[
\rho(A) = U\phi(A)U^* = U\rho(A)P^{-1}U^* = (UP\phi(A)(UP)^{-1}.
\]
Set \( Y = UP \) and the result is proved.

Remark 1. This theorem holds for an isomorphism \( \phi: \mathcal{A}_1 \to \mathcal{A}_2 \), where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are not necessarily reflexive, but are assumed only to contain masas. In fact, one need not even assume that \( \mathcal{A}_1, \mathcal{A}_2 \) are closed. Consequently, in such cases, to prove continuity or the spatial character of \( \phi \), one needs only do so for \( \rho \). The fact that \( \rho(E) = E \) for all \( E \in \mathcal{L} \) makes \( \rho \) much easier to deal with than \( \phi \).

Remark 2. There is a theorem due to Wermer [20] which might be used in place of the theorem of Dixmier employed in the proof of Theorem 2.1. Specifically the theorem says that if \( \mathcal{S} \) is a \( \sigma \)-complete bounded Boolean algebra of projections, then there is a bounded invertible operator \( S \) so that \( SQS^{-1} \) is self-adjoint for each \( Q \in \mathcal{S} \). We are indebted to the referee for calling this theorem to our attention.

**Corollary 2.2.** Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be commutative subspace lattices and suppose that \( \text{Alg } \mathcal{L}_1 \) and \( \text{Alg } \mathcal{L}_2 \) are algebraically isomorphic. If \( \mathcal{L}_1 \) is completely distributive, then so is \( \mathcal{L}_2 \).

It should be observed that this corollary can be proved directly, without recourse to Theorem 2.1. One simply observes that if \( E \in \mathcal{L}_1 \), then \( \phi(E) \) is an idempotent in \( \text{Alg } \mathcal{L}_2 \), and the projection onto the range of \( \phi(E) \) is invariant for all \( A \in \text{Alg } \mathcal{L}_2 \); consequently, the projection lies in \( \mathcal{L}_2 \), and the rest is simple verification.

### 3. Automatic Continuity

In [2], Christensen proves that derivations on CSL algebras are automatically norm-continuous. A variation of this proof works well for isomorphisms.

**Theorem 3.1.** Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be commutative subspace lattices on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, and let \( \phi: \text{Alg } \mathcal{L}_1 \to \text{Alg } \mathcal{L}_2 \) be an algebraic isomorphism. Then \( \phi \) is uniformly bicontinuous.

**Proof.** Let \( \mathcal{M} \) be a masa in \( \text{Alg } \mathcal{L}_1 \), containing \( \mathcal{L}_1 \). From Theorem 2.1 we immediately assume that \( \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L} \) and \( \rho(M) = M \) for each \( M \in \mathcal{M} \). For each projection \( E \) in \( \mathcal{M} \), define the map \( \phi_E: \text{Alg } \mathcal{L} \to \text{Alg } \mathcal{L} \) by \( \phi_E(A) = \phi(A)E \). For some projections \( E \) in \( \mathcal{M} \), it may be that \( \phi_E \) is continuous; in particular if \( \phi_E \) is continuous the theorem is proved. Let
\( d = \{ E \in \mathcal{M} : E \text{ is a projection and } \phi_E \text{ is norm-continuous} \}. \) \( d \) is nonempty because of the zero projection.

Observe that if \( A \in \text{Alg } \mathcal{L}_1 \) is fixed, then

\[
\| \phi_E(A) \| \leq \| \phi(A) \|
\]

so the family of maps \( \{ \phi_E : E \in d \} \) is a pointwise bounded family of bounded maps. By the uniform boundedness principle, there exists \( K > 0 \) such that

\[
\| \phi_E(A) \| \leq K \| A \|
\]

for all \( A \in \text{Alg } \mathcal{L}'_1 \), and all \( E \in d \). It is now easy to see that the set \( d \) is a strongly closed family of projections.

If \( E, F \in d \) then obviously \( EF \in d \); moreover, \( E \vee F = E + F - EF \) and thus \( \phi_{E \vee F} = \phi_E + \phi_F - \phi_{EF}. \) Consequently \( E \vee F \in d \) and \( d \) is a strongly closed lattice of projections. Therefore, \( d \) contains a largest element, say \( E^\perp_0. \) We assert that \( E^\perp_0 \) is a finite-dimensional projection. Suppose not; then since \( \mathcal{M} \) is a masa there exists a sequence \( \{ F_i \} \) of pairwise orthogonal projections in \( \mathcal{M} \) such that \( \sum F_i = E^\perp_0 \). For each \( i, \) the map \( \phi_{F_i} \) cannot be bounded, so there is an operator \( A_i \in \text{Alg } \mathcal{L}'_1 \) with \( \| A_i \| \leq 2^{-i} \) and \( \| \phi_{F_i}(A_i) \| \geq i. \)

Let \( A = \sum A_i F_i. \) Then \( \| A \| \leq 1 \) and

\[
i \leq \| \phi_{F_i}(A_i) \| = \| \phi(A_i) F_i \| = \| \phi(A) F_i \|
\]

which provides the desired contradiction.

Likewise, by considering the maps \( \phi^F(A) = E \phi(A) E^\perp_0, \) we can obtain a maximal \( E_i \) for which \( \phi^F \) is continuous, and show that \( E_i \) is finite-dimensional. Thus, there exist finite collections of rank-one projections in \( \mathcal{M}, \) say \( E^0_0, E^0_1, \ldots, E^0_n, \) and \( E^1_1, E^1_2, \ldots, E^1_m, \) such that \( E^0_i = \sum_{j=0}^i E^0_j \) and \( E^1_i = \sum_{j=1}^m E^1_j. \) On the other hand, the map

\[
A \mapsto E^i_i \phi(A) E^i_0 = \sum_{i,j} E^i_j \phi(A) E^0_i
\]

fails to be continuous. Hence, at least one of the maps, say \( A \mapsto E^i_j \phi(A) E^0_i, \) also fails. Let \( E^i_j = e \otimes e \) and \( F^0_i = f \otimes f, \) \( \| e \| = \| f \| = 1. \) Then

\[
\| E^i_j \phi(A) E^0_i \| = \| \phi(E^i_j A E^0_i) \|
\]

\[
= \| \phi((Ae, f) e \otimes f) \|
\]

\[
= \| (Ae, f) \| \| \phi(e \otimes f) \|
\]

\[
\leq \| A \| \| \phi(e \otimes f) \|.
\]
This shows that the map \( A \mapsto E_1, \phi(A) E_0 \) is indeed continuous and provides the contradiction which finishes the proof.

**Remark.** For CDC algebras, a different argument, making use of rank-one operators, shows that isomorphisms are also continuous, where the topology in question is the weak operator topology on both algebras. The facts concerning weak–weak continuity for the more general class of CSL algebras are not known. The above proof also works for some nonreflexive algebras as well when the algebras contain masas, so that the decomposition of Theorem 2.1 applies.

### 4. Spatial Implementation

It is clear that a spatially implemented isomorphism preserves the property of being rank-one. The fact that the converse holds in the nest algebra and atomic Boolean situations provides the key ingredient in the work of Ringrose [19] and Lambrou [13]. As we have mentioned, an isomorphism between two CSL algebras with completely distributive lattices may fail to preserve rank-oneness, and thus fail to be spatial. However, quasi-spatial implementation and preservation of every finite rank are equivalent; this fact is the central result of the paper.

**Theorem 4.1.** Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be commutative subspace lattices on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, and let \( \mathcal{L}_1 \) be completely distributive. Let

\[
p: \text{Alg } \mathcal{L}_1 \rightarrow \text{Alg } \mathcal{L}_2
\]

be an algebraic isomorphism. The following are equivalent:

(i) \( p \) is quasi-spatial, implemented by a closed, injective linear transformation \( T: \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) whose range and domain are dense.

(ii) \( p \) preserves the rank of every finite-rank operator: that is, \( \text{rank}(p(R)) = \text{rank } R \) for all finite-rank \( R \).

**Lemma 4.2.** Let \( \mathcal{L} \) be a commutative subspace lattice. There exists a vector \( y_0 \) such that for every \( F \) in \( \mathcal{L} \), \( \{ AF y_0 : A \in \text{Alg } \mathcal{L} \} \) is dense in \( F \).

**Proof.** This lemma follows from Arveson's representation theorem [1, Theorem 1.3.1]; however, we shall present a direct proof. \( \mathcal{L} \) is contained in a maximal abelian von Neumann algebra \( \mathcal{M} \), which has a separating vector \( y_0 \). Let \( y_F = F y_0 \) and suppose that for \( G \in \mathcal{L} \), \( G \leq F \) and \( y_F \in G \). Then \( G y_0 = GF y_0 = Gy_F = y_F = F y_0 \). Since \( F \) and \( G \) lie in \( \mathcal{M} \), and \( y_0 \) separates \( \mathcal{M} \),
$G = F$. Now let $F_0$ be the closure of the linear manifold $\{Ay_F : A \in \text{Alg } \mathcal{L}\}$.
$F_0$ is invariant for each $B$ in $\text{Alg } \mathcal{L}$, and thus $F_0 \in \mathcal{L}$ ($\mathcal{L}$ being reflexive).
On the other hand, $F_0$ is obviously the smallest projection containing $y_F$
and invariant under $\text{Alg } \mathcal{L}$. Thus $F_0 = F$, as required.

Remark. We could just as easily have made this argument using the lattice
$\mathcal{L}^\perp = \{F^\perp : F \in \mathcal{L}\}$ and the algebra $\text{Alg}(\mathcal{L}^\perp) = (\text{Alg } \mathcal{L})^\ast = \{A^\ast : A \in \text{Alg } \mathcal{L}\}$. We would then have a vector $x_0$ with the property that
for each $G \in \mathcal{L}$, $\{A^\ast G \cdot x_0 : A \in \text{Alg } \mathcal{L}\}$ is dense in $G^\perp$. For reasons that will
become clear, we denote by $x_F$ not the vector $F^\perp x_0$, but rather $F^\perp x_0$.
Thus, for each $F \in \mathcal{L}$, $\{A^\ast x_F : A \in \text{Alg } \mathcal{L}\}$ is dense in $F^\perp$ (if $F^\perp = 0$,
$x_F = 0$).

For the remainder of this section, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\rho$ will be fixed and will
satisfy the hypotheses of Theorem 4.1, and $y_F$ and $x_F$ will be as above. By
Corollary 2.2 we have that $\mathcal{L}_2$ is completely distributive and by Theorem
2.1 we assume that $\mathcal{L}_1 = \mathcal{L}_2$ ( = $\mathcal{L}$, for convenience) and $\rho(M) = M$ for all
$M$ in the masa $\mathcal{M}$. Since the implication (i) $\Rightarrow$ (ii) is easy, we focus attention
on proving (ii) $\Rightarrow$ (i). Thus we assume that $\rho$ preserves the rank of
every finite-rank operator.

Note first that for $y \in F$, $x \in F^\perp$, $\rho(x \otimes y)$ has rank one and so there exist
$f$, $g$ in $K$ such that $\rho(x \otimes y) = f \otimes g$. The next lemma establishes that $f$
can be chosen to depend only on $x$, and $g$ only on $y$.

**Lemma 4.3.** Let $F \in \mathcal{L}$ be fixed, with $F \neq I$. Then there exist bounded
operators $T$, $S$, with $T : F \to F$, $S : F^\perp \to F^\perp$ such that for all $x \in F^\perp$ and $y \in F$,
$\rho(x \otimes y) = Sx \otimes Ty$.

**Proof.** Let $x_F$, $y_F$ be as in Lemma 4.2 and let $\rho(x_F \otimes y_F) = f \otimes g$. ($f$ and
g are determined up to reciprocal scalar multiples. Choose any one.)
Then $f \otimes g = \rho(x_F \otimes y_F) = \rho(x_F \otimes Fy_F) = \rho(F(x_F \otimes y_F)) = \rho(F)(f \otimes g) = f \otimes \rho(F)g = f \otimes Fg$. Thus $Fg = g$ and so $g \in F$. Similarly, $f \in F^\perp$.

We now define $T$ on vectors of the form $Ay_F$, with $A \in \text{Alg } \mathcal{L}$, by con-
sidering the following equalities:

$$\rho(x_F \otimes Ay_F) = \rho(A(x_F \otimes y_F)) = \rho(A)(f \otimes g) = f \otimes \rho(A)g.$$ 

The equalities show that if $Ay_F = By_F$, then $\rho(A)g = \rho(B)g$. Thus, if
$y = Ay_F$ we write $Ty = \rho(A)g$ and $T$ is well defined. The same equalities
now show that $\rho(x_F \otimes y) = f \otimes Ty$, for $y$ of the form $Ay_F$. Now $\rho$ is con-
tinuous and therefore bounded by Theorem 3.1, and so

$$\|Ty\| \leq \frac{1}{\|f\|} \|f \otimes Ty\| = \frac{1}{\|f\|} \|\rho(x_F \otimes y)\| \leq \frac{\|\rho\|}{\|f\|} \|x_F\| \|y\|,$$
so \( T \) is bounded and can thus be extended to the closure of \( \{ A y_F; A \in \text{Alg} \mathcal{L} \} \), that is, to \( F \). The equation

\[
\rho(x_F \otimes y) = f \otimes Ty
\]

will continue to hold by continuity of \( \rho \) and \( T \). Furthermore, since \( g \in F \) and \( \rho(A) \in \text{Alg} \mathcal{L} \), \( T \) maps \( F \) into \( F \).

Now let \( A \in \text{Alg} \mathcal{L} \) and define \( S A * x_P = p(A) * $ \) in exactly the same way we show that \( S \) is well defined, bounded, and extends to all of \( F^\perp \).

We have, for \( A, B \in \text{Alg} \mathcal{L} \),

\[
\rho(B^* x_F \otimes A y_F) = \rho(A(x_F \otimes y_F) B) = \rho(A)(f \otimes g) \rho(B) = \rho(B)^* f \otimes \rho(A) g = SB^* x_F \otimes T A y_F.
\]

The equation \( \rho(x \otimes y) = S x \otimes T y \) now follows by continuity.

**Lemma 4.4.** With \( F \) still fixed, the vectors \( f \) and \( g \) can be chosen so that there is a positive number \( x \) such that \( 1/||\rho|| \leq x \leq ||\rho^{-1}|| \) and

(i) \( (x/||\rho^{-1}||) ||y|| \leq ||Ty|| \leq x ||\rho|| ||y|| \) for all \( y \in F \), and

(ii) \( (x/||\rho^{-1}||) ||x|| \leq ||Sx|| \leq x ||\rho|| ||x|| \).

Furthermore, the operator \( T \) maps \( F \) onto \( F \) and \( S \) maps \( F^\perp \) onto \( F^\perp \).

**Proof.** We have used only the density properties of \( x_F \) and \( y_F \). Thus, those vectors may be chosen with any nonzero norm. Furthermore, as noted, the vectors \( f \) and \( g \) may be adjusted by reciprocal constants, since for any real number \( \lambda \),

\[
f \otimes g = \left( \frac{1}{\lambda} f \right) \otimes (\lambda g).
\]

We have already seen that \( ||Ty|| \leq ||\rho||(x_F/||f||) ||y|| \). We also have that \( \rho^{-1}(f \otimes Ty) = x_F \otimes y \), so that

\[
||\rho^{-1}|| ||f|| ||Ty|| \geq ||x_F|| ||y||
\]

and thus \( ||Ty|| \geq (1/||\rho^{-1}||) (||x_F||/||f||) ||y|| \).

Similarly, for each \( x \in F^\perp \),

\[
\frac{1}{||\rho^{-1}||} \frac{||y_F||}{||g||} ||y|| \leq ||Sx|| \leq ||\rho|| \frac{||y_F||}{||g||} ||y||.
\]

It is clear that \( f \) and \( g \) can be adjusted so that \( ||y_F||/||g|| = ||x_F||/||f|| \). We denote this number by \( \alpha \). Since \( ||x_F \otimes y_F||/||f \otimes g|| = ||x_F|| ||y_F||/||f|| ||g|| = \alpha^2 \), we know that \( 1/||\rho|| \leq \alpha^2 \leq ||\rho^{-1}|| \) and conditions (i) and (ii) are verified.

To prove that \( T \) maps \( F \) onto \( F \) we note that since \( T \) is bounded below we need only show that the range of \( T \) is dense in \( F \). Let \( P \) be the projec-
tion onto the closure of the set \( \{ TA_y, A \in \text{Alg } \mathcal{L} \} \). It suffices to show that \( P = F \). Since \( TA_y = \rho(A) g \) and \( \rho(\text{Alg } \mathcal{L}) = \text{Alg } \mathcal{L} \), \( P \) is invariant for each operator in \( \text{Alg } \mathcal{L} \) and so \( P = F_0 \) for some \( F_0 \in L \). In fact it is clear that \( F_0 \) is the smallest projection in \( \mathcal{M} \) containing \( g \). Since \( g \in F, F_0 \leq F \). On the other hand, \( F_0 = \rho(F_0) \), so \( g = \rho(F_0) g = T F_0 y_f \) and since \( T \) is one-to-one, \( F_0 y_f = y_f \). Thus \( y_f \in F_0 \), but \( F \) is the smallest projection in \( \mathcal{L} \) containing \( y_f \). Thus \( F \leq F_0 \) and thus \( F_0 = F \). The operator \( S \) is treated in the same way.

The operators \( T \) and \( S \) were constructed with the projection \( F \) fixed. We now refer to them as \( T_F \) and \( S_F \) and try to fit together the \( T_F \)'s and \( S_F \)'s into operators \( T \) and \( S \) defined on the whole space \( \mathcal{H} \) such that

\[
\rho(x \otimes y) = S x \otimes T y
\]

whenever \( x \otimes y \in \text{Alg } \mathcal{L} \). Note that if \( F_0 \neq I \), if \( E < F \), and if \( y \in E \) and \( x \in F_0 \), then also \( y \in F \) and \( \rho(x \otimes y) = S_F x \otimes T_F y \). On the other hand, \( x \in F_0 \leq F \), so \( \rho(x \otimes y) = S_F x \otimes T_F y \). Thus there exists a complex number \( \lambda \) such that \( T_F y = \lambda T_F y \) and \( S_F x = \bar{\lambda} S_F x \). Since \( x \) and \( y \) may vary independently, \( \lambda \) does not depend on \( x \), \( y \) but only on \( E \) and \( F \). We call it \( \hat{\lambda}_{E,F} \) and have

\[
T_F = \hat{\lambda}_{E,F} T_F | E \quad \text{and} \quad S_F = \bar{\hat{\lambda}}_{E,F} S_F | F_0.
\]

Let \( \mathcal{F} \) be the collection of all projections \( F \) in \( \mathcal{L} \) such that \( F \neq 0 \) and \( F_0 \neq I \). Suppose that \( E \) and \( F \) lie in \( \mathcal{F} \) and that \( E \) and \( F \) are comparable; that is, either \( E \leq F \) or \( F \leq E \). In the former case, \( \hat{\lambda}_{E,F} \) has already been defined; in the latter, define

\[
\hat{\lambda}_{E,F} = \frac{1}{\hat{\lambda}_{F,E}}.
\]

Thus \( \hat{\lambda}_{E,F} \) is defined whenever \( E \) and \( F \) are comparable, and it is easy to check that \( \lambda_{E,G} = \hat{\lambda}_{E,F} \lambda_{F,G} \) whenever each pair from \( \{ E, F, G \} \) is comparable.

We define a chain from \( E \) to \( F \) to be a finite sequence of projections \( F_0, F_1, \ldots, F_n \), each in \( \mathcal{F} \), such that \( F_0 = E \), \( F_n = F \), and such that \( E_k \) is comparable to \( E_{k+1} \) for each \( k = 0, 1, \ldots, n - 1 \). If \( E, F \in \mathcal{F} \), and if there is a chain from \( E \) to \( F \), we would like to define \( \hat{\lambda}_{E,F} \) to be \( \hat{\lambda}_{E,F} = \hat{\lambda}_{E_0} \hat{\lambda}_{E_1} \hat{\lambda}_{E_2} \cdots \hat{\lambda}_{E_{n-2} E_{n-1}} \hat{\lambda}_{E_{n-1} E_n} \). Since there may be more than one chain from \( E \) to \( F \), we need to ask whether such a product is well defined; the following lemma is the necessary one. If \( \{ E_0, E_1, \ldots, E_n \} \) is a chain and if \( E_0 = E_n \) we call the chain a cycle of length \( n \). We refer to the product

\[
\hat{\lambda}_{E_0} \hat{\lambda}_{E_1} \hat{\lambda}_{E_2} \cdots \hat{\lambda}_{E_{n-2} E_{n-1}} \hat{\lambda}_{E_{n-1} E_n}
\]

as the \( \lambda \)-product of the cycle (or chain).
Lemma 4.5. Let \( \{E_0, E_1, \ldots, E_n\} \) be a cycle in \( \mathcal{F} \). Then

\[
\lambda_{E_0} E_1 \lambda_{E_2} E_3 \cdots \lambda_{E_{n-2}} E_{n-1} \lambda_{E_{n-1}} E_n = 1.
\]

Proof. We proceed by contradiction. Suppose that the product above is not always 1 for every cycle, and let \( n \) be the smallest integer for which there is a cycle of length \( n \) yielding a \( \lambda \)-product which is not 1. It is easy to check that \( n \) must be larger than 3. Let \( \{E_0, \ldots, E_n\} \) be a cycle whose \( \lambda \)-product is not 1; any shorter cycle has \( \lambda \)-product 1.

First, we observe that the inclusions in the chain must alternate, that is, if \( E_{j-1} < E_j \), then \( E_j > E_{j+1} \). If, on the contrary, \( E_{j-1} < E_j < E_{j+1} \), or if \( E_{j-1} > E_j > E_{j+1} \), we would have

\[
\lambda_{E_{j-1}} E_j \lambda_{E_j} E_{j+1} = \lambda_{E_{j-1}} E_{j+1},
\]

so that the product \( \prod_{i=0}^{n-1} \lambda_{E_i} E_{i+1} \) can be written as a \( \lambda \)-product of a shorter chain, and would therefore be 1. Thus, the inclusions alternate, and as a result, the integer \( n \) must be even. We can represent the situation by a lattice drawing, Fig. 1 (here \( n = 8 \), \( E_8 = E_0 \), and

![Figure 1](image-url)

means that \( F < E \). Since the inclusions alternate we can assume (by cyclically re-numbering if necessary) that each odd-numbered projection is less than the two adjacent ones, and each even-numbered projection is larger than the two adjacent ones:

\[
E_{2k+1} < E_{2k}, \quad E_{2k+1} < E_{2k+2}
\]

for all appropriate \( k \), and of course \( E_{n-1} < E_n = E_0 \).
Now suppose that two odd-numbered projections have nonzero intersection; again cyclically renumbering if necessary, we suppose that one of the projections is $E_1$:

$$E_1 \cap E_m = E \neq 0, \quad 3 \leq m \leq n - 1, \ m \text{ odd}.$$ 

The lattice picture has the form depicted in Fig. 2. The cycles $\{E_2, E_3, \ldots, E_{m-1}, E_m, E, E_2\}$ and $\{E_0, E, E_{m+1}, E_{m+2}, \ldots, E_{n-1}, E_0\}$ each have length less than $n$ (the lengths are $m$ and $n - m + 1$, respectively), and thus

$$\lambda_{E_1 E_2} \lambda_{E_2 E_3} \cdots \lambda_{E_{m-1} E_m} \lambda_{E_m E} \lambda_{E E_2} = 1$$

and

$$\lambda_{E_0 E} \lambda_{E E_{m+1}} \lambda_{E_{m+2} E} \cdots \lambda_{E_{n-1} E_0} = 1.$$

Since $\lambda_{E E_2} = \lambda_{E E_3} \lambda_{E_1 E_2}$, $\lambda_{E_0 E} = \lambda_{E_0 E_3} \lambda_{E_1 E_2}$, and $\lambda_{E_{m+1} E} = \lambda_{E_{m+1} E_m} \lambda_{E_m E}$, we have

$$\lambda_{E_1 E_3} \cdots \lambda_{E_{m-1} E_m} \lambda_{E_m E} \lambda_{E E_1} \lambda_{E_1 E_2} = 1$$

and

$$\lambda_{E_0 E_1} \lambda_{E_1 E_3} \cdots \lambda_{E_{m-1} E_m} \lambda_{E_m E} \lambda_{E E_{m+1}} \lambda_{E_{m+2} E} \cdots \lambda_{E_{n-1} E_0} = 1.$$

It follows easily from these equations and the fact that $\lambda_{F G} = \lambda_{G F}^{-1}$, that

$$\lambda_{E_0 E_1} \lambda_{E_1 E_2} \cdots \lambda_{E_{n-1} E_0} = 1,$$

in contradiction to our assumption. Thus, every pair of odd-numbered projections has zero intersection.

Next, suppose that two even-numbered projections have span $F$, and that $F \neq I$ (i.e., $F \in \mathcal{F}$). An argument analogous to the one above demonstrates that the $\lambda$-product of $\{E_0, E_1, \ldots, E_{n-1}, E_0\}$ is 1, again a contradiction. Thus, $(E_{2k} \vee E_{2m}) = I$ for each appropriate $k, m$. By Lemma 3
of [9], \((E_{2k}) \vee (E_{2m}) = I\) and thus \((E_{2k})^\perp \cap (E_{2m})^\perp = 0\). Thus, for all appropriate \(k, m\) we assume that

\[ E_{2k+1} \cap E_{2m+1} = 0 \]

\((\ast)\)

and

\[ (E_{2k})^\perp \cap (E_{2m})^\perp = 0. \]

\((\ast\ast)\)

We can now proceed with the proof of the lemma, by forming a finite-rank operator as follows. For each even-numbered projection \(E_{2k}\), let \(0 \neq x_{2k} \in (E_{2k})^\perp\). For each \(E_{2k+1}\), let \(0 \neq y_{2k+1} \in E_{2k+1}^\perp\). Since \(E_{2k+1} < E_{2k}\), \(y_{2k+1} \in E_{2k}\) and so \(x_{2k} \otimes y_{2k+1}\) is a rank-one operator in Alg \(\mathcal{L}\); likewise, so is \(x_{2k} \otimes y_{2k+1}\). Now let

\[
R = x_0 \otimes (y_{n-1} - y_1) + x_2 \otimes (y_1 - y_3)
+ x_4 \otimes (y_3 - y_5) + \cdots + x_{n-4} \otimes (y_{n-5} - y_{n-3})
+ x_{n-2} \otimes (y_{n-3} - y_{n-1}).
\]

Because of conditions \((\ast)\), and because \(\mathcal{L}\) is commutative, \(E_{2k+1} < E_{2m+1}^\perp\) if \(k \neq m\). Thus the collection \(\{y_1, y_3, \ldots, y_{n-1}\}\) is an orthogonal set of vectors; likewise, so is \(\{x_0, x_2, \ldots, x_{n-2}\}\). Because of the latter fact, the rank of \(R\) is the dimension of the space spanned by the \(n/2\) vectors \(\{y_{n-1} - y_1, y_1 - y_3, y_3 - y_5, \ldots, y_{n-3} - y_{n-1}\}\), which is the same as the rank of the matrix

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\]

the rank is easily seen to be \((n/2) - 1\), by row reduction.

Recall that the isomorphism \(\rho\) preserves rank; hence \(\rho(R)\) has rank \((n/2) - 1\) as well. On the other hand, since \(E_{2k+1} < E_{2k}\), we have

\[
\rho(x_{2k} \otimes (y_{2k-1} - y_{2k+1})) = S_{E_{2k}} x_{2k} \otimes T_{E_{2k}} (y_{2k-1} - y_{2k+1})
= S_{E_{2k}} x_{2k} \otimes (T_{E_{2k}} y_{2k-1} - T_{E_{2k}} y_{2k+1})
= S_{E_{2k}} x_{2k} \otimes (\lambda_{E_{2k}} T_{E_{2k}} y_{2k-1} - \lambda_{E_{2k}} T_{E_{2k}} y_{2k+1})
= S_{E_{2k}} x_{2k} \otimes (\lambda_{E_{2k}} T_{E_{2k}} y_{2k-1} - \lambda_{E_{2k}} T_{E_{2k}} y_{2k+1}).
\]
Since $T_{E_{2k}}, y_{2k} \in E_{2k-1}$, we see that the collections $\{T_{E_{1}}, y_{1}, T_{E_{3}}, y_{3}, \ldots, T_{E_{n-1}}, y_{n-1}\}$ and likewise $\{S_{E_{0}}, x_{0}, \ldots, S_{E_{n-2}}, x_{n-2}\}$ are orthogonal sets of non-zero vectors (recall that $T_{E}$ and $S_{E}$ are one-to-one). For notational convenience, let $T_{E_{2k+1}}, y_{2k+1} = v_{2k+1}$, $S_{E_{2k}}, x_{2k} = u_{2k}$, and $\lambda_{E_{i}E_{j}} = \lambda_{ij}$ if $|i - j| = 1$. Thus the collection of $u$'s and the collection of $v$'s are each orthogonal sets and since $\rho(R)$ has rank $(n/2) - 1$ it must be that the vectors

$$\{\lambda_{0,n-1} v_{n-1} - \lambda_{01} v_{1}, \lambda_{21} v_{1} - \lambda_{23} v_{3}, \ldots, \lambda_{n-2,n-1} v_{n-1}\}$$

span a space of dimension exactly $(n/2) - 1$. Consequently the matrix

$$
\begin{pmatrix}
\lambda_{0,n-1} & -\lambda_{01} & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_{21} & -\lambda_{23} & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & \vdots & \vdots \\
0 & 0 & \lambda_{n-4,n-2} & -\lambda_{n-4,n-3} & \cdots & 0 & \lambda_{n-2,n-1} \\
-\lambda_{n-2,n-1} & 0 & \cdots & 0 & \lambda_{n-4,n-2} & \cdots & \lambda_{n-2,n-1}
\end{pmatrix}
$$

has determinant 0. Thus

$$0 = \lambda_{0,n-1} \lambda_{21} \cdots \lambda_{n-2,n-1} - \lambda_{01} \lambda_{23} \cdots \lambda_{n-2,n-1},$$

or

$$\lambda_{01} \lambda_{23} \lambda_{45} \cdots \lambda_{n-2,n-1} = \lambda_{0,n-1} \lambda_{21} \cdots \lambda_{n-2,n-1}.$$ 

Since $\lambda_{ij} = \lambda_{ji}^{-1}$, we have

$$\lambda_{01} \lambda_{12} \lambda_{23} \lambda_{34} \cdots \lambda_{n-2,n-1} \lambda_{n-1,n-2} = 1$$

which is the desired result.

With this lemma in hand, we now define $\lambda_{E_{1}E_{2}} = \lambda_{E_{2}E_{1}} \lambda_{E_{1}E_{2}} \cdots \lambda_{E_{n-1}E_{n-1}} \lambda_{E_{n-1}F}$, whenever there is a chain connecting $E$ to $F$. We know that $\lambda_{E_{1}E_{2}}$ will turn out to be the same for any other chain $E, E_{1}, E_{2}, \ldots, E_{n-1}, F$ from $E$ to $F$, by applying the lemma to the cycle $E, E_{1}, \ldots, E_{n-1}, F, E_{n-1}, \ldots, E_{2}, E_{1}, E$. In a nest, any two projections can be connected by a chain of length $n = 1$. Other lattices, unfortunately, may require chains of arbitrary length to connect two projections. Indeed, there may be no chain connecting certain pairs. For instance, consider the lattice $\{0, E, E_{0}^{\perp}, I\}$, where $E$ is any projection. It is clear that no chain connects $E$ to $E_{0}^{\perp}$. Notice that in this case the algebra decomposes as a direct sum, and any isomorphism acts, in effect, on each summand separately. It turns out that
the only way in which two projections can fail to be connectable by a chain is for this sort of direct sum to appear (see Lemma 4.8).

For later reference, we want to keep up with the size of \( \lambda_{EF} \), as the length of chain necessary to connect \( E \) to \( F \) varies. Recall that \( \mathcal{F} \) denoted the collection of all \( E \) in \( \mathcal{L} \) such that \( E \neq 0 \) and \( E \neq I \). Fix \( E \) in \( \mathcal{F} \) and let \( \mathcal{G}^n_E = \{ F \in \mathcal{F} : F \text{ can be connected to } E \text{ by a chain of length } k, \text{ where } k \leq n \} \). Let \( \mathcal{G}_E = \bigcap_n \mathcal{G}^n_E \).

**Lemma 4.6.** If \( F \in \mathcal{G}^n_E \), then

\[
(\| \rho \| \| \rho^{-1} \| )^{3n/2} \leq | \lambda_{EF} | \leq (\| \rho \| \| \rho^{-1} \| )^{3n/2}.
\]

**Proof.** Suppose that \( E_1 \supseteq E \) and that \( y \in E_i \). By Lemma 4.4 we have

\[
\| \rho \|^{-1/2} \| \rho^{-1} \|^{-1} \| y \| \leq \| T_{E_i} y \| \leq \| \rho^{-1/2} \| \rho^{-1} \|^{-1} \| y \|,
\]

and the same inequality for \( T_{E_j} \). Thus,

\[
| \lambda_{E_i, E_j} | = \frac{\| T_{E_i} y \|}{\| T_{E_j} y \|} \leq \frac{\| \rho^{-1/2} \| \rho^{-1} \|^{-1} \| y \|}{\| \rho^{-1/2} \| \rho^{-1} \|^{-1} \| y \|} = (\| \rho \| \| \rho^{-1} \| )^{3/2}.
\]

Likewise \( \lambda_{E_i, E_j} \geq (\| \rho \| \| \rho^{-1} \| )^{-3/2} \). The inequality in the lemma is now obtained from the fact that

\[
\lambda_{EF} = \lambda_{E_1 E_2} \cdots \lambda_{E_{n-1} E_n}.
\]

The quantity \( \| \rho \| \| \rho^{-1} \| \) is no smaller than 1, so we do not expect the \( \lambda_{EF} \) to be universally bounded over all of \( \mathcal{G}_E \); indeed, later examples will show that they need not be.

**Lemma 4.6.** Let \( E \in \mathcal{F} \). Then there exist linear transformations \( T_0 \), with range and domain both dense in \( \bigvee \{ F : F \in \mathcal{G}_E \} \), and \( S_0 \), with range and domain both dense in \( \bigvee \{ F^\perp : F \in \mathcal{G}_E \} \), such that \( \rho(x \otimes y) = S_0 x \otimes T_0 y \) whenever there is \( F \in \mathcal{G}_E \) for which \( y \in F \) and \( x \in F^\perp \).

**Proof.** To each \( F \) in \( \mathcal{G}_E \) we have associated operators \( T_F \) and \( S_F \) such that

\[
\rho(x \otimes y) = S_F x \otimes T_F y,
\]

whenever \( x \in F^\perp \) and \( y \in F \).

Let \( \tilde{T}_F = \lambda_{EF} T_F \) and \( \tilde{S}_F = \lambda_{EF} S_F \). Since \( \lambda_{EF} \lambda_{EF} = 1 \) we have \( \rho(x \otimes y) = \tilde{S}_F x \otimes \tilde{T}_F y \) for appropriate \( x \) and \( y \). Observe that if \( F, G \in \mathcal{G}_E \) and \( G \leq F \), then \( \lambda_{EG} = \lambda_{EF} \lambda_{FG} \). Reason: if \( \{ E, E_1, ..., E_n, F \} \) is a chain from \( E \) to \( F \) then \( \{ E, E_1, ..., E_n, F, G \} \) is a chain from \( E \) to \( G \). Thus, if \( y \in G \), we have \( \tilde{T}_G y = \lambda_{EG} T_G y = \lambda_{EF} \lambda_{FG} T_G y \). On the other hand, by definition of
\[ \lambda_{GF} \text{ we have } T_G y = \lambda_{GF} T_F y, \text{ so } \tilde{T}_G y = \lambda_{EF} \lambda_{GF} T_F y = \lambda_{EF} T_F y = \tilde{T}_F y. \]

Thus, if \( G < F \), \( \tilde{T}_G \) and \( \tilde{T}_F \) agree on \( G \). Let \( \mathcal{M} \) be the nonclosed linear span of the \( G \) in \( \mathcal{G}_E \); that is, \( \mathcal{M} = \{ y_1 + \cdots + y_n : \text{each } y_i \text{ lies in some } F \in \mathcal{G}_E \} \).

Define a linear transformation \( T_0 \) on \( \mathcal{M} \) by \( T_0(F) = \tilde{T}_F \). \( T_0 \) is well defined by the coherence of the \( \tilde{T}_F \). Since \( \tilde{T}_F \) maps \( F \) onto \( F \), \( T_0 \) maps \( \mathcal{M} \) onto \( \mathcal{M} \).

Similarly, define \( S_0(F) = \tilde{S}_F \). If \( y \in F \), \( x \in F_\perp \) for \( F \in \mathcal{G}_E \), we have \( \rho(x \otimes y) = S_0 x \otimes T_0 y \), and the proof is complete.

**Lemma 4.8.** For each \( E \in \mathcal{F} \), let \( G_E = \sqrt{\{ F : F \in \mathcal{G}_E \} } \). If \( E, E' \in \mathcal{F} \), then either \( G_{E'} = G_E \) or \( G_{E'} \cap G_E = \emptyset \).

**Proof.** Note that if \( F \in \mathcal{G}_E \) then \( \mathcal{G}_{E'} = \mathcal{G}_E \) (if there is a chain connecting \( E \) to \( E' \) and one connecting \( E' \) to \( F \), then there is one connecting \( E \) to \( F \)), whence \( G_{E'} = G_E \). Suppose that \( E' \notin \mathcal{G}_E \); then, for any \( F \in \mathcal{G}_{E'} \), \( F \notin \mathcal{G}_E \). For such an \( F' \), consider the projection \( EF' \). If \( EF' \neq 0 \), then \( EF' \in \mathcal{G}_{E'} \cap \mathcal{G}_E = \mathcal{G}_{E'} \cap \mathcal{G}_E \). Since this is impossible it must be that \( EF' = 0 \).

Likewise, if \( F \in \mathcal{G}_{E'} \), \( FF' = 0 \) and hence \( G_{E'} G_E = G_E G_{E'} = 0 \).

**Proposition 4.9.** If \( \text{Alg } \mathcal{L} \) is irreducible, then there is a closed, densely defined linear transformation \( T \) such that the domain \( I \) of \( T \) is an invariant linear manifold for \( \text{Alg } \mathcal{L} \), \( \ker T = \{ 0 \} \), and such that

\[ \rho(A) T y = T A y, \]

for all \( y \) in the domain of \( T \).

**Proof.** The irreducibility and Lemma 4.8 guarantee that there is only one \( G_E \). By Lemma 4.7, there exist densely defined transformations \( T_0 \) and \( S_0 \) such that \( \rho(x \otimes y) = S_0(x \otimes T_0 y) \) whenever \( x \otimes y \in \text{Alg } \mathcal{L} \). Recall that the domain of \( T_0 \), which we denote by \( \mathcal{D}(T_0) \), is the (nonclosed) linear span of the projections \( F \) for which \( F \neq I \). Let \( F \in \mathcal{F} \) and suppose that \( y \in F \), \( x \in F_\perp \). Then for any \( A \in \text{Alg } \mathcal{L} \), \( A y \in F \) and we have

\[ S_0 x \otimes \rho(A) T_0 y = \rho(A)(S_0 x \otimes T_0 y) = \rho(A) \rho(x \otimes y) \]

\[ = \rho(A(x \otimes y)) = \rho(x \otimes A y) = S_0 x \otimes T_0 A y. \]

Since, for \( x \neq 0 \), \( S_0 x \neq 0 \), we have

\[ \rho(A) T_0 y = T_0 A y, \]

for any \( y \) in \( F \), and thus for any \( y \) in \( \mathcal{D}(T_0) \). In particular, the equation

\[ (v, x) T_0 y = (x \otimes T_0 y) v = T_0(x \otimes y) v = \rho(x \otimes y) T_0 v \]

\[ = (S_0 x \otimes T_0 y) T_0 v = (T_0 v, S_0 x) T_0 y \]

reveals that \( (v, x) = (T_0 v, S_0 x) \) for all \( v \in \mathcal{D}(T_0) \), \( x \in \mathcal{D}(S_0) \). In consequence, \( T_0 \) and \( S_0 \) are closable and \( \mathcal{D}(T_0^*) \) contains the range of \( S_0 \), which is dense.
in \( \mathcal{H} \). Let \( T = T_0^* \) be the closure of \( T_0 \). Then \( \mathcal{D}(T) \) is dense and the fact that \( \ker T_0^* = (\text{ran } T_0^*)^\perp = \{0\} \) shows that \( T \) is one-to-one. We must show that \( I \) also implements \( \phi \). Suppose that \( y \in \mathcal{D}(T) \). Then there is a sequence of vectors \( y_n \in \mathcal{D}(T_0) \) such that \( y_n \to y \), and \( \{ T_0 y_n \} \) is a Cauchy sequence, which, by definition, converges to \( Ty \). If \( A \in \text{Alg } \mathcal{L} \) then \( Ay_n \to Ay \), \( Ay_n \in \mathcal{D}(T_0) \), and

\[
\lim T_0 Ay_n = \lim \rho(A)T_0 y_n = \rho(A) Ty.
\]

Consequently, \( Ay \in \mathcal{D}(T) \) and, by the closure of the operator \( T \), \( \rho(A) Ty = TAy \). This concludes the proof.

We are finally in a position to prove the central result, Theorem 4.1. Recall that we have tacitly assumed the hypothesis that \( \mathcal{L} \) and \( \mathcal{M} \) are commutative subspace lattices, with \( \mathcal{L} \) completely distributive, and that \( \rho : \text{Alg } \mathcal{L} \to \text{Alg } \mathcal{M} \) is an algebraic isomorphism.

**Proof of Theorem 4.1.** (ii) \( \Rightarrow \) (i). As usual, we can immediately suppose that \( \mathcal{L}_1 = \mathcal{L}_2 \) and that \( \rho(M) = M \) for all \( M \) in a masa \( \mathcal{M} \) which contains \( \mathcal{M} \). Let \( E \in \mathcal{L} \) be a projection with \( E^\perp \neq 0 \), and consider the collection \( \{ G_{E} \} \) defined before Lemma 4.6. If \( G_{E} = \mathcal{F} \) then Proposition 4.9 is in effect. If \( G_{E} \neq \mathcal{F} \) then there exists \( F \in \mathcal{F} \) with \( F \notin G_{E} \), such that \( G_F G_E = G_E G_F = 0 \).

We proceed in this way, creating a sequence \( \{ G_{E} \} \) of mutually orthogonal projections in \( \mathcal{L} \). We shall suppress the \( E \) and write simply \( G_i \). The separability of \( \mathcal{H} \) guarantees that there are no more than countably many \( G_E \), and, because \( \bigvee \{ F : F \in \mathcal{F} \} = 1 \), \( \bigvee G_i = 1 \). Thus, \( G_i^\perp = \bigvee_{i < j} G_j \) is also in \( \mathcal{L} \), and the algebra \( \text{Alg } \mathcal{L} \) can be written as the direct sum \( \bigoplus_{i} \text{Alg}(G_i, \mathcal{L} G_i) \). (This is a slight abuse of notation; the projections in \( G_i \mathcal{L} G_i \) are meant to act on the range of \( G_i \), not on \( \mathcal{H} \).) Since \( \rho(G_i) = G_i \), the algebra \( \text{Alg}(G_i, \mathcal{L} G_i) \) is invariant under \( \rho \) and \( \rho \) can also be written as a direct sum: \( \rho = \bigoplus_{i} \rho_i \), where \( \rho_i : \text{Alg}(G_i, \mathcal{L} G_i) \to \text{Alg}(G_i, \mathcal{L} G_i) \) is an isomorphism of \( \text{Alg}(G_i, \mathcal{L} G_i) \). Clearly, each \( \rho_i \) preserves rank and by Proposition 4.9, there is a closed map \( T_i \) with domain and range dense in \( G_i \), such that \( \rho(A) T_i = T_i A \) for all \( A \in \text{Alg}(G_i, \mathcal{L} G_i) \). Set \( T = \bigoplus T_i \). It is a simple matter to check that \( T \) satisfies all the requirements, and the proof is complete.

One would also like to know under what conditions the implementing operator can be chosen to be bounded, with bounded inverse. For each \( F \in \mathcal{F} \) the operator \( T_F \) is bounded and can in fact be chosen to have any norm whatever. Since \( \tilde{T}_F = \lambda_{EF} T_F \), it is therefore necessary to know when the set \( \{ \lambda_{EF} \} \) is bounded both above and below. By Lemma 4.6, this will be the case provided there is some \( n \) for which \( \mathcal{W}_E^n = \mathcal{W}_E \). If \( \text{Alg } \mathcal{L} \) splits as a direct sum, there is no harm in adjusting the norm of \( T \) on each irreducible
piece. In order to state the next theorem conveniently, we define the following "distance" function on \( \mathcal{F} \): if \( E, F \in \mathcal{F} \), let \( \delta(E, F) \) be the smallest \( n \) such that \( F \in \mathcal{F}_n \); otherwise, let \( \delta(E, F) = \infty \).

**Theorem 4.10.** Assume the conditions of Theorem 4.1 and suppose in addition that there is a positive number \( K \) such that, for all \( E, F \in \mathcal{F} \), either \( \delta(E, F) \leq K \) or \( \delta(E, F) = \infty \). Then \( \rho \) is implemented by a bounded invertible operator.

We omit the details.

5. **Examples**

In this section we present a class of examples to show that some ranks may be preserved while others are not, and that isomorphisms may be quasi-spatial without being spatial.

The algebras \( \mathcal{A}_{2n} \) and \( \mathcal{A}_\infty \) have been discussed at some length in [5, 7]; we refer the reader to those papers for a careful definition. For our purposes it suffices to say that the algebras \( \mathcal{A}_{2n} \) are tridiagonal matrices, of size \( 2n \times 2n \), of the form

\[
\begin{bmatrix}
* & * \\
& * \\
& & * \\
& & & \ddots \\
& & & & * \\
& & & & & * \\
\end{bmatrix}
\]

where all nonstarred entries are 0. The algebra \( \mathcal{A}_{2n} \) is reflexive; the lattice consists of certain diagonal projections and is commutative and completely distributive. The algebra \( \mathcal{A}_\infty \) consists of infinite matrices of the form

\[
\begin{bmatrix}
* & * & * & \cdots \\
& * & * & \ddots \\
& & * & \ddots \\
& & & \ddots \\
\end{bmatrix}
\]
and, once again, the lattice is commutative and completely distributive. In fact, these lattices are the join of two commuting nests (pairwise). Such lattices are defined as width-2 by Arveson [1] and the algebra $\text{Alg } \mathcal{L}$ is then the intersection of two nest algebras. It is somewhat surprising that many properties which hold for nest algebras do not extend to this natural class of CSL algebras.

**Example 5.1.** An isomorphism need not preserve rank. For this example, let $n = 2$ and let $\rho: \mathcal{A}_4 \to \mathcal{A}_4$ be defined by

\[
\begin{bmatrix}
  a & b & 0 & h \\
  0 & c & 0 & 0 \\
  0 & d & e & f \\
  0 & 0 & 0 & g
\end{bmatrix} \mapsto \begin{bmatrix}
  a & b & 0 & -h \\
  0 & c & 0 & 0 \\
  0 & d & e & f \\
  0 & 0 & 0 & g
\end{bmatrix}.
\]

It is easy to check that $\rho$ is an isomorphism. However, the rank of the matrix

\[
\begin{bmatrix}
  0 & 1 & 0 & -1 \\
  0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

is two, whereas the rank of $\rho(A)$ is one.

**Example 5.2.** An isomorphism may preserve rank 1, but not all finite ranks. For this, let $n = 3$ and let $\rho: \mathcal{A}_6 \to \mathcal{A}_6$ be defined similarly:

\[
\begin{bmatrix}
  a & b & 0 & 0 & 0 & m \\
  0 & c & 0 & 0 & 0 & 0 \\
  0 & d & e & f & 0 & 0 \\
  0 & 0 & 0 & g & 0 & 0 \\
  0 & 0 & h & i & j & k \\
  0 & 0 & 0 & 0 & 0 & k
\end{bmatrix} \mapsto \begin{bmatrix}
  a & b & 0 & 0 & 0 & -m \\
  0 & c & 0 & 0 & 0 & 0 \\
  0 & d & e & f & 0 & 0 \\
  0 & 0 & 0 & g & 0 & 0 \\
  0 & 0 & h & i & j & 0 \\
  0 & 0 & 0 & 0 & 0 & k
\end{bmatrix}.
\]

Any rank-one matrix in $\mathcal{A}_6$ must have all its non-zero entries confined to a single row or column; in either case its image under the isomorphism $\rho$ has rank one. However, the matrix
Remark. While it is true that, in CDC algebras, every finite-rank operator can be written as a sum of rank-one operators, it may be necessary to use more terms in the sum than the rank of the operator. For example, the matrix of the last example cannot be written as the sum of two rank-one operators in $\mathcal{A}_0$. For further discussion of this phenomenon, we refer to [9].

Example 5.3. An isomorphism may be quasi-spatial but not spatial. Consider the algebra $\mathcal{A}_\infty$ and let $\rho$ be implemented by the unbounded operator

$$
\begin{bmatrix}
1 \\
2 \\
3 \\
4 \\
0
\end{bmatrix}
$$

In other words, we have

$$
\rho: \begin{bmatrix}
a & b & 0 & 0 & 0 & 0 & \ldots \\
0 & c & 0 & 0 & 0 & 0 & \ldots \\
0 & d & e & f & 0 & 0 & \ldots \\
0 & 0 & 0 & h & 0 & 0 & \ldots \\
0 & 0 & 0 & i & j & k
\end{bmatrix} \to \begin{bmatrix}
a \frac{1}{2}b & 0 & 0 & 0 & 0 & \ldots \\
0 & c & 0 & 0 & 0 & 0 & \ldots \\
0 & \frac{3}{4}d & e & \frac{3}{4}f & 0 & 0 & \ldots \\
0 & 0 & 0 & h & 0 & 0 & \ldots \\
0 & 0 & 0 & \frac{5}{6}i & j & \frac{5}{6}k
\end{bmatrix}.
$$

It is easy to check that no bounded operator can implement $\rho$. Moreover, using arguments like those in [6] one can easily see that any automorphism $\rho: \mathcal{A}_\infty \to \mathcal{A}_\infty$ preserves rank and by Theorem 4.1 must therefore be quasi-spatial. Finally, using arguments as in [6] one can compute the quotient of $\text{Aut } \mathcal{A}_\infty$ by the spatial automorphisms.
6. Guaranteeing Preservation of Rank

In this section we use Theorem 4.1 to provide further sufficient conditions on the lattice \( \mathcal{L} \) to ensure that all isomorphisms are quasi-spatial. These results directly generalize Ringrose's theorem. It may be quite difficult to find necessary and sufficient conditions; some examples of Lambrou [14] suggest that the situation, even for completely distributive lattices, can be very complicated. In view of Theorem 2.1 we restrict attention to masa-fixing automorphisms. The first lemma shows when preservation of rank-one operators guarantees preservation of rank for many other finite-rank operators.

**Lemma 6.1.** Let \( \mathcal{L} \) be a commutative, completely distributive lattice, let \( \mathcal{M} \) be a masa contained in \( \text{Alg} \mathcal{L} \), and let \( \rho: \text{Alg} \mathcal{L} \to \text{Alg} \mathcal{L} \) be an automorphism such that \( \rho(A) = A \) for all \( A \) in \( \mathcal{M} \). Suppose also that for all rank-one \( S \in \text{Alg} \mathcal{L} \), \( \rho(S) \) has rank one. Let \( E \in \mathcal{L} \), with \( E \neq 1 \), and let \( R \) be any finite-rank operator in \( \text{Alg} \mathcal{L} \), such that \( R = RE \). Then the ranks of \( \rho(R) \) and of \( R \) are equal.

*Proof.* Suppose that \( \rho \) preserves the property of being rank-one. Then by Lemma 4.3 there are operators \( T_E \) and \( S_E \), defined on \( E \) and \( E_\perp \), respectively, such that

\[
\rho(x \otimes y) = S_E x \otimes T_E y
\]

whenever \( y \in E \) and \( x \in E_\perp \). \( T_E \) and \( S_E \) are injective, \( T_E \) maps \( E \) onto \( E \), and \( S_E \) maps \( E_\perp \) onto \( E_\perp \). For \( x \in E_\perp \) and \( y \in E \) we have

\[
S_E x \otimes \rho(R) T_E y = \rho(R)(S_E x \otimes T_E y) = \rho(R) \rho(x \otimes y) = \rho(x \otimes R y) = S_E x \otimes T_E R y,
\]

since \( E \) is invariant for \( R \). If \( x \neq 0 \), \( S_E x \neq 0 \), and so for any \( y \in E \), \( \rho(R) T_E y = T_E R y \). Since \( R = RE \) and \( T_E \) is one-to-one, we see that the ranks of \( \rho(R) \) and of \( R \) must be equal.

**Theorem 6.2.** Let \( \rho: \text{Alg} \mathcal{L}_1 \to \text{Alg} \mathcal{L}_2 \) be an automorphism, where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are commutative and \( \mathcal{L}_1 \) is completely distributive. Suppose that \( \mathcal{L}_1 \) contains a net \( \{E_\alpha\} \) of projections such that \( E_\alpha \to 1 \) (strongly) and, for each \( \alpha \), \( E_\alpha \neq 1 \). Then \( \rho \) is quasi-spatial.

*Proof.* As usual, we reduce immediately to the case that \( \mathcal{L}_1 = \mathcal{L}_2 \) and \( \rho \) fixes each operator in a masa \( \mathcal{M} \). In [14], Lemma 3.2, Lambrou shows
that if $E_x \neq I$, and if $R$ has rank one, then $\rho(RE)$ has rank no more than one. Since $E_x \neq 1$, if $R$ has rank one then $\rho(RE_x)$ has rank one when $RE_x \neq 0$.

Let $R = x \otimes y$; then $RE_x = (E_x x) \otimes y$ and $\|R - RE_x\| = \|(x - E_x x) \otimes y\| \to 0$. Thus $\{RE_x\}$ converges to $R$ in norm, and because $\rho$ is norm-continuous, $\rho(RE_x) \to \rho(R)$. Hence $\rho(R)$ has rank one.

We now use Lemma 6.1. Suppose that $R$ is now a finite-rank operator; then the ranks of $\rho(RE_x)$ and of $RE_x$ are equal. Since rank can only be an integer, there must be an $x$ for which rank(RE, $= \text{rank}(R)$ and rank(\rho(RE_x)) = \text{rank}(\rho(R))$. (Note that $\|E - RE_x\| \to 0$ in the same way as before.) Consequently, $\rho$ preserves all finite ranks and, by Theorem 4.1, $\rho$ is quasi-spatial.

We remark that the sufficient condition of Theorem 6.2 is by no means necessary. For instance, an atomic Boolean lattice admits only quasi-spatial automorphisms, but does not in general contain a net $\{E_x\}$ as above.

We show below that this result applies to lattices generated by finitely many commuting independent nests. Recall that an interval from a nest $\mathcal{L}$ is a projection of the form $E - F$ where $E, F \in \mathcal{L}$ and $F \subseteq E$. A collection of nests $\mathcal{L}_1, \ldots, \mathcal{L}_n$ is called independent if, whenever $I_j$ is an interval from $\mathcal{L}_j$, then $\bigcap_{j=1}^n I_j \neq 0$. For example, the two nests used to determine $\mathcal{J}_1$ and $\mathcal{J}_2$ are not independent. Yet for $\mathcal{J}_\infty$ each automorphism is quasi-spatial as noted in Example 5.3. On the other hand, if $\mathcal{L}$ is the tensor product of nests $\mathcal{L}_1$ and $\mathcal{L}_2$ then $\mathcal{L}$ is generated by the independent nests $\mathcal{L}_1 \otimes \mathcal{L}_2$ and $I_1 \otimes \mathcal{L}_2$ [6].

We first require a lemma, which is a sort of converse of Lemma 1.1.

**Lemma 6.3.** Let $\mathcal{L}$ be a commutative subspace lattice, let $E \in \mathcal{L}$, and suppose that $x$ is a vector with the property that, for all $y \in E$, $x \otimes y \in \text{Alg } \mathcal{L}$. Then $x \in F_{\perp}$.

**Proof.** Suppose that $F \in \mathcal{L}$ and that $F \nsubseteq E$. Then there exists a vector $y \in E \setminus F_{\perp}$, $y \neq 0$. Since $x \otimes y \in \text{Alg } \mathcal{L}$ we have

$$F_{\perp}(x \otimes y) F = (Fx) \otimes (F_{\perp} y) = 0.$$  

Since $F_{\perp} y = y \neq 0$, it must be that $Fx = 0$. Now $E_\perp = \bigvee \{F \in \mathcal{L} : F \nsubseteq E\}$, so $E_\perp x = 0$, that is, $x \in E_{\perp}$.

Recall that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two commutative subspace lattices, then $\text{Alg}(\mathcal{L}_1 \lor \mathcal{L}_2) = \text{Alg } \mathcal{L}_1 \cap \text{Alg } \mathcal{L}_2$. The proof of the next lemma is due to Alan Hopenwasser.

**Lemma 6.4.** Let $\{\mathcal{L}_i\}_{i=1}^n$ be a finite collection of commuting independent
nests, and let $\mathcal{L}$ be the lattice generated by $\{\mathcal{L}_i\}_{i=1}^n$. Suppose that $E_i \in \mathcal{L}_i$, $0 \neq E_i \neq I$, and let $E = E_1 \wedge E_2 \wedge \cdots \wedge E_n$. Then
\[ E = (E_1)_- \vee (E_2)_- \vee \cdots \vee (E_n)_-, \]
where the subscript $\_-$ refers to the lattice $\mathcal{L}$ for $E$, and to the lattice $\mathcal{L}_i$ for each $E_i$.

**Proof.** For simplicity we present the proof in the case $n = 2$ and omit the obvious modifications for the general case. So, we have
\[ E = E_1 \wedge E_2, \]
\[ E = \vee \{ F \in \mathcal{L}_1 \vee \mathcal{L}_2 : F \ni E \}, \]
\[ (F_1)_- = \vee \{ F_1 \in \mathcal{L}_1 : F_1 \ni F_i \} = \vee \{ F_1 \in \mathcal{L}_1 : F_1 \prec E_1 \}, \]
\[ (E_2)_- = \vee \{ F_2 \in \mathcal{L}_2 : F_2 \ni E_2 \} = \vee \{ F_2 \in \mathcal{L}_2 : F_2 \prec E_2 \}. \]
First, observe that if $F_1 \in \mathcal{L}_1$ and $F_1 \prec E_1$, then, by independence, the intervals $(E_1 - F_1)$ and $(E_1 - 0)$ intersect nontrivially. The intersection is a subspace of $E_1 \wedge E_2$ but not of $F_1$; consequently, $F_1 \ni E_1 \wedge E_2$. Thus, we have
\[ E = (E_1 \wedge E_2)_-. \]
Similarly, $E \geq (E_2)_-$ and so $E \geq (E_1)_- \vee (E_2)_-.$

The other inclusion follows easily from the preceding lemma. Suppose that $x \in (E_1)_- \wedge (E_2)_- \wedge$ and that $y \in E_1 \wedge E_2.$ In particular, $x \in (E_1)_- \wedge$ and $y \in E_1,$ so $x \otimes y \in \text{Alg } \mathcal{L}_1.$ But, by the same token, $x \otimes y \in \text{Alg } \mathcal{L}_2.$ Thus $x \otimes y \in (\text{Alg } \mathcal{L}_1) \cap (\text{Alg } \mathcal{L}_2) = \text{Alg} (\mathcal{L}_1 \vee \mathcal{L}_2).$ Now apply the lemma to see that $x \in (E_1 \wedge E_2)_-.$ Since $x$ was arbitrary, it follows that
\[ (E_1)_- \wedge (E_2)_- \wedge \leq (E_1 \wedge E_2)_-, \]
which, together with the first inclusion completes the proof.

**Theorem 6.5.** Let $\mathcal{L}_1$ be generated by finitely many commuting independent nests, let $\mathcal{L}_2$ be any commutative subspace lattice, and let $\phi : \text{Alg } \mathcal{L}_1 \to \text{Alg } \mathcal{L}_2$ be an algebraic isomorphism. Then $\phi$ is spatially implemented.

**Proof.** As usual we assume that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ and that $\phi$ is an automorphism whose restriction to a masa $\mathcal{M}$ is the identity. Let
\( \mathcal{L} = \mathcal{L}_1 \vee \cdots \vee \mathcal{L}_n \) where \( \mathcal{L}_j \) is a nest, and, from each \( \mathcal{L}_j \), choose a sequence \( E_j^{(i)} \) converging strongly to the identity, with \( E_j^{(i)} \neq 1 \). Let \( E^{(0)} = E_1^{(0)} \wedge \cdots \wedge E_n^{(0)} \). Then \( (E^{(i)})^\perp = (E_1^{(i)})^\perp \wedge \cdots \wedge (E_n^{(i)})^\perp \), which is nonzero because of the independence of the nests. Moreover, \( E^{(i)} \to 1 \) because multiplication is strongly continuous on bounded sets. Thus we can invoke Theorem 6.2 to see that \( \phi \) is quasispatially implemented.

To finish the proof and show that \( \phi \) is in fact spatially implemented, we use Theorem 4.10. Let \( H = \{E_1 \wedge E_2 \wedge \cdots \wedge E_n: E_i \in \mathcal{L}_i \} \). Then \( H \) generates \( \mathcal{L} \). Consequently, if \( E \in \mathcal{L} \) then there is a projection \( F \in H \) such that \( E \wedge F \neq 0 \), and so \( \delta(E, F) \leq 2 \) (see Theorem 4.10 for the definition of \( \delta \)), and we may assume that \( F = E_1 \wedge \cdots \wedge E_n \) with \( E_j \neq I \). It is easy to see that if \( F, F' \in H \), then \( \delta(F, F') \leq 2 \) (i.e., \( E_1 \wedge E_2 \) and \( E'_1 \wedge E'_2 \) both contain \((E_1 \wedge E') \wedge (E_2 \wedge E'_2)\), which is nonzero by independence of the nests). Hence, if \( E, E' \in \mathcal{L} \) we have \( \delta(E, E') \leq 6 \) and the proof is complete.

### 7. Derivations and Automorphisms

For Banach algebras, derivations and automorphisms are always related. In particular, if \( \delta: \mathcal{A} \to \mathcal{A} \) is a bounded derivation, then the exponential \( e^{\delta} \) is an automorphism, as can be verified by consideration of the infinite series. Suppose now that \( \mathcal{A} = \text{Alg} \mathcal{L} \), with \( \mathcal{L} \) commutative, and such that every isomorphism is quasi-spatially implemented. For each complex number \( z \) the exponential \( e^{z\delta} \) is an automorphism, so there is a possibly unbounded but closed and densely defined linear transformation \( T_z \) so that for each \( A \in \text{Alg} \mathcal{L} \) and each \( x \) in the domain of \( T_z \),

\[
e^{z\delta}(A) T_z x = T_z Ax, \quad (*)\]

If \( \mathcal{L} \) is completely distributive, then the domain of every \( T_z \) contains \( \mathcal{D} \), the (nonclosed) span of \( \{F \in \mathcal{L}: F \neq I\} \). Thus Eq. (*) holds for every \( x \in \mathcal{D} \) and every \( z \in \mathbb{C} \). Assume temporarily that the function \( z \mapsto \delta \) is differentiable, that is, for every \( x \in \mathcal{D} \), \( y \in \mathcal{M} \) the map \( z \mapsto (T_z x, y) \) is an analytic function from \( \mathbb{C} \) into \( \mathbb{C} \). We then take derivatives, holding \( A \) and \( x \) fixed,

\[
(\delta e^{z\delta})(A) T_z x + e^{z\delta}(A)(T_z)'x = (T_z)'Ax.
\]

Evaluate when \( z = 0 \) and observe that we can take \( T_0 = 1 \):

\[
\delta(A) x + AT_0' x = T_0 Ax
\]

where we have denoted \((d/dz)T_z)_{z=0}\) by \( T_0 \). Note that Eq. (*) implies that \( T_0 Ax = AT_0 x \) for all \( x \in \mathcal{D} \). Thus

\[
\delta(A) = T_0 A - AT_0'
\]
and assuming the analyticity of $z \to T_z$, the derivation $\delta$ has been shown to be quasi-spatial, namely $\delta = \text{Ad}(T_0)$.

The transformation $T_z$ is not uniquely determined by Eq. (*) in that $T_z$ can always be adjusted by multiplying by a scalar-valued function; if $T_z$ satisfies (*), so does $f(z) T_z$, where $f: \mathbb{C} \to \mathbb{C}$ is any function at all—not necessarily analytic. We need to find one function $f$ so that the map $z \mapsto f(z) T_z$ is analytic.

To simplify the situation, we look at rank-one operators $A$. Let $EE \in \mathcal{L}$, and let $u \in E^\perp$, $y \in E$. Suppose that some assignment $z \mapsto T_z$ is given; we want to find a function $f: \mathbb{C} \to \mathbb{C}$ so that the map $z \mapsto f(z) T_z$ is analytic. Equation (*) becomes

$$e^{z\delta}(u \otimes y) = T_z(u \otimes y) T_z^{-1}$$

$$= ((T_z^{-1})^* u) \otimes (T_z y).$$

Note that $T_z^*$ makes sense since $T_z$ is closed and densely defined; note also that the domains of $T_z^*$ and $(T_z^{-1})^*$ each contain every $F^\perp$, with $F \in \mathcal{L}$, $F \neq I$. If $u, y$ are held fixed, the function $z \mapsto e^{z\delta}(u \otimes y)$ is clearly analytic. Thus the right-hand side $(T_z^{-1})^* u \otimes T_z y$ is analytic. We want to find a function $f: \mathbb{C} \to \mathbb{C}$ so that $z \mapsto f(z) T_z y$ is an analytic function for each $y$. In fact, such a function exists, but the argument above uses only the fact that the map $z \mapsto f(z) T_z$ is differentiable at $z = 0$, which will allow us to simplify the proof.

**Lemma 7.1.** Let $u: \mathbb{C} \to \mathcal{H}$ and $y: \mathbb{C} \to \mathcal{H}$ be vector-valued functions and suppose that the map from $\mathbb{C}$ into $\mathcal{B}(\mathcal{H})$ defined by $z \mapsto u(z) \otimes y(z)$ is analytic. Suppose that $u(0)$ and $y(0)$ are both nonzero. Then there is a neighborhood $\mathcal{N}$ of the origin in $\mathbb{C}$ and a function $f: \mathcal{N} \to \mathbb{C}$ such that the map $z \mapsto f(z) y(z)$ is analytic on $\mathcal{N}$, and the map $z \mapsto u(z)/f(z)$ is conjugate analytic.

**Proof.** For any $h, k \in \mathcal{H}$ the function

$$z \mapsto ((u(z) \otimes y(z)) h, k) = (h, u(z))(y(z), k)$$

is analytic. In particular, for $h = u(0)$, $k = y(0)$ we have that $\phi(z) = (u(0), u(z))(y(z), y(0))$ is analytic and $\phi(0) = \|u(0)\|^2 \|y(0)\|^2 \neq 0$. Thus there is a neighborhood $\mathcal{N}$ of $z = 0$ in which $(u(0), u(z)) 
eq 0$ and $(y(z), y(0)) \neq 0$.

Let $h$ be any nonzero vector and define

$$m(z) = \frac{(h, u(z))(y(z), y(0))}{(u(0), u(z))(y(z), y(0))};$$
$m$ is the quotient of analytic functions and the denominator is nonzero for $z \in \mathcal{N}$. Thus, for $z \in \mathcal{N}$,

$$m(z) = \left( h, \frac{u(z)}{(u(0), u(z))} \right)$$

is analytic. Since $h$ is arbitrary, the map

$$z \mapsto \frac{u(z)}{(u(0), u(z))}$$

is conjugate analytic. [A map $x: \mathbb{C} \to \mathcal{H}$ is conjugate analytic if, for all $h \in \mathcal{H}$, the function $(h, x(z))$ is analytic.]

Likewise, the map $z \mapsto y(z)/(y(z), y(0))$ is analytic on $\mathcal{N}$. Let $f(z) = 1/(y(z), y(0))$. Then $z \mapsto f(z) y(z)$ is analytic on $\mathcal{N}$ and we have to show that $u(z)/f(z)$ is conjugate analytic. But if $h \in \mathcal{H}$ we have

$$(h, u(z)/f(z)) = \frac{((u(z) \otimes y(z)) h, y(0))}{(f(z) y(z), y(0))}.$$

and the right-hand side is the quotient of analytic functions with denominator nonzero in $\mathcal{N}$. This completes the proof.

Thus we combine the remarks preceding this lemma with the lemma to obtain our final result.

**Proposition 7.2.** For a CDC algebra $\mathcal{A}$, if every automorphism on $\mathcal{A}$ is quasi-spatial, then every derivation of $\mathcal{A}$ into $\mathcal{A}$ is quasi-inner.

**References**