JOURNAL OF FUNCTIONAL ANALYSIS 18, 15–26 (1975)

Asymptotic Convergence of Nonlinear Contraction Semigroups in Hilbert Space*

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Communicated by Tosio Kato

Received January 31, 1974

DEDICATED TO THE MEMORY OF MY SISTER, SUSAN GAIL BRUCK BOPPEL, AND TO HER SON, PRYCE.

Let S be a contraction semigroup on a closed convex subset C of a Hilbert space. If the generator of S satisfies a strengthened monotonicity condition then the weak $\lim_{t\to\infty} S(t)x$ exists for all x in C. As one consequence, the method of steepest descent converges weakly for convex functions in Hilbert space; and it converges strongly for *even* convex functions.

INTRODUCTION

Let S be a nonlinear contraction semigroup on a closed convex subset C of a real Hilbert space H. In this note we introduce a simple condition on the generator A of S (which we call *demipositivity*) sufficient to guarantee the existence of a weak $\lim_{t\to\infty} S(t)x = S(\infty)x$ for each $x \in C$. The mapping $S(\infty)$ so defined satisfies $S(t)\, S(\infty) = S(\infty)\, S(t) = S(\infty)$ for $0 \leqslant t \leqslant \infty$, so that S can in a sense be extended to have an idempotent right end point; $S(\infty)$ is a nonexpansive retraction of C onto the fixed-point set of S. The maximal monotone operators usually cited as examples—$\partial\varphi$ if $\varphi : H \to (-\infty, +\infty]$ is a proper l.s.c. (lower semicontinuous) convex function, and $I - T$ if $T : H \to H$ is nonexpansive—turn out to be demipositive if φ assumes a minimum and T has a fixed point, and hence generate weakly asymptotically convergent contraction semigroups.

Dafermos and Slemrod [6] have recently investigated the asymptotic

* Partially supported by NSF Grant GP-38516.

behavior of S (which may be more complex than simple convergence) under the hypothesis that the ω-limit sets

$$\omega(x) = \bigcap_{s>0} \mathrm{Cl}\{S(t)x : t \geqslant s\}$$

are nonempty. Their proofs rely on compactness arguments. Brézis [1, 2] has investigated the asymptotic convergence of S if S is generated by a subdifferential; his proofs rely on careful estimates of $\|(d/dt)\, S(t)x\|$. Although the questions raised in [1] inspired the present paper, our methods of proof are quite distinct from those of [1, 2, 6].

One of the consequences of our main theorem is that the method of steepest descent for convex functions converges weakly. In the classical sense this means that if $\varphi\colon H \to (-\infty, \infty)$ is a differentiable convex function which assumes a minimum in H and $x(t)$ is a solution of

$$\dot{x}(t) = -\operatorname{grad} \varphi(x(t))$$

on $[0, \infty)$, then the weak $\lim_{t\to\infty} x(t)$ exists and is a minimum point of φ. We actually prove this result in the more general case where $\varphi\colon H \to (-\infty, +\infty]$ is only assumed to be convex and l.s.c.; if φ assumes a finite minimum, then the absolutely continuous solutions of

$$\dot{x}(t) \in -\partial\varphi(x(t))$$

converge weakly to minimum points of φ.

It is apparently unknown, even in the classical case, whether the method of steepest descent for convex functions converges *strongly*. Conditions such as compactness of level sets or the uniform convexity of φ are usually imposed to guarantee strong convergence, but these are unsatisfactory in view of the weak convergence. It is therefore of interest that we establish the strong convergence if φ is also *even*.

1. Weak Asymptotic Convergence

Throughout this paper H will denote a real Hilbert space with inner product $(\cdot, \cdot)$, C a nonempty closed convex subset of H, and S a contraction semigroup on C. (That is, $S = \{S(t)\colon 0 \leqslant t < \infty\}$ is a family of nonexpansive self-mappings of C such that $S(0) = I$, $S(t_1 + t_2) = S(t_1)\, S(t_2)$ for all $t_1, t_2 \in [0, \infty)$, and $S(\cdot)x$ is strongly continuous for each $x \in C$.) "lim" and "$\to$" refer to convergence in the norm topology, while "w-lim" and "$\rightharpoonup$" denote weak convergence. A will be multivalued operator on H, i.e., a subset of $H \times H$.

The following conditions seems to be crucial in guaranteeing weak asymptotic convergence.

DEFINITION. A is *firmly positive* if

(i) $(v, x - y) \geqslant 0$ for all $[x, v] \in A$ and $y \in A^{-1}(0)$, and

(ii) there exists $y_0 \in A^{-1}(0)$ such that $0 \in A(x)$ whenever $v \in A(x)$ and $(v, x - y_0) = 0$.

A is *demipositive* if (i) and (ii) hold and y_0 also satisfies

(iii) the conditions $x_n \rightharpoonup x$, $v_n \in A(x_n)$, (v_n) bounded, and $\lim_n(v_n, x_n - y_0) = 0$ imply $0 \in A(x)$.

THEOREM 1. *Suppose A is demipositive and $x\colon [0, \infty) \to H$ is absolutely continuous and satisfies*

$$x(t) \in D(A) \qquad \textit{for all} \quad t \geqslant 0, \tag{1.1}$$

$$\dot{x}(t) \in -A(x(t)) \text{ a.e.}, \tag{1.2}$$

$$\| \dot{x}(t)\| \in L^{\infty}(0, \infty). \tag{1.3}$$

Then $w\text{-}\lim_{t\to\infty} x(t)$ exists and belongs to $A^{-1}(0)$.

Proof. Choose $M > 0$ and a null set $N \subset [0, \infty)$ such that $\dot{x}(t)$ exists, $\dot{x}(t) \in -A(x(t))$, and $\| \dot{x}(t)\| \leqslant M$, for all $t \in [0, \infty)\backslash N$. For any $y \in A^{-1}(0)$,

$$(d/dt)\, \tfrac{1}{2} \| x(t) - y \|^2 = (\dot{x}(t), x(t) - y) \leqslant 0, \tag{1.4}$$

since $-\dot{x}(t) \in A(x(t))$ and part (i) of the definition is satisfied. Since x is absolutely continuous, so is $\frac{1}{2} \| x(t) - y \|^2$, and (1.4) implies $\| x(t) - y \|$ is decreasing in t. Thus a finite $\lim_{t\to\infty} \| x(t) - y \|$ exists for all $y \in A^{-1}(0)$.

Fix an element y_0 which satisfies part (iii) of the definition and set

$$\begin{aligned} h(t) &= -(\dot{x}(t), x(t) - y_0) \qquad && \text{if} \quad t \in [0, \infty)\backslash N, \\ &= 0 && \text{if} \quad t \in N. \end{aligned}$$

By (1.4), $h \geqslant 0$, and since $\lim_{t\to\infty} \| x(t) - y_0 \|$ exists and $h(t) = -(d/dt)\, \frac{1}{2} \| x(t) - y_0 \|^2$ a.e., we have also $h \in L^1(0, \infty)$.

By a (*)-sequence we shall mean a sequence $(t_n) \subset [0, \infty)\backslash N$ such that $t_n \uparrow +\infty$ and $\lim_n h(t_n) = 0$. Since $h \in L^1(0, \infty)$, (*)-sequences certainly exist. Now we claim the existence (and, necessarily, the uniqueness) of a point x^* such that $x^* \in A^{-1}(0)$ and $x(t_n) \rightharpoonup x^*$ for every (*)-sequence (t_n). Indeed, since $\lim_{t\to\infty} \| x(t) - y_0 \|$ exists, $\{x(t): t \geqslant 0\}$ is weakly sequentially precompact, so it is enough to show that the conditions

$$x(t_n) \rightharpoonup x^*, \qquad x(s_n) \rightharpoonup x^{**} \tag{1.5}$$

for (*)-sequences (s_n) and (t_n) imply $x^* = x^{**} \in A^{-1}(0)$.

Certainly (1.5) implies $x^*, x^{**} \in A^{-1}(0)$. Indeed, $h(t_n) = (v_n, x(t_n) - y_0)$ where $v_n = -\dot{x}(t_n) \in A(x(t_n))$, and (v_n) is bounded (since $\| \dot{x}(t_n)\| \leqslant M$); $\lim_n (v_n, x(t_n) - y_0) = 0$ because (t_n) is a (*)-sequence, so by part (iii) of the definition the w-$\lim_n x(t_n) = x^*$ is in $A^{-1}(0)$. Similarly, $x^{**} \in A^{-1}(0)$.

Now we appeal to a lemma of Opial [8]: if $(w_n) \subset H$, $w_n \rightharpoonup w$, and $w' \neq w$, then

$$\liminf_n \| w_n - w \| < \liminf_n \| w_n - w' \|.$$

In the present situation this means that if $x^* \neq x^{**}$ then

$$\liminf_n \| x(t_n) - x^* \| < \liminf_n \| x(t_n) - x^{**} \|. \tag{1.6}$$

But $x^*, x^{**} \in A^{-1}(0)$ so by our previous remarks both $\lim_{t\to\infty} \| x(t) - x^*\|$ and $\lim_{t\to\infty} \| x(t) - x^{**} \|$ exist. Hence (1.6) implies

$$\lim_{t\to\infty} \| x(t) - x^* \| < \lim_{t\to\infty} \| x(t) - x^{**} \|.$$

This is impossible since the roles of x^* and x^{**} are interchangeable, hence the assumption $x^* \neq x^{**}$ is false. We have shown the existence of $x^* \in A^{-1}(0)$ such that $x(t_n) \rightharpoonup x^*$ for every (*)-sequence (t_n).

Call a sequence $(s_n) \subset [0, \infty)$ an almost-(*)-sequence if $s_n \uparrow +\infty$ and there exists a (*)-sequence (t_n) such that $\lim_n (s_n - t_n) = 0$. Since $\| \dot{x} \| \leqslant M$ a.e., and x is absolutely continuous, $\| x(s_n) - x(t_n)\| \leqslant M \mid s_n - t_n \mid$, so $\lim_n \| x(s_n) - x(t_n)\| = 0$. Thus $x(s_n) \rightharpoonup x^*$ whenever (s_n) is an *almost*-(*)-sequence.

Now let (s_n) be any sequence in $[0, \infty)$ such that $s_n \uparrow +\infty$; we claim that (s_n) has an almost-(*)-subsequence. Given $\delta > 0$ put $P_\delta = \{t \in [0, \infty): t \notin N \text{ and } h(t) < \delta\}$; since $h \in L^1(0, \infty)$, the complement of P_δ has finite measure, hence only finitely many of the open intervals $(s_n - \delta, s_n + \delta)$ can fail to intersect P_δ. That is, for each $\delta > 0$ there exists an integer $m = m(\delta)$ such that for all $n \geqslant m$, there exists $t \in P_\delta$ with $\mid t - s_n \mid < \delta$. The existence of an almost-(*)-subsequence of (s_n) is now obvious.

Finally, suppose w-$\lim_{t\to\infty} x(t) \neq x^*$. Since $\{x(t): t \geqslant 0\}$ is weakly sequentially precompact, there must exist a sequence $s_n \uparrow +\infty$ such that $(x(s_n))$ converges weakly to some vector other than x^*. But (s_n)

has an almost-(*)-subsequence (s_{n_i}), for which $(x(s_{n_i}))$ converges weakly to x^*; the contradiction proves $w\text{-}\lim_{t\to\infty} x(t) = x^*$. We have already noted that $x^* \in A^{-1}(0)$. Q.E.D.

The greatest interest in Theorem 1 attaches to the case where A is maximal monotone, since A then generates a contraction semigroup which is weakly asymptotically convergent.

THEOREM 2. *If the generator of a contraction semigroup S on C is demipositive then for each $x \in C$ the $w\text{-}\lim_{t\to\infty} S(t)x = S(\infty)x$ exists and the operator $S(\infty)$ is a nonexpansive retraction of C onto the fixed-point set of S. Furthermore, $S(\infty)\, S(t) = S(t)\, S(\infty) = S(\infty)$ for all $0 \leqslant t \leqslant \infty$.*

Proof. It is known (e.g., [5]) that there is a one-to-one correspondence between maximal monotone operators A on H and contraction semigroups on closed convex subsets of H; if A is the generator of S then A is maximal monotone, $D(A)$ is dense in C, and for all $x_0 \in D(A)$, $x(t) = S(t)\, x_0$ is the unique absolutely continuous solution of (1.1) and (1.2) for which $x(0) = x_0$. x also satisfies (1.3). If A is demipositive then, in view of Theorem 1, $w\text{-}\lim_{t\to\infty} S(t)\, x_0$ exists and belongs to $A^{-1}(0)$ for each $x_0 \in D(A)$.

For any $x \in C$ and $w \in H$ with $\| w \| = 1$,

$$\begin{aligned} &|(S(t)x - S(s)x, w)| \\ &\quad \leqslant \| S(t)x - S(t)x_0 \| + |(S(t)x_0 - S(s)x_0 , w)| + \| S(s)x_0 - S(s)x \| \\ &\quad \leqslant 2 \| x - x_0 \| + |(S(t)x_0 - S(s)x_0 , w)|. \end{aligned}$$

Thus

$$\limsup_{s,t\to\infty} |(S(t)x - S(s)x, w)| \leqslant 2 \| x - x_0 \|$$

for every $x_0 \in D(A)$. Since $D(A)$ is dense in C, this implies that $\{S(t)x \colon t \geqslant 0\}$ is weakly Cauchy, and hence weakly convergent as $t \to \infty$.

Define $S(\infty)$ by $S(\infty)x = w\text{-}\lim_{t\to\infty} S(t)x$. Clearly $S(\infty)$ maps C into C; since $\| \cdot \|$ is weakly l.s.c. and each $S(t)$ is nonexpansive, $S(\infty)$ is also nonexpansive. We have already noted that $S(\infty)$ maps $D(A)$ into $A^{-1}(0)$; since $A^{-1}(0)$ is closed and convex ([5], Lemma 2.2]) and $S(\infty)$ is continuous, therefore $S(\infty)$ maps C into $A^{-1}(0)$. But $A^{-1}(0)$ is the fixed-point set of S, and $S(\infty)$ acts as the identity on this set; therefore $S(\infty)$ is a nonexpansive retraction of C onto the fixed-point set of S. This proves $S(s)\,S(\infty) = S(\infty)$ for $0 \leqslant s \leqslant \infty$. If $0 \leqslant s < \infty$ then

$$\begin{aligned} S(\infty)\,S(s)x &= w\text{-}\lim_{t\to\infty} S(t)\,S(s)x \\ &= w\text{-}\lim_{t\to\infty} S(t+s)x \\ &= w\text{-}\lim_{t\to\infty} S(t)x = S(\infty)x, \end{aligned}$$

so $S(\infty)\,S(s) = S(\infty)$. Q.E.D.

Fortunately some very useful mappings are demipositive, or are demipositive once they are known to be firmly positive.

THEOREM 3. *If any of the following conditions is satisfied then A is demipositive.*

(a) *A is the subdifferential $\partial\varphi$ of a proper* l.s.c. *convex function $\varphi\colon H \to (-\infty, +\infty]$ which assumes a minimum in H;*

(b) *$A = I - T$, where $T\colon H \to H$ is nonexpansive and has a fixed point;*

(c) *A is maximal monotone, odd, and firmly positive;*

(d) *A is maximal monotone and* int $A^{-1}(0) \neq \varnothing$;

(e) *A is maximal monotone, firmly positive, and weakly closed* (i.e., $x_n \rightharpoonup x$, $v_n \rightharpoonup v$, $v_n \in A(x_n)$ *imply* $v \in A(x)$).

Proof. Operators of any of these types are monotone and hence satisfy part (i) of the definition.

(a) Recall that φ is *proper* if it is not identically $+\infty$, and that $\partial\varphi$ is the multivalued operator defined by

$$\partial\varphi(x) = \{w \in H\colon \varphi(y) \geqslant \varphi(x) + (w, y - x) \text{ for all } y \in H\}.$$

Normalizing, we may assume $0 = \min \varphi$. If y_0 is any minimum point of φ then $0 \in \partial\varphi(y_0)$. Let $x_n \rightharpoonup x$, $v_n \in \partial\varphi(x_n)$, and suppose $\lim_n(v_n, x_n - y_0) = 0$. Then $\lim_n \varphi(x_n) = 0$ because $\varphi(x_n) \geqslant 0$ and

$$0 = \varphi(y_0) \geqslant \varphi(x_n) + (v_n, y_0 - x_n).$$

φ is weakly l.s.c. because it is convex and strongly l.s.c., hence $\varphi(x) \leqslant \liminf_n \varphi(x_n) = 0$. It follows that x is a minimum point of φ, so that $0 \in \partial\varphi(x)$. Part (iii) of the definition (and a fortiori part (ii)) is satisfied, so $\partial\varphi$ is demipositive.

(b) If $T: H \to H$ is nonexpansive then for any $x, y \in H$

$$0 \leqslant \| x - y \|^2 - \| Tx - Ty \|^2 = (x - y + Tx - Ty, (x - y) - (Tx - Ty))$$
$$= (2(x - y) - (Ax - Ay), Ax - Ay),$$

for $A = I - T$, so that

$$(Ax - Ay, x - y) \geqslant \tfrac{1}{2} \| Ax - Ay \|^2.$$

Since $A^{-1}(0)$ is the fixed point set of T, there exists $y_0 \in A^{-1}(0)$ and

$$(Ax, x - y_0) \geqslant \tfrac{1}{2} \| Ax \|^2 \tag{1.7}$$

for all $x \in H$. Let $x_n \rightharpoonup x$ and suppose $\lim_n(Ax_n, x_n - y_0) = 0$; (1.7) implies $\lim_n Ax_n = 0$, and the theorem of Browder [4] that $I - T$ is demiclosed shows $Ax = 0$. Thus A is demipositive.

(c) We may as well assume $y_0 = 0$ in part (ii) of the definition. First, $A^{-1}(0)$ is nonempty, convex, and symmetric about 0 (since A is odd), so $0 \in A^{-1}(0)$. Second, if $v \in A(x)$ and $(v, x) = 0$ then $(v, y_0) \leqslant 0$ for all $y_0 \in A^{-1}(0)$ because the monotonicity of A implies $(v, x - y_0) \geqslant 0$. But since A is odd, also $-y_0 \in A^{-1}(0)$, so that we actually have $(v, y_0) = 0$. Hence $(v, x - y_0) = 0$. If A is firmly positive with respect to y_0 in part (ii) of the definition, then we must have $0 \in A(x)$; so that A is also firmly positive with respect to 0.

Let $x_n \rightharpoonup x$, $v_n \in A(x_n)$, and suppose (v_n) is bounded and $\lim_n(v_n, x_n) = 0$. Without loss of generality we may suppose $v_n \rightharpoonup v$. Since A is odd, $-v_n \in A(-x_n)$, and the monotonicity of A therefore implies

$$(v_n \pm v_m, x_n \pm x_m) \geqslant 0,$$

so that

$$(v_n, x_n) + (v_m, x_m) \geqslant |(v_n, x_m) + (v_m, x_n)|.$$

Letting first $n \to \infty$, then $m \to \infty$, we conclude that $(v, x) = 0$. The conditions $x_n \rightharpoonup x$, $v_n \in A(x_n)$, $v_n \rightharpoonup v$, $(v_n, x_n) \to (v, x)$ $(= 0)$ imply that $v \in A(x)$ (see [3, Lemma 1.2]). Since A is firmly positive with respect to 0 and $(v, x) = 0$ this, in turn, implies $0 \in A(x)$. Therefore A is demipositive with respect to $y_0 = 0$.

(d) Let $y_0 \in \operatorname{int} A^{-1}(0)$, and let $\rho > 0$ be so small that $y \in A^{-1}(0)$ whenever $\| y - y_0 \| \leqslant \rho$. Let $x_n \rightharpoonup x$ and $v_n \in A(x_n)$ and suppose (v_n) is bounded and $\lim_n(v_n, x_n - y_0) = 0$. Without loss of generality we may suppose that $v_n \neq 0$ for all n. Now $y = y_0 + \rho v_n/\| v_n \| \in A^{-1}(0)$ and A is monotone, so $(v_n, x_n - y) \geqslant 0$. Hence

$$(v_n, x_n - y_0) = (v_n, x_n - y) + \rho \| v_n \| \geqslant \rho \| v_n \|.$$

Since $\lim_n(v_n, x_n - y_0) = 0$, we see that $v_n \to 0$ strongly. But since A is maximal monotone the conditions $x_n \rightharpoonup x$, $v_n \to 0$, $v_n \in A(x_n)$ imply $0 \in A(x)$, i.e., A is demipositive.

(e) Let A be firmly positive and suppose $x_n \rightharpoonup x$, $v_n \in A(x_n)$, (v_n) is bounded, and $\lim_n(v_n, x_n - y_0) = 0$ where y_0 is the vector in part (ii) of the definition. Without loss of generality we may suppose $v_n \rightharpoonup v$, where (since A is weakly closed) $v \in A(x)$. Thus

$$\lim_n (v_n, x_n) = (v, y_0) \leqslant (v, x),$$

with $v \in A(x)$; by [3, Lemma 1.2], $\lim_n(v_n, x_n) = (v, x)$, so that $(v, x - y_0) = 0$. Since A is firmly positive with respect to y_0, $0 \in A(x)$, i.e., A is demipositive. Q.E.D.

Remark. Theorem 2 and Theorem 3(d) imply that if S is generated by A and int $A^{-1}(0) \neq \varnothing$, then the w-$\lim_{t\to\infty} S(t)x$ exists. Actually, Brézis [1, Theor. 26] has shown that in this case the *strong* limit exists.

If A is maximal monotone and $(I + A)^{-1}$ is compact then A is easily seen to be weakly closed (in fact, the conditions $x_n \rightharpoonup x$, $v_n \in A(x_n)$, $v_n \rightharpoonup v$ imply $x_n \to x$). Thus if A is also firmly positive, then Theorem 2 implies the existence of a w-$\lim_{t\to\infty} S(t)x$; but since Dafermos and Slemrod [6] have shown that $\{S(t)x: t \geqslant 0\}$ is strongly precompact in this case, the strong $\lim_{t\to\infty} S(t)x$ actually exists.

2. Steepest Descent

Theorem 2, Theorem 3, and a result of Brézis [1] imply that the method of steepest descent converges weakly for convex functions.

THEOREM 4. *Let $\varphi: H \to (-\infty, +\infty]$ be a proper* l.s.c. *convex function which assumes a minimum in H. Then for any $x_0 \in \mathrm{Cl}\, D(\varphi)$ there exists a unique function $x: [0, \infty) \to H$ which is absolutely continuous on $[\delta, \infty)$ for all $\delta > 0$ and which satisfies*

$$x(t) \in D(\partial\varphi) \qquad \text{for all} \quad t > 0, \tag{2.1}$$

$$\dot{x}(t) \in -\partial\varphi(x(t)) \text{ a.e.}, \tag{2.2}$$

$$x(0) = x_0, \tag{2.3}$$

and w-$\lim_{t\to\infty} x(t)$ exists and is a minimum point of φ.

Proof. $\partial\varphi$ is maximal monotone [7] and therefore generates a contraction semigroup on $\text{Cl } D(\partial\varphi)$. But $\text{Cl } D(\partial\varphi) = \text{Cl } D(\varphi)$ [2], where $D(\varphi) = \{x: \varphi(x) < +\infty\}$, and Brézis [1] has shown that $x(t) = S(t)\, x_0$ is the unique continuous solution of (2.1)–(2.3) for $x_0 \in \text{Cl } D(\partial\varphi)$ (not just $x_0 \in D(\partial\varphi)$). By Theorem 2 and Theorem 3(a), $w\text{-}\lim_{t\to\infty} x(t)$ therefore exists and belongs to $(\partial\varphi)^{-1}(0)$ (and hence is a minimum point of φ). Q.E.D.

THEOREM 5. *Let the hypotheses of Theorem* 4 *hold and suppose in addition that* φ *is even. Then the strong* $\lim_{t\to\infty} x(t)$ *exists and is a minimum point of* φ.

Proof. Temporarily fix $t_0 > 0$ and define a function $g: [0, t_0] \to (-\infty, \infty)$ by

$$g(t) = \| x(t)\|^2 - \| x(t_0)\|^2 - \tfrac{1}{2} \| x(t) - x(t_0)\|^2.$$

Clearly $g'(t) = (\dot{x}(t), x(t) + x(t_0))$ a.e.

Since $\varphi(x(s))$ is decreasing in $s > 0$ [2, p. 517], φ is even, and $-\dot{x}(t) \in \partial\varphi(x(t))$ a.e., we have

$$\begin{aligned}\varphi(x(t)) &\geqslant \varphi(x(t_0)) = \varphi(-x(t_0))\\ &\geqslant \varphi(x(t)) + (-\dot{x}(t), -x(t_0) - x(t))\\ &= \varphi(x(t)) + g'(t),\end{aligned}$$

a.e., in $[0, t_0]$. Thus $g'(t) \leqslant 0$ a.e.

Now $x(t) = S(t - \delta)\, x(\delta)$ if $t \geqslant \delta > 0$, and since $x(\delta) \in D(\partial\varphi)$ by (2.1), $x(t)$ is absolutely continuous on $[\delta, t_0]$. Therefore $g(t)$ is absolutely continuous on $[\delta, t_0]$ for every $\delta > 0$, and since $g'(t) \leqslant 0$ a.e., g is decreasing on $(0, t_0]$. Thus $g(t) \geqslant g(t_0) = 0$ if $0 < t \leqslant t_0$.

We have proven

$$\| x(t) - x(t_0)\|^2 \leqslant 2 \| x(t)\|^2 - 2 \| x(t_0)\|^2 \tag{2.4}$$

whenever $0 < t \leqslant t_0$. This implies, first, that $2 \| x(t)\|^2$ is decreasing in t, and hence is convergent as $t \to \infty$; and second, that $\{x(t): t \geqslant 0\}$ is a Cauchy net, and hence converges strongly to some $x^* \in H$. Of course $x^* = w\text{-}\lim_{t\to\infty} x(t)$, which is by Theorem 4 a minimum point of φ. Q.E.D.

Remark. More generally, suppose $0 < t_1 < t_2 < \cdots < t_n$, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset (0, \infty)$, and $\sum \lambda_j = 1$; then by the same reasoning

$$\left(\dot{x}(t), x(t) + \sum \lambda_j x(t_j)\right) \leqslant 0,$$

a.e., in $[0, t_1]$, so $\| x(t) + \sum \lambda_j x(t_j)\|$ is a decreasing function of t on $[0, t_1]$. It follows by induction that $\| \sum \lambda_j x(t_j)\| \geqslant \| x(t_n)\|$. Passing to the limit, we see that for each $T > 0$ the point of $\mathrm{Cl}\,\mathrm{co}\{x(t): 0 \leqslant t \leqslant T\}$ which is closest to 0 is $x(T)$. (2.4) is a special case of this fact since it is equivalent to

$$\| x(t_0)\| \leqslant \| \tfrac{1}{2}x(t) + \tfrac{1}{2}x(t_0)\|.$$

COROLLARY 1. *Suppose* $K: D(K) \subset H \to H$ *is positive and self-adjoint. Then for each* $x_0 \in H$ *the initial value problem*

$$x(t) \in D(K) \qquad \text{for all} \quad t > 0, \tag{2.5}$$

$$\dot{x}(t) = -K(x(t)) \qquad \text{for all} \quad t > 0, \tag{2.6}$$

$$x(0) = x_0 \tag{2.7}$$

has a unique continuous solution $x: [0, \infty) \to H$ *and* $\lim_{t\to\infty} x(t)$ *exists and* $= P_{N(K)}(x_0)$ *(where* $P_{N(K)}$ *is the orthogonal projection of* H *onto the null space of* K*).*

Proof. Put $\varphi(x) = \frac{1}{2}(Kx, x)$ for $x \in D(K)$, $\varphi(x) = +\infty$ for $x \notin D(K)$. Then it is easy to see that φ is an even, proper, l.s.c. convex function on H with $\partial\varphi = K$. By Theorem 5 the problem (2.1)–(2.3) has for each $x_0 \in \mathrm{Cl}\, D(K) = H$ a unique solution x and $\lim_{t\to\infty} x(t)$ exists and belongs to $(\partial\varphi)^{-1}(0) = N(K)$. Equations (2.1)–(2.3) imply (2.5)–(2.7) (the fact that (2.6) is satisfied for *all* $t > 0$ is a well-known feature of linear semigroup theory). Finally,

$$\begin{aligned}(x(t) - x_0, y) &= \int_0^t (\dot{x}(s), y)\, ds = \int_0^t -(K(x(s)), y)\, ds \\ &= \int_0^t -(x(s), Ky)\, ds = 0,\end{aligned}$$

for all $y \in N(K)$, so $x(t) - x_0 \in N(K)^\perp$ for all $t \geqslant 0$. Letting $x^* = \lim_{t\to\infty} x(t)$, we therefore find $x^* - x_0 \in N(K)^\perp$ and $x^* \in N(K)$; so $x^* = P_{N(K)}(x_0)$. Q.E.D.

It is still an open question whether the method of steepest descent converges strongly for convex functions which are not even. Since $w\text{-}\lim_{t\to\infty} x(t)$ exists, one might expect the means $(1/T)\int_0^T x(t)\, dt$ to converge strongly as $T \to \infty$, even if $x(T)$ does not; but in fact these are equivalent.

THEOREM 6. *If the hypotheses of Theorem 4 hold, then*

$$\lim_{T\to\infty} \left\| x(T) - \frac{1}{T}\int_0^T x(t)\,dt \right\| = 0.$$

Proof. Brézis [1] has shown that the solution $x(t)$ of (2.1)–(2.3) satisfies $t \| \dot{x}(t)\|^2 \in L^1(0, \infty)$, and that d^+x/dt exists for all $t > 0$ and $\| d^+x/dt \|$ is decreasing on $(0, \infty)$ (it may be unbounded near 0). We shall write $\dot{x}(t)$ instead of d^+x/dt. Since $\| \dot{x} \|$ is decreasing,

$$\int_{T/2}^T t \| \dot{x}(t)\|^2\,dt \geqslant \left(\int_{T/2}^T t\,dt\right) \| \dot{x}(T)\|^2 = \frac{3}{8} T^2 \| \dot{x}(T)\|^2.$$

Since $t \| \dot{x}(t)\|^2 \in L^1(0, \infty)$, therefore $\lim_{T\to\infty} T \| \dot{x}(T)\| = 0$. Integrating by parts,

$$x(T) - \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^T t\dot{x}(t)\,dt.$$

Since $\lim_{t\to\infty} t\dot{x}(t) = 0$, this implies

$$\lim_{T\to\infty} \left\| x(T) - \frac{1}{T}\int_0^T x(t)\,dt \right\| = 0.$$

Q.E.D.

Note added in proof. The Yosida approximations of a contraction semigroup which has a fixed point are always weakly asymptotically convergent. That is, if S has generator A, for $\lambda > 0$ put $A_\lambda = \lambda^{-1}[I - (I + \lambda A)^{-1}]$ and let S_λ be the semigroup generated by A_λ. It follows from Theorem 3(b) that A_λ is semipositive, and hence $w\text{-}\lim_{t\to\infty} S_\lambda(t)x$ exists and is a fixed point of S for each x in H.

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